STRONG GEODETIC PROBLEM IN NETWORKS

PAUL MANUEL[1], SANDI KLAVŽAR[2,3,4], ANTONY XAVIER[5]

ANDREW AROKIARAJ[5] AND ELIZABETH THOMAS[5]

[1]*Department of Information Science*
College of Computing Science and Engineering
Kuwait University, Kuwait

[2]*Faculty of Mathematics and Physics*
University of Ljubljana, Slovenia

[3]*Faculty of Natural Sciences and Mathematics*
University of Maribor, Slovenia

[4]*Institute of Mathematics, Physics and Mechanics*
Ljubljana, Slovenia

[5]*Department of Mathematics*
Loyola College, Chennai, India

e-mail: pauldmanuel@gmail.com
sandi.klavzar@fmf.uni-lj.si
dantonyxavierlc@gmail.com
andrewarokiaraj@gmail.com
elizathomas.25@gmail.com

Abstract

In order to model certain social network problems, the strong geodetic problem and its related invariant, the strong geodetic number, are introduced. The problem is conceptually similar to the classical geodetic problem but seems intrinsically more difficult. The strong geodetic number is compared with the geodetic number and with the isometric path number. It is determined for several families of graphs including Apollonian networks. Applying Sierpiński graphs, an algorithm is developed that returns a minimum path cover of Apollonian networks corresponding to the strong geodetic number. It is also proved that the strong geodetic problem is NP-complete.

Keywords: geodetic problem, strong geodetic problem, Apollonian networks, Sierpiński graphs, computational complexity.

2010 Mathematics Subject Classification: 05C12, 05C70, 68Q17.

1.   Introduction

Shortest paths play a significant role in graph theory due to its strategic applications in several domains such as transportation problems, communication problems, etc. In the literature, shortest paths are also known as geodesics as well as isometric paths.

The following social network problem was considered in [20]. A vertex represents a member of the social network and an edge represents direct communication between two members of the social network. Communication among the members is restricted to only along shortest path (geodesic). Members who are lying along a geodesic are grouped together. Two coordinators supervise groups of members who lie on geodesics between the two coordinators. The problem is to identify minimum number of coordinators in such a way that each member of the social network lies on some geodesic between two coordinators. Then they modeled the above social network problem in terms of graphs as follows: Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. Let $g(x, y)$ be a geodesic between x and y and let $V\big(g(x, y)\big)$ denote the set of vertices lying on $g(x, y)$. If $S \subseteq V$, then let $I(S)$ be the set of all geodesics between vertices of S and let $V\big(I(S)\big) = \bigcup_{P \in I(S)} V(P)$. If $V\big(I(S)\big) = V$, then the set S is called a *geodetic set* of G. The *geodetic problem* is to find a minimum geodetic set S of G whose cardinality is denoted with $\mathrm{g}(G)$.

In [20] the authors claimed that the geodetic problem is NP-complete for general graphs, but the reduction given is in the wrong direction. (Additional pitfalls of [20] are described in [19].) A sound proof of the fact that the calculation of the geodetic number is an NP-hard problem for general graphs was given in [1]. Dourado *et al.* [10] extended this result by establishing that the problem is difficult already for chordal graphs and bipartite weakly chordal graphs, while on the other hand it is polynomial on co-graphs and split graphs. Ekim *et al.* [11] further showed that the problem is polynomially solvable for proper interval graphs. The geodetic problem was also studied in product graphs [3, 35], block-cactus graphs [38], and in line graphs [18], while Chartrand *et al.* [9] investigated it in oriented graphs.

Some new concepts were introduced combining geodetic and domination theory such as geodomination [6] and geodetic domination problem [24]. The hull problem which was introduced by Everett *et al.* [13] is similar to the geodetic problem. The relationship between hull problem and geodetic problem was explored by several authors [14, 27]. Steiner set is another concept which is similar to geodetic set. Hernando [26] and Tong [36] probed the role of geodetic problem in hull and Steiner problems. For further results of the geodetic problem see [5, 7, 8] as well as the comprehensive survey [4]. We also refer to the book [33] for related convexity aspects.

In another situation of social networks, a set of coordinators needs to be identified in such a way that each member of a social network will lie on a geodesic between two coordinators and one pair of coordinators will be able to supervise the members of only one geodesic of the social network. This situation is stronger than in the previous case. Following the geodetic problem set up, we model this social network problem as follows. Let $G = (V, E)$ be the graph corresponding to the social network. If $S \subseteq V$, then for each pair of vertices $x, y \in S$, $x \neq y$, let $\widetilde{g}(x, y)$ be a **selected fixed** shortest x, y-path. Then we set

$$\widetilde{I}(S) = \{\widetilde{g}(x, y) : x, y \in S\},$$

and let $V\big(\widetilde{I}(S)\big) = \bigcup_{P \in \widetilde{I}(S)} V(P)$. If $V\big(\widetilde{I}(S)\big) = V$ for some $\widetilde{I}(S)$, then the set S is called a *strong geodetic set*. The *strong geodetic problem* is to find a minimum strong geodetic set S of G. Clearly, the collection $\widetilde{I}(S)$ of geodesics consists of exactly $\big(|S| \times (|S| - 1)\big)/2$ elements. The cardinality of a minimum strong geodetic set is the *strong geodetic number* of G and denoted by $\mathrm{sg}(G)$. (For the edge version of the strong geodetic problem see [31].)

The rest of the paper is organized as follows. In the next section we relate the strong geodetic number with the geodetic number and with the isometric path number. We also determine the strong geodetic number of several families of graphs. In Section 3 we first determine determine the strong geodetic number of Apollonian networks and then develop an algorithm that returns paths arising from a minimum strong geodetic set of these networks. The algorithm is based on a connection between the Apollonian networks and Sierpiński graphs. In Section 4 we prove that the strong geodetic problem is NP-complete.

2. EXAMPLES AND BASIC PROPERTIES

In this section we give connections between the strong geodetic number and two related invariant: the geodetic number and the isometric path number. Along the way several examples are provided for which the strong geodetic number is determined. At the end of the section we discuss why the strong geodetic problem appears more difficult than the geodetic problem.

A vertex of a graph is *simplicial* if its neighborhood induces a clique. Clearly, a simplicial vertex necessarily lies in any strong geodetic set. This simple fact will be utmost useful in the rest of the paper. The fact in particular implies that $\mathrm{sg}(K_n) = n$. Actually, K_n is the unique graph of order n with the strong geodetic number equal to n.

A graph is *geodetic* if any two vertices are joined by a unique shortest path. For instance, trees, block graphs, and k-trees are families of geodetic graphs. The concept of geodetic graphs goes back to Ore [32], see also [2] for an early survey

and [37] for some recent developments. The strong geodetic number relates to the geodetic number in the following obvious way.

Lemma 2.1. *If G is a connected graph, then $\mathrm{sg}(G) \geq \mathrm{g}(G)$. Moreover, the equality holds if G is a geodetic graph.*

In Figure 1(a) a geodetic graph G is shown for which $\mathrm{sg}(G) = \mathrm{g}(G) = 3$. Figure 1(b) shows a non-geodetic graph H for which we also have $\mathrm{sg}(H) = \mathrm{g}(H) = 3$. Hence not only geodetic graphs attain the equality in Lemma 2.1.
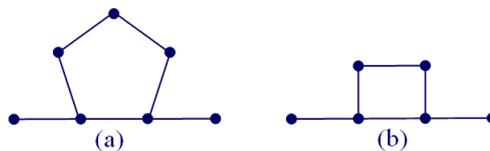

Figure 1. (a) Geodetic graph G. (b) Non-geodetic graph H.

The *isometric path problem* is to find a minimum number of isometric paths (alias shortest paths) that cover all the vertices of a given graph [15]. For a graph G, this invariant is called the *isometric path number* and denoted $\mathrm{ip}(G)$. The strong geodetic number and the isometric path number are related as follows.

Lemma 2.2. *If G is a connected graph, then*

$$\left\lceil \frac{1 + \sqrt{8 \cdot \mathrm{ip}(G) + 1}}{2} \right\rceil \leq \mathrm{sg}(G) \leq 2 \cdot \mathrm{ip}(G).$$

Proof. Let S be a strong geodetic set with $|S| = \mathrm{sg}(G)$. Since $\left|\widetilde{I}(S)\right| = \binom{\mathrm{sg}(G)}{2}$, we infer that $\mathrm{ip}(G) \leq \binom{\mathrm{sg}(G)}{2}$ and hence $\mathrm{sg}(G)^2 - \mathrm{sg}(G) - 2 \cdot \mathrm{ip}(G) \geq 0$. Since one zero of the corresponding quadratic equation is negative and because $\mathrm{sg}(G)$ is an integer, the first inequality follows.

To establish the inequality $\mathrm{sg}(G) \leq 2 \cdot \mathrm{ip}(G)$, consider a smallest set of geodesics that cover $V(G)$. Then the set of end-vertices of these geodesics forms a strong geodetic set. ∎

The bounds of Lemma 2.2 are sharp. A sporadic example for the first (in)equality is the Petersen graph P. Using the fact that the diameter of P is 2, it is not difficult to deduce that $\mathrm{sg}(P) = \mathrm{ip}(P) = 4$ and then the equality holds. An infinity such family is the following. Let $k \geq 4$ and let G_{4k} be the graph constructed as follows. First take a $4k$-cycle on vertices $v_1, v_2, \ldots, v_{4k}$, and attach a leaf at each of the vertices $v_1, v_{k+1}, v_{2k+1}, v_{3k+1}$, respectively. Finally,

connect v_1 and v_{2k+1} with a new path of length $2k$, and do the same for the vertices v_{k+1} and v_{3k+1}. Since the leaves of G_{4k} are simplicial vertices, $\text{sg}(G_{4k}) \geq 4$. On the other hand, the leaves also form a strong geodetic set, hence $\text{sg}(G_{4k}) = 4$. It is also not difficult to verify that $\text{ip}(G_{4k}) = 4$, hence the equality.

For the equality in the second inequality of Lemma 2.2 observe that $\text{sg}(K_{2n}) = 2n = 2 \cdot \text{ip}(K_{2n})$ and that $\text{sg}(K_{1,2n}) = 2n = 2 \cdot \text{ip}(K_{1,2n})$.

To conclude the section we point out that after the present paper has been submitted, several developments on the strong geodetic problem followed. In particular, the paper [28] brings the following result. If $n \geq 6$, then

$$
\text{sg}(K_{n,n}) = \begin{cases} 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil; & 8n - 7 \text{ is not a perfect square}, \\ 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil - 1; & 8n - 7 \text{ is a perfect square}. \end{cases}
$$

As could be guessed from the formulation, the proof of this result is quite technical. Moreover, the problem to determine $\text{sg}(K_{m,n})$ for all m and n is still open. This situation should be compared with the easy result that $\text{g}(K_{m,n}) = \min\{m, n, 4\}$.

A reason for this difference between the geodetic problem and the strong geodetic problem is that when we select a "good candidate" for a strong geodetic set, we still need to determine specific geodesics among the pairs of vertices, while in the geodetic problem this is a routine task because we just need to consider all related geodesics. This point is demonstrated in the next section on Apollonian networks.

3. COMPLETE APOLLONIAN NETWORKS

In this section we consider complete Apollonian networks. We first determine their strong geodetic number, a task that is not difficult. In the rest of the section we then give an explicit construction of geodesics that arise from a strong geodetic set consisting of simplicial vertices.

Apollonian networks were investigated from different points of view [17, 34, 40] and are constructed as follows. Start from a single triangle $\triangle(a, b, c)$. A new vertex $(0, 0)$ is added inside the $\triangle(a, b, c)$ and the vertex $(0, 0)$ is connected to a, b, and c. The vertex $(0, 0)$ is called 0-level vertex. Inductively, r-level vertices are constructed from $(r-1)$-level vertices. At r-th level, a new vertex v is added inside a triangular face $\triangle(x, y, z)$ and the new vertex v is connected to x, y, and z. At each inductive step, if all $(r-1)$-level triangular faces are filled by r-level vertices, then the constructed graph is called a *complete Apollonian network*, otherwise it is an *incomplete Apollonian network*. An r-level complete Apollonian network

is denoted by $A(r)$. In particular, $A(0)$ is isomorphic to the complete graph K_4 on 4 vertices, while $A(3)$ (together with their subgraphs $A(0)$, $A(1)$, and $A(2)$) is shown in Figure 2. The k-level vertices of Apollonian network will be denoted by $(k,1), (k,2), \ldots, (k,3^k)$; in the figure we have left out the brackets in order to make the figure more transparent.
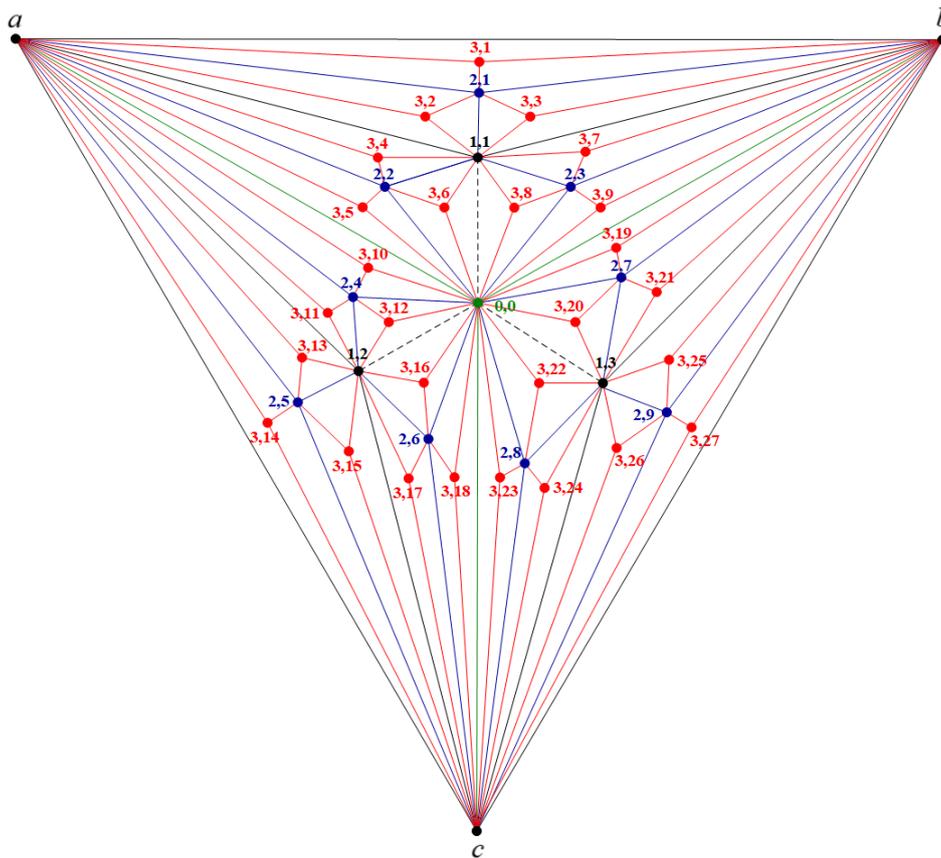

Figure 2. A complete Apollonian network $A(3)$. The 3-level vertices T_3 are simplicial. A vertex (x, y) is written as x, y due to lack of space in the diagram.

Proposition 3.1. *For Apollonian networks $A(r)$ we have* $\mathrm{sg}(A(0)) = \mathrm{sg}(A(1)) = 4$, *and if* $r \geq 2$, *then* $\mathrm{sg}(A(r)) = 3^r$.

Proof. We first observe that $\mathrm{sg}(A(0)) = \mathrm{sg}(A(1)) = 4$ and $\mathrm{sg}(A(2)) = 9$. Indeed, $A(0)$ and $A(2)$ contain 4 and 9 simplicial vertices, respectively. It is straightforward to verify that these sets are also strong geodetic sets. On the other hand, $A(1)$ contains only 3 simplicial vertices, but with them we can cover at most 6

vertices of $A(1)$. Since the $\left|V\big(A(1)\big)\right| = 7$, we have $\mathrm{sg}\big(A(1)\big) \geq 4$. It is clear that $\mathrm{sg}\big(A(1)\big) \leq 4$.

Suppose now that $r \geq 3$. We have already observed that the r-level vertices T_r of $A(r)$ are simplicial. Since $\left|T_r\right| = 3^r$ it follows that $\mathrm{sg}\big(A(r)\big) \geq 3^r$.

Consider the embedding of $A(r)$ in the plane as presented in Figure 2. Then the subgraph $A_1(r)$ induced by the triangle $(0,0) - a - b$ and all the vertices that lie inside its face is isomorphic to $A(r - 1)$. Similarly, the triangles $(0,0) - a - c$ and $(0,0) - b - c$ together with their interiors induce subgraphs $A_2(r)$ and $A_3(r)$ isomorphic to $A(r - 1)$. By the induction hypothesis, $\mathrm{sg}\big(A_1(r-1)\big) = \mathrm{sg}\big(A_2(r - 1)\big) = \mathrm{sg}\big(A_3(r - 1)\big) = 3^{r-1}$. Since A_r is the union of $A_1(r)$, $A_2(r)$, and $A_3(r)$, we also have $\mathrm{sg}\big(A(r)\big) \leq 3^r$. ∎

The proof of Proposition 3.1 is inductive and hence non-constructive in the sense that, knowing that the set of simplicial vertices T_r of $A(r)$ is a smallest strong geodetic set, we do not know explicitly a set of paths $\widetilde{I}(T_r)$ that cover $V\big(A(r)\big)$. In the rest of the section we develop a related algorithm that is based on a connection between the Apollonian networks and Sierpiński graphs.

If G is a plane graph (that is, a planar graph together with a drawing in the plane), then the *inner dual* $\mathrm{inn}(G)$ of G is the graph obtained by putting a vertex into each of the inner faces of G and by connecting two vertices if the corresponding faces share an edge. (So the inner dual is just like the dual, except that no vertex is put into the infinite face.)

The Sierpiński graphs S_p^n were introduced in [29]; see the recent survey [23] for a wealth of information on the Sierpiński graphs and [12] for their generalization. Here we need the base-3 Sierpiński graphs S_3^n (alias Hanoi graphs H_3^n, see [22]) which can be described as follows. $S_3^1 = K_3$ with $V(S_3^1) = \{0, 1, 2\}$. For $n \geq 2$, the Sierpiński graph S_3^n can be constructed from 3 copies of S_3^{n-1} as follows. For each $j \in \{0, 1, 2\}$, concatenate j to the left of the vertices in a copy of S_3^{n-1} and denote the obtained graph with jS_3^{n-1}. Then for each $i \neq j$, join copies iS_3^{n-1} and jS_3^{n-1} by the single edge between vertices ij^{n-1} and ji^{n-1}. The vertices 0^n, 1^n, and 2^n are called the *extremal vertices* of S_3^n.

In order to design an algorithm that constructs $\widetilde{I}(S)$ for Apollonian networks $A(r)$, we adopt the technique first used by Zhang, Sun, and Xu [39] to enumerate spanning trees of Apollonian networks and then followed by Liao, Hou, and Shen [30] to calculate the Tutte polynomial. Their finding can be stated as follows.

Lemma 3.2 [30, 39]**.** *If $r \geq 0$, then $\mathrm{inn}\big(A(r)\big)$ is isomorphic to S_3^{r+1}.*

Lemma 3.2 in particular yields the following information that suits the construction of a minimum strong geodetic set of Apollonian networks.

Corollary 3.3. *Let T_r denote the set of r-level vertices of $A(r)$. Then there is a 1-1 map between the vertices of T_r and the vertices of S_3^r and, in addition, between the vertices of $V(A(r)) \setminus (T_r \cup \{a, b, c\})$ and the inner faces of S_3^r.*

Figures 2 and 3 illustrate the 1-1 map between $A(3)$ and S_3^3 which is defined in Corollary 3.3.

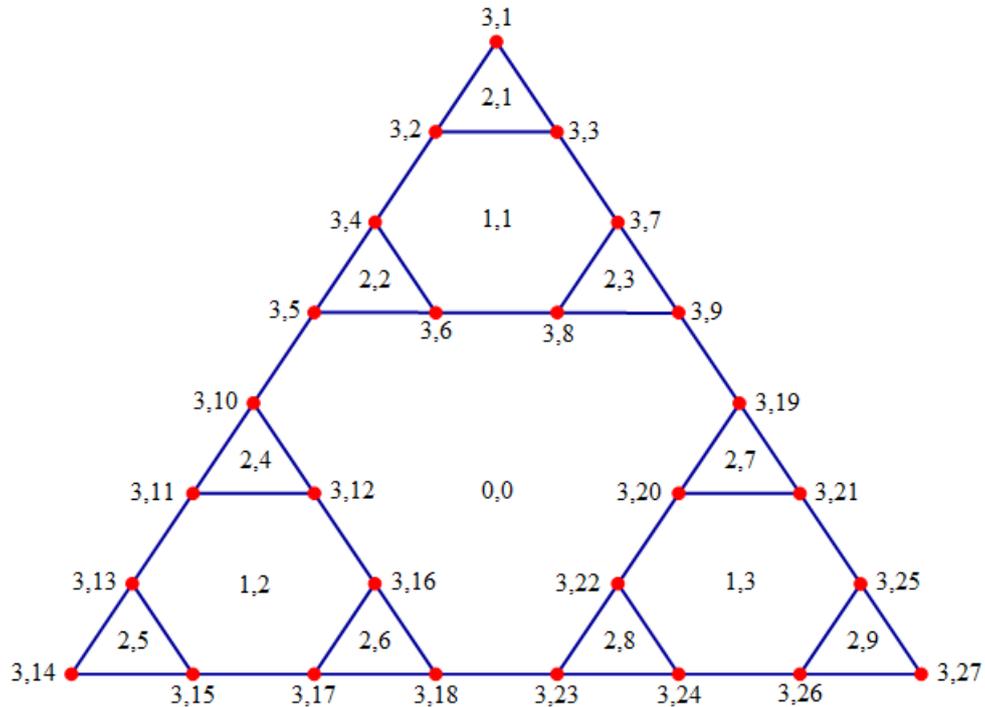

Figure 3. The Sierpiński graph S_3^3. An 1-1 map between $A(3)$ and S_3^3 defined in Corollary 3.3 is illustrated here. The vertices of S_3^3 carry the labels of 3-level vertices T_3 of $A(3)$ and the inner faces of S_3^3 carry the labels of the remaining vertices $V(A(3)) \setminus T_3$ of $A(3)$. A vertex (x, y) is written as x, y due to lack of space in the diagram.

The drawing of S_3^3 as shown in Figure 3 is based on the embeddings $f_n : V(S_3^n) \to \mathbb{R}^2$, where $f_n(x) = (f_1(x), f_2(x))$ and $n \geq 1$, as described in [21, Section 4]. Hence we say that an edge xy of S_3^n is *horizontal*, if $f_2(x) = f_2(y)$. With this terminology in hand we have the following:

Observation 3.4. *For each inner face F of the Sierpiński graph S_3^n, there is a unique horizontal edge xy of S_3^n such that the inner face F sits on the horizontal edge xy.*

For instance, in Figure 3 the inner face $(1,1)$ sits on the horizontal edge $(3,6)$–$(3,8)$ and the inner face $(2,9)$ sits on the horizontal edge $(3,26)$–$(3,27)$.

Based on Corollary 3.3 and Observation 3.4 we now describe an algorithm to construct an $\widetilde{I}(T_r)$ of $A(r)$ as follows:

(1) By Observation 3.4, for each inner face z of S_3^r, there is a unique horizontal edge xy of S_3^r such that the inner face z sits on the edge xy. In other words, for each vertex z of $V\big(A(r)\big) \setminus T_r$ of $A(r)$, there is a unique pair of simplicial vertices x and y of T_r such that geodesic xzy of $A(r)$ covers z. So, we define $\widetilde{I}(T_r)$ as a collection of geodesic xzy of $A(r)$ where z is the inner face of S_3^r that sits on the horizontal edge xy of S_3^r.

(2) Each pair of vertices from the three extremal vertices of S_3^r contributes a geodesic of length 2 to $\widetilde{I}(T_r)$ to cover the vertices of a, b and c. In our example of Figure 3, $(3,1)$, $(3,14)$ and $(3,27)$ are the three extremal vertices of S_3^{r+1}. A pair $(3,1)$ and $(3,14)$ of vertices contributes a geodesic $(3,1)$-a-$(3,14)$ of length 2 to $\widetilde{I}(T_r)$.

The proof of correctness of the above algorithm is simple. As already mentioned, Corollary 3.3 implies that there is a 1–1 map between the vertices of $V\big(A(r)\big) \setminus \big(T_r \cup \{a,b,c\}\big)$ and the inner faces of S_3^r. In the same way, Observation 3.4 implies that there is a 1-1 map between the inner faces of S_3^r and the horizontal edges of S_3^r. Thus the above constructed $\widetilde{I}(T_r)$ consisting of geodesics of length 2 covers all the vertices of $V\big(A(r)\big) \setminus T_r$.

We have thus presented an explicit construction/algorithm for a set of paths $\widetilde{I}(T_r)$ that cover $V\big(A(r)\big)$. If a recursive description of it suffices, then based on Proposition 3.1, the following recursive algorithm can be used.

def find-paths($A(r)$):
    if $r < 3$:
        return the paths for the case $r < 3$
    else:
        let $A_1(r-1)$, $A_2(r-1)$, and $A_3(r-1)$ be as in Proposition 3.1
        P_1 = find-paths$\big(A_1(r-1)\big)$
        P_2 = find-paths$\big(A_2(r-1)\big)$
        P_3 = find-paths$\big(A_3(r-1)\big)$
        remove from P_2 paths that cover vertices covered by paths in P_1
        remove from P_3 paths that cover vertices covered by paths in $P_1 \cup P_2$
        return $P_1 \cup P_2 \cup P_3$

4. Complexity of the Strong Geodetic Problem

In this section we prove that the strong geodetic problem is NP-complete. The proof's reduction will be from the dominating set problem which is a well-known

NP-complete problem [16]. A set D of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex from $V \setminus D$ has a neighbor in D. The *dominating set problem* asks whether for a given graph G and integer k, the graph G contains a dominating set of cardinality at most k.

Let $G = (V, E)$ be a graph. Then construct the graph $\bar{G} = (\bar{V}, \bar{E})$ as follows. The vertex set $\bar{V}$ is

$$\bar{V} = V \cup V' \cup V'', \text{ where } V' = \{x' : x \in V\} \text{ and } V'' = \{x'' : x \in V\} .$$

The vertex set V' induces a clique and V'' induces an independent set. The edge set of $\bar{G}$ is $\bar{E} = E \cup E' \cup E''$, where E' contains the edges of the complete graph induced by the vertices of V', while $E'' = \{xx' : x \in V\} \cup \{x'x'' : x \in V\}$.

The graph $\bar{G}$ can be considered as composed of three layers: the top layer consists of G itself, the middle layer forms a clique of order $|V|$, and the bottom layer is an independent set of order $|V|$. An example of the construction is presented in Figure 4.
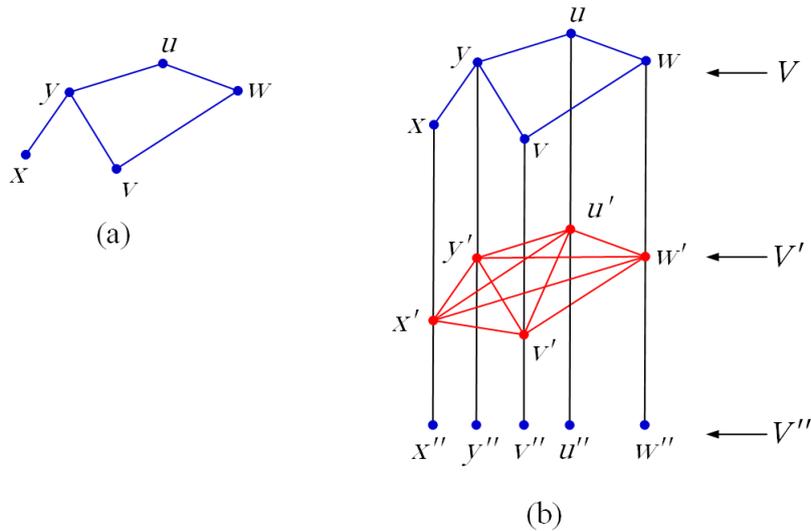

Figure 4. (a) Graph $G = (V, E)$.   (b) $\bar{G} = (\bar{V}, \bar{E})$.

We first observe the following fact that holds true since a pendent vertex belongs to any strong geodetic set. (Alternatively, a pendant vertex is a simplicial vertex.)

Property 4.1. *The vertex set V'' of $\bar{G}$ is a subset of any strong geodetic set of $\bar{G}$.*

Property 4.2. *If X is a strong geodetic set of $\bar{G}$, then there exists a strong geodetic set Y with $|Y| \leq |X|$, such that $Y = S \cup V''$ and $S \subseteq V$.*

Proof. X is a strong geodetic set of $\bar{G}$. Consider a geodesic $g(y', x)$ of $\widetilde{I}(X)$ such that $y' \in V'$ and $x \in V$. The geodesic $g(y', x)$ is of length 2 and is of the form either $y'x'x$ or $y'yx$. The geodesic $y'x'x$ covers the vertex x' of V' and the geodesic $y'yx$ covers the vertex y of V. The vertices of V' are covered by geodesics $h(u'', v'')$ where $u'', v'' \in V''$ and $h(u'', v'') \in \widetilde{I}(X)$ by Property 4.1. Thus geodesic $g(y', x)$ is only of the form $y'yx$. This geodesic $y'yx$ can be replaced by $y''y'yx$ which is already in $\widetilde{I}(X)$. Thus the vertices of V' are redundant in X.

Set $Y = X \setminus V'$. As discussed above, Y is still a strong geodetic set of $\bar{G}$. Clearly, $Y = S \cup V''$ where $S \subseteq V$ and $|Y| \leq |X|$. ∎

We can now prove the key fact for our reduction.

Property 4.3. *$S \subseteq V$ is a dominating set of G if and only if $S \cup V''$ is a strong geodetic set of $\bar{G}$.*

Proof. Suppose S is a dominating set of G. Given the vertex set $S \cup V''$ in $\bar{G}$, we define the set of paths $\widetilde{Y} = \{xyy'y'' : x \in S, xy \in E\}$.

Note first that each path from $\widetilde{Y}$ is a geodesic. In addition, from the definition of the dominating set it easily follows that the geodesics from $\widetilde{Y}$ cover all the vertices of V. Next we define $\widetilde{Z} = \{u''u'v'v'' : u'', v'' \in V''\}$.

It is straightforward to observe that the geodesics from $\widetilde{Z}$ cover all the vertices of $V' \cup V''$. Now it is clear that any $\widetilde{I}(S \cup V'')$ that includes $\widetilde{Y} \cup \widetilde{Z}$ is a strong geodetic set of $\bar{G}$.

Conversely, suppose that $S \cup V''$ is a strong geodetic set of $\bar{G}$. (We may assume that the geodetic set is of this form by Property 4.2.) Then there exists a set $\widetilde{I}(S \cup V'')$ of geodesics such that these geodesics cover all the vertices of $\bar{G}$.

Consider an arbitrary vertex $u \in V \setminus S$ and let $P \in \widetilde{I}(S \cup V'')$ be a geodesic that covers u. Then the endpoints of P, say x and y, lie in $S \cup V''$. By symmetry, there are two cases to be considered. If $x \in S$ and $y \in V''$, then necessarily $P = xuu'y$. In the second case $x, y \in S$. Clearly, the distance between x and y in $\bar{G}$ is at most 3. Hence all the vertices of P lie in V and the vertex u must be adjacent to x or y on P. In both cases P thus yields a neighbor of u in S. This in turn means that S is a dominating set of G. ∎

Note that Property 4.3 also implies that $S \subseteq V$ is a minimum dominating set of G if and only if $S \cup V''$ is a minimum strong geodetic set of $\bar{G}$. Combining this result with the fact that the graph $\bar{G}$ can clearly be constructed from G in a polynomial time, we have arrived at the main result of this section.

Theorem 4.4. *The strong geodetic problem is NP-complete.*

5.   Further Research

Even though the strong geodetic problem and the isometric path problem [15] seem to be similar, they are two different graph combinatorial problems. While the first problem minimizes the number of vertices, the second problem minimizes the number of geodesics. In this paper we have shown that the strong geodetic problem is NP-complete. To our knowledge, the complexity status of the isometric path problem is not known. Moreover, the isometric path number is known for a few graphs such as grids and block graphs but is not known even for multi-dimensional grids and other grid-like architectures. In any case, it would be useful to study the relationship between the strong geodetic problem and the isometric path problem that we initiated in Lemma 2.2.

We have introduced the strong geodetic problem following the classical geodetic problem from [20]. We have proved that the strong geodetic problem is NP-complete. The complexity status of this problem is unknown for chordal graphs, bipartite graphs, Cayley graphs, intersection graphs, permutation graphs, etc.

We have solved the strong geodetic problem for complete Apollonian networks. Further research is to investigate the strong geodetic problem for (multi-dimensional) grids, grid-like architectures, cylinders and torus. The approach from Section 3 also indicates that it would be on interest to determine the strong geodetic number of Sierpiński graphs.

Acknowledgments

References

[1] M. Atici, *Computational complexity of geodetic set*, Int. J. Comput. Math. **79** (2002) 587–591.
doi:10.1080/00207160210954

[2] J. Bosák, *Geodetic graphs*, in: Combinatorics Proc. Fifth Hungarian Colloq., Keszthely (1976) Vol. I, Colloq. Math. Soc. János Bolyai **18** (1978) 151–172.

[3] B. Brešar, S. Klavžar and A. Tepeh Horvat, *On the geodetic number and related metric sets in Cartesian product graphs*, Discrete Math. **308** (2008) 5555–5561.
doi:10.1016/j.disc.2007.10.007

[4] B. Brešar, M. Kovše and A. Tepeh, *Geodetic sets in graphs*, in: Structural Analysis of Complex Networks, Birkhäuser/Springer, New York (2011) 197–218.
doi:10.1007/978-0-8176-4789-6_8

[5] G.J. Chang, L.-D. Tong and H.-T. Wang, *Geodetic spectra of graphs*, European J. Combin. **25** (2004) 383–391.
doi:10.1016/j.ejc.2003.09.010

[6] G. Chartrand, F. Harary, H.C. Swart and P. Zhang, *Geodomination in graphs*, Bull. Inst. Combin. Appl. **31** (2001) 51–59.

[7] G. Chartrand, F. Harary and P. Zhang, *Geodetic sets in graphs*, Discuss. Math. Graph Theory **20** (2000) 129–138.
doi:10.7151/dmgt.1112

[8] G. Chartrand, F. Harary and P. Zhang, *On the geodetic number of a graph*, Networks **39** (2002) 1–6.
doi:10.1002/net.10007

[9] G. Chartrand and P. Zhang, *The geodetic number of an oriented graph*, European J. Combin. **21** (2000) 181–189.
doi:10.1006/eujc.1999.0301

[10] M.C. Dourado, F. Protti, D. Rautenbach and J.L. Szwarcfiter, *Some remarks on the geodetic number of a graph*, Discrete Math. **310** (2010) 832–837.
doi:10.1016/j.disc.2009.09.018

[11] T. Ekim, A. Erey, P. Heggernes, P. van't Hof and D. Meister, *Computing minimum geodetic sets of proper interval graphs*, Lecture Notes in Comput. Sci. **7256** (2012) 279–290.
doi:10.1007/978-3-642-29344-3_24

[12] A. Estrada-Moreno, J.A. Rodríguez-Velázquez and E.D. Rodríquez-Bazan, *On generalized Sierpiński graphs*, Discuss. Math. Graph Theory **37** (2017) 547–560.
doi:10.7151/dmgt.1945

[13] M.G. Everett and S.B. Seidman, *The hull number of a graph*, Discrete Math. **57** (1985) 217–223.
doi:10.1016/0012-365X(85)90174-8

[14] A. Farrugia, *Orientable convexity, geodetic and hull numbers in graphs*, Discrete Appl. Math. **148** (2005) 256–262.
doi:10.1016/j.dam.2005.03.002

[15] S.L. Fitzpatrick, *The isometric path number of the Cartesian product of paths*, Congr. Numer. **137** (1999) 109–119.

[16] M.R. Garey and D.S. Johnson, Computers and Intractability; A Guide to the Theory of NP-Completeness (W.H. Freeman & Co., New York, NY, 1990).

[17] A. Goldner and F. Harary, *Note on a smallest non-Hamiltonian maximal planar graph*, Bull. Malays. Math. Sci. Soc. **6** (1975) 41–42.

[18] V.M. Goudar, K.S. Ashalatha, Venkatesha and M.H. Muddebihal, *On the geodetic number of line graph*, Int. J. Contemp. Math. Sci. **7** (2012) 2289–2295.

[19] P. Hansen and N. van Omme, *On pitfalls in computing the geodetic number of a graph*, Optim. Lett. **1** (2007) 299–307.
doi:10.1007/s11590-006-0032-3

[20] F. Harary, E. Loukakis and C. Tsouros, *The geodetic number of a graph*, Math. Comput. Modelling **17** (1993) 89–95.
doi:10.1016/0895-7177(93)90259-2

[21] A.M. Hinz, K. Klavžar, U. Milutinović, D. Parisse and C. Petr, *Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence*, European J. Combin. **26** (2005) 693–708.
doi:10.1016/j.ejc.2004.04.009

[22] A.M. Hinz, S. Klavžar, U. Milutinović and C. Petr, The Tower of Hanoi—Myths and Maths (Birkhäuser/Springer, Basel, 2013).

[23] A.M. Hinz, S. Klavžar and S.S. Zemljič, *A survey and classification of Sierpiński-type graphs*, Discrete Appl. Math. **217** (2017) 565–600.
doi:10.1016/j.dam.2016.09.024

[24] A. Hansberg and L. Volkmann, *On the geodetic and geodetic domination numbers of a graph*, Discrete Math. **310** (2010) 2140–2146.
doi:10.1016/j.disc.2010.04.013

[25] P. Hansen and N. van Omme, *On pitfalls in computing the geodetic number of a graph*, Optim. Lett. **1** (2007) 299–307.
doi:10.1007/s11590-006-0032-3

[26] C. Hernando, T. Jiang, M. Mora, I.M. Pelayo and C. Seara, *On the Steiner, geodetic and hull numbers of graphs*, Discrete Math. **293** (2005) 139–154.
doi:10.1016/j.disc.2004.08.039

[27] J.-T. Hung, L.-D. Tong and H.-T. Wang, *The hull and geodetic numbers of orientations of graphs*, Discrete Math. **309** (2009) 2134–2139.
doi:10.1016/j.disc.2008.04.034

[28] V. Iršič, *Strong geodetic number of complete bipartite graphs and of graphs with specified diameter*, Graphs Combin. **34** (2018) 443–456.

[29] S. Klavžar and U. Milutinović, *Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem*, Czechoslovak Math. J. 47(122) (1997) 95–104.

[30] Y. Liao, Y. Hou and X. Shen, *Tutte polynomial of the Apollonian network*, J. Stat. Mech., October (2014) P10043.
doi:10.1088/1742-5468/2014/10/P10043

[31] P. Manuel, S. Klavžar, A. Xavier, A. Arokiaraj and E. Thomas, *Strong edge geodetic problem in networks*, Open Math. **15** (2017) 1225–1235.
doi:10.1515/math-2017-0101

[32] O. Ore, Theory of Graphs (Amer. Math. Soc., Providence, R.I., 1962).
doi:10.1090/coll/038

[33] I.M. Pelayo, Geodesic Convexity in Graphs, Springer Briefs in Mathematics (Springer, New York, 2013).
doi:10.1007/978-1-4614-8699-2

[34] G.L. Pellegrini, L. de Arcangelis, H.J. Herrmann and C. Perrone-Capano, *Modelling the brain as an Apollonian network* (27 Jan 2007).
arXiv:q-bio/0701045 [q-bio.NC]

[35] J.A. Soloff, R.A. Márquez and L.M. Friedler, *Products of geodesic graphs and the geodetic number of products*, Discuss. Math. Graph Theory **35** (2015) 35–42.
doi:10.7151/dmgt.1774

[36] L.-D. Tong, *Geodetic sets and Steiner sets in graphs*, Discrete Math. **309** (2009) 4205–4207.
doi:10.1016/j.disc.2008.10.010

[37] V.A. Voblyĭ, *Enumeration of labeled geodetic planar graphs*, Mat. Zametki **97** (2015) 336–341.
doi:10.4213/mzm10549

[38] F.-H. Wang, Y.-L. Wang and J.-M. Chang, *The lower and upper forcing geodetic numbers of block-cactus graphs*, European J. Oper. Res. **175** (2006) 238–245.
doi:10.1016/j.ejor.2005.04.026

[39] J. Zhang, W. Sun and G. Xu, *Enumeration of spanning trees on Apollonian networks*, J. Stat. Mech., September (2013) P09015.
doi:10.1088/1742-5468/2013/09/P09015

[40] Z. Zhang, B. Wu and F. Comellas, *The number of spanning trees in Apollonian networks*, Discrete Appl. Math. **169** (2014) 206–213.
doi:10.1016/j.dam.2014.01.015