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NEIGHBOR SUM DISTINGUISHING TOTAL CHOOSABILITY OF IC-PLANAR GRAPHS

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Abstract

Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane such that every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short. A proper total-k-coloring of a graph G is a mapping $c : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ denote the sum of the color of a vertex v and the colors of all incident edges of v. A total-k-neighbor sum distinguishing-coloring of G is a total-k-coloring of G such that for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$. The least number k needed for such a coloring of G is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. In this paper, it is proved that if G is an IC-planar graph with maximum degree $\Delta(G)$, then $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$, where $ch''_{\Sigma}(G)$ is the neighbor sum distinguishing total choosability of G.

Keywords: neighbor sum distinguishing total choosability, maximum degree, IC-planar graph, Combinatorial Nullstellensatz.

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1. Introduction

All graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For planar graph G, $F(G)$ denotes its face set, $d(v)$ denotes the *degree* of a vertex v in G. The *length* or *degree* of a face f, denoted by $d(f)$, is the length of the boundary walk of f in G. We call v a k-*vertex*, or a k^+-*vertex*, or a k^--*vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call f a k-*face*, or a k^+-*face*, or a k^--*face* if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A proper total-k-coloring of a graph G is a mapping $c : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ be the sum of the color of a vertex v and the colors of all edges incident with v. If for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$, then we say such total-k-coloring a *neighbor sum distinguishing total-k-coloring*, denoted by tnsd-k-coloring for short. The least number k needed for such a coloring of G is the *neighbor sum distinguishing total chromatic number*, denoted by $\chi''_{\Sigma}(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture proposed by Pilśniak and Woźniak [11].

Conjecture 1. *For any graph G, $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$.*

Loeb and Tang [10] proved that this bound was asymptotically correct by showing that $\chi''_{\Sigma}(G) \leq \Delta(G)(1 + o(1))$. Pilśniak and Woźniak [11] proved that Conjecture 1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. With the Combinatorial Nullstellensatz, neighbor sum distinguishing total coloring have been studied widely, see [4–6, 8, 9, 12, 19]

For a given graph G, let $L_x(x \in V \cup E)$ be a set of lists of real numbers and each of size k. The neighbor sum distinguishing total choosability of G is the least number k for which for any specified collection of such lists, there exists a neighbor sum distinguish total coloring with colors from L_x for each $x \in V \cup E$, and we denote it by $ch''_{\Sigma}(G)$. We call such a coloring of G *list neighbor sum distinguish total-k-coloring* and denote it by *ltnsd-k-coloring*. Ding *et al.* [4] proved that for any graph G, $ch''_{\Sigma}(G) \leq 2\Delta(G) + col(G) - 1$, where $col(G)$ is the coloring number of G. Later Ding *et al.* [5] improved the bound to $ch''_{\Sigma}(G) \leq 2\Delta(G) + col(G) - 2$. Recently, Lu *et al.* [20] improved the bound to $ch''_{\Sigma}(G) \leq \max\{\Delta(G) + \left\lfloor \frac{3col(G)}{2} \right\rfloor - 1, 3col(G) - 2\}$. The list neighbor sum distinguish total-k-coloring of some special classes of graphs were also investigated. Graphs with bounded maximum average degree (Yao and Kong [16]); d-degenerate graphs (Yao *et al.* [18]); planar graphs (Qu *et al.* [13], Wang *et al.* [15]).

In this paper, we consider IC-planar graphs and prove the following result.

Theorem 2. *Let G is an IC-planar graph with maximum degree $\Delta(G)$. Then $ch''_{\Sigma}(G) \leq \max\{\Delta(G)+3, 17\}$.*

An *IC-plane graph* is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph G has a drawing in the plane in which every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Albertson [1] in 2008. Setting a conjecture of Albertson [1], Král and Stacho [7] showed that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graph. We call G a 1-planar graph if it can be drawn on a plane such that each edge is crossed by at most one other edge.

2. PRELIMINARIES

Every IC-planar graph G in this paper has been embedded on a plane such that all its crossings are independent and the number of crossings is as small as possible. In other words, we call G an IC-plane graph. *The associated plane graph* $G^{\times}$ of G is obtained by turning all crossings of G into new 4-vertices on a plane. For convenience, a vertex in $G^{\times}$ is called *false* if it is not a vertex of G and *real* otherwise. For a vertex $v \in V(G)$, we use $d_i(v)$ to denote the number of i-vertices which are adjacent to v. One can see that every real vertex in $G^{\times}$ is adjacent to at most one false vertex and incident with at most two false faces in $G^{\times}$.

Lemma 3 [17]**.** *Let G be a 1-plane graph and $G^{\times}$ be its associated plane graph. If $d_G(u) = 3$ and v is a crossing vertex in $G^{\times}$, then either $uv \notin E(G^{\times})$ or uv is not incident with two 3-faces.*

We define that a graph G' is *smaller* than a graph G if $|E(G')| < |E(G)|$. We call a graph *minimal* for a property when no smaller graph satisfies it. Let from now on $G = (V, E)$ be a minimal counterexample to Theorem 2. We set $k = \max\{\Delta(G)+3, 17\}$. For each 5^{-}-vertex $v \in V(G)$, it is obvious that v has at most five neighbors and five incident edges, so v has at most 15 forbidden colors. Since $k \geq 17$, we can first erase the color of vertex v and finally recolor it after arguing. In other words, we may omit the coloring for all 5^{-}-vertices of G in the following discussion.

Theorem 4 (Combinatorial Nullstellensatz [2])**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose the degree $deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ in P is non-zero. Then if $S_1, S_2, \ldots, S_n$*

are subsets of $\mathbb{F}$ *with* $|S_i| > k_i$*, there are* $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ *so that* $P(s_1, s_2, \ldots, s_n) \neq 0$.

Lemma 5 [14]**.** *If* $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ *is of degree* $\leq s_1 + s_2 + \cdots + s_n$*, where* $s_1, s_2, \ldots, s_n$ *are nonnegative integers, then*

$$\left(\frac{\partial}{\partial x_1}\right)^{s_1}\left(\frac{\partial}{\partial x_2}\right)^{s_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{s_n} P(x_1, x_2, \ldots, x_n)$$
$$= \sum_{x_1=0}^{s_1}\cdots\sum_{x_n=0}^{s_n}(-1)^{s_1+x_1}\binom{s_1}{x_1}\cdots(-1)^{s_n+x_n}\binom{s_n}{x_n}P(x_1, x_2, \ldots, x_n).$$

Lemma 6 [13]**.** *Let* L_i *be the sets of real numbers, with* $|L_i| = l_i$*, where* $i = 1, 2, \ldots, p$*. Let* $L = \left\{\sum_{i=1}^{p} x_i \,|\, x_i \in L_i \text{ and } \prod_{1\leq i<j\leq p}(x_i - x_j) \neq 0\right\}$*. Then* $|L| \geq \sum_{i=1}^{p}(l_i - p + 1) - (p - 1) = \sum_{i=1}^{p} l_i - p^2 + 1$.

3. Proof of Theorem 2

3.1. Unavoidable configurations

In the following, we will often delete some edges to get a proper subgraph G' of G, then by the minimality of G, there exists an ltnsd-k-coloring c of G'. Let $W_G(v) = \sum_{e\ni v, e\in E(G)} c(e) + c(v)$. We may extend this coloring c to the whole graph G. For any $x \in V(G) \cup E(G)$, the *available* colors are the remaining colors after excluding the colors of its adjacent edges and vertices in G' from L_x.

Claim 7. *For any vertex* $v \in V(G)$*, it holds that*

$$\sum_{j=1}^{t}[d_j(v)(\Delta(G) + 4 - d(v) - j)] \leq d(v) - 1, \quad (1 \leq t \leq 5).$$

Claim 8. *For any vertex* $v \in V(G)$*,* $d_{2^-}(v) \leq \frac{d_{6^+}(v)-1}{\Delta(G)-d(v)+1}$*. Moreover, if* $d(v) = \Delta(G)$*, then* $d_{2^-}(v) \leq d_{6^+}(v) - 1$*.*

The proof of Claim 7 and 8 are similar to that of Claim 3.1 and Claim 3.2 in [13], we omit it here. By Claim 7, we can easily get the following Corollaries.

Corollary 9. *If* $d(v) = 8$*, then* $d_{5^-}(v) \leq 1$*.*

Corollary 10. *If* $d(v) = 9$*, then* $d_{5^-}(v) \leq 2$*.*

Corollary 11. *If* $d(v) = 10$*, then* $d_{5^-}(v) \leq 3$*.*

Claim 12. *If* $d(v) = 11$ *and* $d_{6^+}(v) \leq 6$*, then* $d_{3^-}(v) \leq 1$*.*

Proof. Suppose to the contrary that v is adjacent to two 3^--vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{11}\}$, $d(v_1) = d(v_2) = 3$ and $d(v_j) \geq 6$, $(6 \leq j \leq 11)$. Consider $G' = G - vv_1 - vv_2$, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_1, vv_2 and recolor vertices v_1, v_2. Let S_1, S_2 be the sets of available colors for vv_1, vv_2, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, $(i = 1, 2)$. By Lemma 6, $|L| \geq |S_1| + |S_2| - 4 + 1 = 7 > 6$. We can choose a pair, say $(x, y) \in S_1 \times S_2$ with $x \neq y$, such that $x + y \notin \{W_G(v_j) - W_G(v) \,|\, 6 \leq j \leq 11\}$. Finally, we can recolor v_1, v_2 to get an ltnsd-k-coloring of G, a contradiction. ■

Claim 13. *If $d(v) = 12$ and $d_{6^+}(v) \leq 6$, then $d_{3^-}(v) \leq 2$.*

Proof. Suppose to the contrary that v is adjacent to three 3^--vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{12}\}$, $d(v_1) = d(v_2) = d(v_3) = 3$ and $d(v_j) \geq 6$, $(7 \leq j \leq 12)$. Consider $G' = G - \{vv_i \,|\, i = 1, 2, 3\}$, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_1, vv_2, vv_3 and recolor vertices v_1, v_2, v_3. Let S_1, S_2, S_3 be the sets of available colors for vv_1, vv_2, vv_3, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, $(1 \leq i \leq 3)$. By Lemma 6, $|L| \geq |S_1| + |S_2| + |S_3| - 9 + 1 = 7 > 6$. We can choose a triple, say $(x, y, z) \in S_1 \times S_2 \times S_3$ with x, y, z distinct colors, such that $x + y + z \notin \left\{W_G(v_j) - W_G(v) | 7 \leq j \leq 12\right\}$. Finally, we can recolor v_1, v_2, v_3 to get an ltnsd-k-coloring of G, a contradiction. ■

By Lemma 5, if $P(x_1, x_2, \ldots, x_n)$ is a polynomial with $deg(P) = n$, $k_1, k_2, \ldots, k_m$ are non-negative integers with $\sum_{i=1}^{m} k_i = n$ and $cp\left(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}\right)$ is the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ in P, then $\frac{\partial^n P}{\partial_{x_1}^{k_1} \cdots \partial_{x_m}^{k_m}} = cp\left(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}\right) \prod_{i=1}^{m} k_i!$. In the following, we use MATLAB to calculate the coefficients of specific monomials. Moreover, we will list the codes in Appendix.

Claim 14. *Every 5^--vertex is not adjacent to 7^--vertex in G.*

Proof. Suppose to the contrary that there exists a 5^--vertex u adjacent to a 7^--vertex v. Without loss of generality, we assume that $d(u) = 5$, $d(v) = 7$, $N(u) = \{v, u_1, \ldots, u_4\}$, $N(v) = \{u, v_1, \ldots, v_6\}$. Consider $G' = G - uv$, then G' admits an ltnsd-k-coloring c. Now we will recolor the vertices u, v and color the edge uv. Let S_1, S_2, S_3 be the sets of available colors for u, uv, v, respectively. Notice that the colors in $\{c(uu_i) \,|\, 1 \leq i \leq 4\} \cup \{c(u_i) \,|\, 1 \leq i \leq 4\}$ are forbidden for u, the colors in $\{c(uu_i) \,|\, 1 \leq i \leq 4\} \cup \{c(vv_i) \,|\, 1 \leq i \leq 6\}$ are forbidden for uv, and the colors in $\{c(vv_i) \,|\, 1 \leq i \leq 6\} \cup \{c(v_i) \,|\, 1 \leq i \leq 6\}$ are forbidden for v. Thus, $|S_1| = 17 - 8 = 9 > 8$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 12 = 5 > 4$. We associate that u, uv, v with the variables x_1, x_2, x_3, respectively. Then we consider the following polynomial.

$$P(x_1,x_2,x_3) = (x_1-x_2)(x_1-x_3)\left(x_2-x_3)(x_1+\sum_{l=1}^{4}c(uu_l)-x_3-\sum_{k=1}^{6}c(vv_k)\right)$$
$$\prod_{i=1}^{4}\left(x_1+x_2+\sum_{l=1}^{4}c(uu_l)-W(u_i)\right)$$
$$\prod_{j=1}^{6}\left(x_2+x_3+\sum_{k=1}^{6}c(vv_k)-W(v_j)\right).$$

We have $cp\left(x_1^6x_2^4x_3^4\right) = -25$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \le i \le 3)$ such that $P(x_1,x_2,x_3) \ne 0$. We color u, uv, v correspondingly. Finally, we can get an ltnsd-k-coloring of the graph G, a contradiction. ■

Claim 15. *Every* 6^-*-vertex is not adjacent to* 6^-*-vertex in* G.

Proof. Suppose to the contrary that there exists a 6^--vertex u adjacent to a 6^--vertex v. Without loss of generality, we assume that $d(u) = 6$, $d(v) = 6$, $N(u) = \{v, u_1, \ldots, u_5\}$, $N(v) = \{u, v_1, \ldots, v_5\}$. Consider $G' = G - uv$, then G' admits an ltnsd-k-coloring c. Now we will recolor the vertices u, v and color the edge uv. Let S_1, S_2, S_3 be the sets of available colors for u, uv, v, respectively. Notice that the colors in $\{c(uu_i) \,|\, 1 \le i \le 5\} \cup \{c(u_i) \,|\, 1 \le i \le 5\}$ are forbidden for u, the colors in $\{c(uu_i) \,|\, 1 \le i \le 5\} \cup \{c(vv_i) \,|\, 1 \le i \le 5\}$ are forbidden for uv, and the colors in $\{c(vv_i) \,|\, 1 \le i \le 5\} \cup \{c(v_i) \,|\, 1 \le i \le 5\}$ are forbidden for v. Thus, $|S_1| = 17 - 10 = 7 > 6$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 10 = 7 > 6$. We associate that u, uv, v with the variables x_1, x_2, x_3, respectively. Then we consider the following polynomial.

$$P(x_1,x_2,x_3) = (x_1-x_2)(x_1-x_3)(x_2-x_3)\left(x_1+\sum_{l=1}^{5}c(uu_l)-x_3-\sum_{k=1}^{5}c(vv_k)\right)$$
$$\prod_{i=1}^{5}\left(x_1+x_2+\sum_{l=1}^{5}c(uu_l)-W(u_i)\right)$$
$$\prod_{j=1}^{5}\left(x_2+x_3+\sum_{k=1}^{5}c(vv_k)-W(v_j)\right).$$

We have $cp\left(x_1^6x_2^4x_3^4\right) = -20$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \le i \le 3)$ such that $P(x_1,x_2,x_3) \ne 0$. We color u, uv, v correspondingly. Finally, we can get an ltnsd-k-coloring of the graph G, a contradiction. ■

Claim 16. *Let* $d(v) = 13$ *and* $d_{6^+}(v) \le 6$, *then* $d_{3^-}(v) \le 5$. *Moreover, if* $d_{2^-}(v) \ge 1$, *then* $d_{3^-}(v) \le 4$.

Proof. Suppose to the contrary that there exists a 13-vertex v adjacent to six 3^--vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \ldots, v_{13}\}$, $d(v_i) = 3$, $(1 \le i \le 6)$ and $d(v_j) \ge 6$, $(8 \le j \le 13)$. Consider $G' = G - \{vv_i \,|\, i = 1, 2, \ldots, 6\}$, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, $(1 \le i \le 6)$. Let S_i, $(1 \le i \le 6)$ be the sets of available colors for vv_i, $(1 \le i \le 6)$, respectively. It is easy to obtain that $|S_i| = 17 - 7 - 1 - 2 = 7 > 6$, $(1 \le i \le 6)$. We associate that vv_i, $(1 \le i \le 6)$ with the variables x_i, $(1 \le i \le 6)$, respectively. Then we consider the following polynomial.

$$P\big(x_1, x_2, x_3, x_4, x_5, x_6\big) = \prod_{1 \le i < j \le 6} (x_i - x_j) \prod_{k=8}^{13} \left(\sum_{t=1}^{6} x_t + \sum_{l=7}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp\big(x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1\big) = 1$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \le i \le 6)$ such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \ne 0$. We color vv_i, $(1 \le i \le 6)$ correspondingly. Finally, we can recolor vertices v_i, $(1 \le i \le 6)$ to get an ltnsd-k-coloring of the graph G, a contradiction.

Moreover, if $d(v_1) = 2$, $d(v_i) = 3$, $(2 \le i \le 5)$ and $d(v_j) \ge 6$, $(8 \le j \le 13)$. Consider $G' = G - \{vv_i \,|\, i = 1, 2, \ldots, 5\}$, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, $(1 \le i \le 5)$. Let S_i, $(1 \le i \le 5)$ be the sets of available colors for vv_i, $(1 \le i \le 5)$, respectively. It is easy to obtain that $|S_1| = 17 - 8 - 1 - 1 = 7 > 6$, $|S_i| = 17 - 8 - 1 - 2 = 6 > 5$, $(2 \le i \le 5)$. We associate that vv_i, $(1 \le i \le 5)$ with the variables x_i, $(1 \le i \le 5)$, respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5) = \prod_{1 \le i < j \le 5} (x_i - x_j) \prod_{k=8}^{13} \left(\sum_{t=1}^{5} x_t + \sum_{l=6}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp\big(x_1^6 x_2^4 x_3^3 x_4^2 x_5^1\big) = -5$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \le i \le 5)$ such that $P(x_1, x_2, x_3, x_4, x_5) \ne 0$. We color vv_i, $(1 \le i \le 5)$ correspondingly. Finally, we can recolor vertices v_i, $(1 \le i \le 5)$ to get an ltnsd-k-coloring of the graph G, a contradiction. ∎

Claim 17. *Let $d(v) = \Delta(G) \ge 14$ and $d_{6^+}(v) \le 6$. If $d_{2^-}(v) \ge 1$, then $d_{3^-}(v) \le 5$.*

Proof. Let $d(v) = d$. Suppose to the contrary that there exists a d-vertex v adjacent to six 3^--vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \ldots, v_d\}$, $d(v_1) = 2$, $d(v_i) = 3$, $(2 \le i \le 6)$ and $d(v_j) \ge 6$, $(d - 5 \le j \le d)$. Consider $G' = G - \{vv_i \,|\, i = 1, 2, \ldots, 6\}$, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, $(1 \le i \le 6)$. Let S_i, $(1 \le i \le 6)$ be the sets of available colors for vv_i $(1 \le i \le 6)$, respectively.

It is easy to obtain that $|S_1| = (\Delta(G)+3) - (\Delta(G)-6) - 1 - 1 = 7 > 6$, $|S_i| = (\Delta(G)+3) - (\Delta(G)-6) - 1 - 2 = 6 > 5$, $(2 \le i \le 6)$. We associate that vv_i, $(1 \le i \le 6)$ with the variables x_i, $(1 \le i \le 6)$, respectively. Then we consider the following polynomial.

$$P(x_1,x_2,x_3,x_4,x_5,x_6) = \prod_{1\le i<j\le 6} (x_i - x_j) \prod_{k=d-5}^{d} \left(\sum_{t=1}^{6} x_t + \sum_{l=7}^{d} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp\big(x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1\big) = 1$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \le i \le 6)$ such that $P(x_1,x_2,x_3,x_4,x_5,x_6) \neq 0$. We color vv_i, $(1 \le i \le 6)$ correspondingly. Finally, we can recolor vertices v_i, $(1 \le i \le 6)$ to get an ltnsd-k-coloring of the graph G, a contradiction. ■

3.2. Discharging process

Let T be the graph obtained by removing all 2^--vertices from the graph G and $T^\times$ be the associated plane graph of T. We have $d_T(v) = d(v) - d_{2^-}(v)$.

Corollary 18. *For any vertex v with $d(v) \ge 7$, it holds that $d_T(v) \ge 7$.*

Proof. If $7 \le d(v) \le 10$, we can easily get $d_T(v) \ge 7$ by Claim 14 and Corollaries 9–11. When $d(v) > 10$, we just consider the situation $d_{6^+}(v) \le 6$. By Claim 8, $d_T(v) = d(v) - d_{2^-}(v) \ge d(v) - \frac{d_{6^+}(v)-1}{\Delta(G)-d(v)+1} \ge 11 - \frac{5}{14-11+1} \ge 9$. ■

We apply the discharging method on associated plane graph $T^\times$ of T and complete the proof by contradiction. Since $T^\times$ is a plane graph, we have

$$\begin{aligned} &\sum_{v\in V(T^\times)} (d_{T^\times}(v) - 6) + \sum_{f\in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= \sum_{v\in V(T)} (d_T(v) - 6) + \sum_{v\in V(T^\times)\setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f\in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= \sum_{v\in V(T)} (d(v) - d_{2^-}(v) - 6) + \sum_{v\in V(T^\times)\setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f\in F(T^\times)} (2d_{T^\times}(f) - 6) \\ &= -12. \end{aligned}$$

Now we define the initial charge function $ch(x)$ of $x \in V(T^\times) \cup F(T^\times)$. Let $ch(v) = d_T(v) - 6 = d(v) - d_{2^-}(v) - 6$ if $v \in V(T)$, $ch(v) = d_{T^\times}(v) - 6$ if $v \in V(T^\times) \setminus V(T)$ and $ch(f) = 2d_{T^\times}(f) - 6$ if $f \in F(T^\times)$. Then we define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(T^\times) \cup F(T^\times)$ such that $ch'(x) \ge 0$ for all $x \in V(T^\times) \cup F(T^\times)$. Notice that our discharging rules only move charge around and do not affect the sum. Thus we have $0 \le \sum_{x\in V(T^\times)\cup F(T^\times)} ch'(x) =$

$\sum_{x\in V(T^\times)\cup F(T^\times)} ch(x) = -12$, a contradiction. Since for every vertex $v \in V(T)$, $ch(v) = d_G(v) - d_{2^-}(v) - 6$, in the discharging process, we use $d_G(v)$ instead of $d_T(v)$. Similarly, for every vertex $v \in V(T)$, when check $ch'(v) \geq 0$, we split the proof into cases depending on the size of $d_G(v)$.

For $v \in V(T^\times)$ and $f \in F(T^\times)$, we define the discharging rules as follows. Note that within all the degree of a real vertex shall refer to its degree in G and the faces and their degrees correspond to the graph $T^\times$.

(R1): If the edge uv belongs to two 3-faces and $d(v) = 3$, then u sends 1 to v.

(R2): If the edge uv belongs to exactly one 3-face and $d(v) = 3$, then u sends $\frac{1}{2}$ to v.

(R3): If the edge uv belongs to two 3-faces and $d(v) = 4$, then u sends $\frac{1}{2}$ to v.

(R4): If the edge uv belongs to two 3-faces and $d(v) = 5$, then u sends $\frac{1}{5}$ to v.

(R5): Every 4-face sends 1 to each incident real 5^--vertex in $T^\times$.

(R6): Every 5^+-face sends 2 to each incident real 5^--vertex in $T^\times$.

(R7): Let v be a false vertex crossed by edge uw and xy in $T^\times$. If $d(u) \geq 7$, then u sends 1 to v. Moreover, if $d(w) = 3$, then u sends $\frac{3}{2}$ to v.

By Corollary 18 and the discharging rules, we obtain the following facts easily.

Fact 1. For any $f \in F(T^\times)$, f is incident with at most $\left\lfloor \frac{d(f)}{2} \right\rfloor$ real 5^--vertices in $T^\times$.

Fact 2. Each vertex v gives at most $\frac{d_{3^+}(v)}{2} + 1$ away.

Let f be a face of $T^\times$. Clearly, if $d(f) = 3$, then $ch'(f) = ch(f) = 2d(f) - 6 = 0$. If $d(f) = 4$, then $ch'(f) \geq ch(f) - 2 = 0$ by Fact 1 and (R5). If $d(f) \geq 5$, then $ch'(f) \geq ch(f) - \left\lfloor \frac{d(f)}{2} \right\rfloor \times 2 = 0$ by Fact 1 and (R6).

We next check the final charge of the vertex $v \in V(T^\times)$. Obviously, $d(v) \geq 3$. Recall that v has an initial weight of $d(v) - d_{2^-}(v) - 6$.

Suppose $d(v) = 3$. If v is incident with three 3-faces, then $ch'(v) \geq ch(v)+3 = 0$ by (R1). If v is incident with two 3-faces, then $ch'(v) \geq ch(v)+1+\frac{1}{2}\times 2+1 = 0$ by (R1), (R2), (R5) and (R6). If v is incident with one 3-face, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 \times 2 = 0$ by (R2), (R5) and (R6). Otherwise, v is incident with three 4^+-faces, then $ch'(v) \geq ch(v) + 1 \times 3 = 0$ by (R5) and (R6).

Suppose $d(v) = 4$ and v is a real vertex. We have $d_{2^-}(v) = 0$. If v is incident with four 3-faces, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 4 = 0$ by (R3). If v is incident with three 3-faces, then $ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 = 0$ by (R3), (R5) and (R6). If v is incident with at most two 3-faces, then $ch'(v) \geq ch(v) + 1 \times 2 = 0$ by (R5) and (R6).

Suppose $d(v) = 4$ and v is a false vertex crossed by edge uw and xy. By Claim 14 and 15, v is adjacent to at most two 6^--vertices.

If $d_{6^-}(v) = 2$, without loss of generality, we assume that $d(u) \leq 6$ and $d(x) \leq 6$. By Claim 15, if $4 \leq d(u) \leq 6$ and $4 \leq d(x) \leq 6$, then $ux \notin E(T)$, v gives no weight away by (R3) and (R4). By the same claim, v is also adjacent to two 7^+-vertices. So v receives at least $1 \times 2 = 2$ from its 7^+-neighbors by (R7). Thus, we have $ch'(v) \geq ch(v) + 2 = 0$. If one of the vertices x, u is a 3-vertex, without loss of generality, we assume that $d(u) = 3$. Then, by Claim 14, w is a 8^+-vertex. v may receives $\frac{3}{2}$ from vertex w and 1 from vertex y by (R7) and gives at most $\frac{1}{2}$ away by (R3) and (R4). Thus, we have $ch'(v) \geq ch(v) + \frac{3}{2} + 1 - \frac{1}{2} = 0$. Otherwise, $d(u) = d(x) = 3$. By Claim 14, v receives at least $\frac{3}{2} \times 2 = 3$ from its 8^+-neighbors by (R7). And v gives at most $\frac{1}{2} \times 2 = 1$ away by Lemma 3, Claim 15 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - 1 = 0$.

If $d_{6^-}(v) = 1$, without loss of generality, we assume that $d(u) \leq 6$. Then v is adjacent to three 7^+-vertices. So v receives at least $1 \times 3 = 3$, and v gives at most $\frac{1}{2}$ away by Lemma 3 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - \frac{1}{2} > 0$.

If v is adjacent to four 7^+-vertices, v receives at least $1 \times 4 = 4$ from its 7^+-neighbors by (R7) and gives no weight away. So we have $ch'(v) \geq ch(v) + 1 \times 4 > 0$.

Suppose $d(v) = 5$. If v is not incident with any 4^+-faces, then by (R4), $ch'(v) \geq ch(v) + \frac{1}{5} \times 5 = 0$. Otherwise, if v is incident with at least one 4^+-faces, then by (R5) and (R6), $ch'(v) \geq ch(v) + 1 = 0$.

Suppose $d(v) = 6$. v gives no weight away to any other vertex by the discharging rules. So $ch'(v) = ch(v) = 0$.

Suppose $d(v) = 7$. v gives at most 1 to the false neighbor in $T^\times$ by (R7), then $ch'(v) = ch(v) - 1 = 0$.

Suppose $d(v) = 8$. By Corollary 9, $ch'(v) = ch(v) - \max\left\{2, \frac{3}{2}\right\} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 9$. By Corollary 10, $ch'(v) = ch(v) - \max\left\{3, 1 + \frac{3}{2}\right\} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 10$. By Corollary 11, $ch'(v) = ch(v) - \max\left\{4, 2 + \frac{3}{2}\right\} \geq 0$ by (R1)–(R4) and (R7).

Next we check the final charge of the vertices with $d(v) \geq 11$. Let w be a false vertex crossed by edge uv and edge xy. According to the discharging rules, if $d(u) \leq 5$, then v gives at most $d_{5^-}(v) + \frac{1}{2}$ away. Otherwise, v gives at most $d_{5^-}(v) + 1$ away. Therefore, for every vertex v with $d_{6^+}(v) \geq 7$, $ch'(v) \geq 0$. In the following discussion, we only consider the vertex with $d(v) \geq 11$ and $d_{6^+}(v) \leq 6$.

Suppose $d(v) = 11$. By Claim 12, we have that $d_{3^-}(v) \leq 1$. If $d_{2^-}(v) = 0$, then $d_3(v) \leq 1$. We have $ch'(v) = ch(v) - \max\left\{1 + d_3(v) + \left(\left\lfloor \frac{11-1}{2} \right\rfloor - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + \left\lfloor \frac{11-1}{2} \right\rfloor \times \frac{1}{2}\right\} = 5 - \max\left\{\frac{7+d_3(v)}{2}, 4\right\} > 0$ by (R1)–(R4) and (R7). If $d_{2^-}(v) = 1$, then $d_3(v) = 0$. We have $ch'(v) = ch(v) - 1 - \left\lfloor \frac{11-1}{2} \right\rfloor \times \frac{1}{2} = 3 - \frac{5}{2} > 0$ by (R3) and (R7).

Suppose $d(v) = 12$. By Claim 13, we have that $d_{3^-}(v) \leq 2$.

If $d_{2^-}(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) -$

$\max\{1+d_3(v)+(\lfloor\frac{12-1}{2}\rfloor - d_3(v)) \times \frac{1}{2}, \frac{3}{2} + (d_3(v)-1)+(\lfloor\frac{12-1}{2}\rfloor - (d_3(v)-1)) \times \frac{1}{2}\} = 6 - \frac{7+d_3(v)}{2} > 0$ by (R1)–(R4) and (R7). If $d_3(v) = 0$, then we have $ch'(v) = ch(v) - 1 - \lfloor\frac{12-1}{2}\rfloor \times \frac{1}{2} > 0$ by (R3) and (R7).

If $d_{2^-}(v) \geq 1$ and $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - \max\Big\{1 + d_3(v) + \Big(\Big\lfloor\frac{12-d_{2^-}(v)-1}{2}\Big\rfloor - d_3(v)) \times \frac{1}{2}, \frac{3}{2} + (d_3(v)-1\Big) + \Big(\Big\lfloor\frac{12-d_{2^-}(v)-1}{2}\Big\rfloor - (d_3(v)-1)\Big) \times \frac{1}{2}\Big\} \geq \frac{9-3d_{3^-}(v)+d_3(v)}{4} > 0$ by (R1)–(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2^-}(v) \leq 2$. So $ch'(v) = ch(v) - 1 - \Big\lfloor\frac{12-d_{2^-}(v)-1}{2}\Big\rfloor \times \frac{1}{2} \geq \frac{9-3d_{2^-}(v)}{4} > 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 13$. By Claim 16, we have that $d_{3^-}(v) \leq 5$. Moreover, if $d_{2^-}(v) \geq 1$, then $d_{3^-}(v) \leq 4$.

If $d_{2^-}(v) = 0$, then $d_3(v) \leq 5$. If $d_3(v) \geq 1$, then we have $ch'(v) = ch(v) - \max\{1 + d_3(v) + (\lfloor\frac{13-1}{2}\rfloor - d_3(v)) \times \frac{1}{2}, \frac{3}{2} + (d_3(v)-1) + (\lfloor\frac{13-1}{2}\rfloor - (d_3(v)-1)) \times\frac{1}{2}\} = 3 - \frac{d_3(v)}{2} > 0$ by (R1)–(R4) and (R7). If $d_3(v) = 0$, then we have $ch'(v) = ch(v) - 1 - \lfloor\frac{13-1}{2}\rfloor \times \frac{1}{2} > 0$ by (R3) and (R7).

If $d_{2^-}(v) \geq 1$, then $d_{3^-}(v) \leq 4$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - \max\Big\{1+d_3(v)+\Big(\Big\lfloor\frac{13-d_{2^-}(v)-1}{2}\Big\rfloor - d_3(v)\Big) \times \frac{1}{2}, \frac{3}{2} + (d_3(v)-1) + \Big(\Big\lfloor\frac{13-d_{2^-}(v)-1}{2}\Big\rfloor - (d_3(v)-1)\Big) \times \frac{1}{2}\Big\} \geq 3 - \frac{3d_{3^-}(v)}{4} + \frac{d_3(v)}{4} > 0$ by (R1)–(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2^-}(v) \leq 4$. So $ch'(v) = ch(v) - 1 - \Big\lfloor\frac{13-1-d_{2^-}(v)}{2}\Big\rfloor \times \frac{1}{2} \geq 3 - \frac{3d_{2^-}(v)}{4} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = \Delta(G) \geq 14$. If $d_{2^-}(v) = 0$, then by Fact 2, we have $ch'(v) = ch(v) - \frac{d_{3^+}(v)}{2} - 1 \geq 0$ by (R1)–(R4) and (R7).

If $d_{2^-}(v) \geq 1$, then by Claim 17, $d_{3^-}(v) \leq 5$. If $d_3(v) \geq 1$, then we have $ch'(v) = ch(v) - \max\Big\{1 + d_3(v) + \Big(\Big\lfloor\frac{14-d_{2^-}(v)-1}{2}\Big\rfloor - d_3(v)\Big) \times \frac{1}{2}, \frac{3}{2} + (d_3(v)-1) + \Big(\Big\lfloor\frac{14-d_{2^-}(v)-1}{2}\Big\rfloor - (d_3(v)-1)\Big) \times \frac{1}{2}\Big\} \geq \frac{15-3d_{3^-}(v)+d_3(v)}{4} > 0$ by (R1)-(R4) and (R7). Otherwise $d_3(v) = 0$, then $d_{2^-}(v) \leq 5$. So $ch'(v) = ch(v) - d_{2^-}(v) - 1 - \Big\lfloor\frac{14-1-d_{2^-}(v)}{2}\Big\rfloor \times \frac{1}{2} \geq \frac{15-3d_{2^-}(v)}{4} \geq 0$ by (R1)–(R4) and (R7).

This completes the proof.

4. Remark

By the definition of IC-planar graphs, we know that every planar graphs are special IC-planar graphs. In [13], the authors proved that $ch''_{\Sigma}(G) \leq \max\{\Delta(G)+3, 16\}$. So we can easily obtain the following question.

Question 1. *Is it true that* $ch''_{\Sigma}(G) \leq \Delta(G) + 3$ *for IC-planar graphs with* $\Delta = 13$*?*

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Appendix A

```
%% The m.file of Matlab to compute the coefficients.
% INPUT
function coefficents ()
syms x1 x2 x3 x4 x5 x6 x7 % Variables used in the following.

% Claim 3.7 %To calculate the coefficient of x1^6x2^4x3^4
P=(x1-x2)*(x2-x3)*(x1-x3)^2*(x1+x2)^4*(x2+x3)^6; % The polynomial
cp1=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)
    /factorial(4)

% Claim 3.8
P=(x1-x2)*(x2-x3)*(x1-x3)^2*(x1+x2)^5*(x2+x3)^5;
cp2=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)
    /factorial(4)

% Claim 3.9
P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x6)*(x2-x3)*(x2-x4)*(x2-x5)
  *(x2-x6)*(x3-x4)*(x3-x5)*(x3-x6),...,*(x4-x5)*(x4-x6)*(x5-x6)
  *(x1+x2+x3+x4+x5+x6)^6;
cp3=diff(diff(diff(diff(diff(diff(P,x1,6),x2,5),x3,4),x4,3),x5,2),
    x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
    /factorial(2)/factorial(1)

P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x2-x3)*(x2-x4)*(x2-x5)*(x3-x4)
  *(x3-x5)*(x4-x5)*(x1+x2+x3+x4+x5+x6)^6;
cp4=diff(diff(diff(diff(diff(P,x1,6),x2,4),x3,3),x4,2),x5,1)
    /factorial(6)/factorial(4)/factorial(3)/factorial(2)
    /factorial(1)

% Claim 3.10
P=(x1-x2)*(x1-x3)*(x1-x4)*(x1-x5)*(x1-x6)*(x2-x3)*(x2-x4)*(x2-x5)
  *(x2-x6)*(x3-x4)*(x3-x5)*(x3-x6)...,*(x4-x5)*(x4-x6)*(x5-x6)
  *(x1+x2+x3+x4+x5+x6)^6;
cp5=diff(diff(diff(diff(diff(diff(P,x1,6),x2,5),x3,4),x4,3),x5,2),
    x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
    /factorial(2)/factorial(1)
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