

LOWER BOUND ON THE NUMBER OF HAMILTONIAN CYCLES OF GENERALIZED PETERSEN GRAPHS

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Abstract

In this paper, we investigate the number of Hamiltonian cycles of a generalized Petersen graph $P(N, k)$ and prove that

$$\Psi(P(N, 3)) \geq N \cdot \alpha_N,$$

where $\Psi(P(N, 3))$ is the number of Hamiltonian cycles of $P(N, 3)$ and α_N satisfies that for any $\varepsilon > 0$, there exists a positive integer M such that when $N > M$,

$$\left((1 - \varepsilon) \frac{(1 - r^3)}{6r^3 + 5r^2 + 3} \right) \left(\frac{1}{r} \right)^{N+2} < \alpha_N < \left((1 + \varepsilon) \frac{(1 - r^3)}{6r^3 + 5r^2 + 3} \right) \left(\frac{1}{r} \right)^{N+2},$$

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where $\frac{1}{r} = \max \left\{ \left| \frac{1}{r_j} \right| : j = 1, 2, \dots, 6 \right\}$ and each r_j is a root of equation $x^6 + x^5 + x^3 - 1 = 0$, $r \approx 0.782$. This shows that $\Psi(P(N, 3))$ is exponential in N and also deduces that the number of 1-factors of $P(N, 3)$ is exponential in N .

Keywords: generalized Petersen graph, Hamiltonian cycle, partition number, 1-factor.

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1. INTRODUCTION

Graphs throughout this paper are simple, finite and connected. All undefined terminology can be found in [4]. A *Hamiltonian cycle* (respectively, *path*) of G is a cycle (respectively, path) which contains all the vertices of G . A graph is *Hamiltonian* if it contains a Hamiltonian cycle. For convenience, we use $\Psi(G)$ to denote the number of Hamiltonian cycles in G . A *matching* in a graph G is a set of pairwise nonadjacent edges. If M is a matching, then the two ends of each edge in M are said to be *matched* under M , and each vertex incident with an edge of M is said to be *covered* by M . Furthermore, if a matching covers every vertex of graph G , then we say that it is a *perfect matching*. Sometimes, a perfect matching is also called a *1-factor* of graph G . A *generalized Petersen graph* $P(N, k)$ for $N \geq 3$ and $1 \leq k < \frac{N}{2}$ is a graph with vertex set $V = \{u_i : i = 1, 2, \dots, N\} \cup \{v_i : i = 1, 2, \dots, N\}$, and edge set $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 1, 2, \dots, N\}$, where subscripts are taken modulo N . Obviously, $P(N, k)$ is 3-regular and Petersen graph is exactly $P(5, 2)$. Then the induced subgraph $G[u_1, u_2, \dots, u_N] = u_1 u_2 \cdots u_N u_1$ is a cycle. We say that such cycle is an *Outer rim*. Similarly, the cycles induced by $G[v_1, v_2, \dots, v_N]$ are called *Inner rims*.

Watkins [14] first defined generalized Petersen graph and conjectured that every $P(N, k)$ other than $P(5, 2)$ and $P(5, 3)$ has a 1-factorization. Meanwhile, Robertson [9] proved that $P(N, 2)$ is non-Hamiltonian if and only if $N \equiv 5 \pmod{6}$. Sometimes, such non-Hamiltonian $P(N, 2)$ can be called Robertson graphs. In 1972, Bondy [3] independently proved Robertson's result and further showed that $P(N, 3)$ is a Hamiltonian graph wherever $N \neq 5$. Finally, Watkins's 1-factorization conjecture was solved by Castagna and Prins [5] in 1972. They gave a positive answer to this problem. As a further research, they found that it was very difficult to find non-Hamiltonian $P(N, k)$ other than Robertson graphs, and thus conjectured that Robertson graphs are the only non-Hamiltonian examples in generalized Petersen graph. Concerning this conjecture, Bannai [2] proved that if N and k are relatively prime, then $P(N, k)$ is Hamiltonian unless $N \equiv 5 \pmod{6}$ and $P(N, k) \cong P(N, 2)$. The most important contribution to the resolution of this conjecture is due to Brian Alspach. In [1], he proved that

$P(N, k)$ is Hamiltonian if and only if it is neither $P(N, 2) \cong P(N, \frac{N-1}{2})$, $N \equiv 5 \pmod{6}$, nor $P(N, \frac{N}{2})$, $N \equiv 0 \pmod{4}$ and $N \geq 8$.

On the other hand, the problem of counting the number of Hamiltonian cycles in a given graph also attracts many researchers' attention. This problem can be considered as an *enumeration problem* (Garey and Johnson [7]) or a *counting problem* (Papadimitriou [8]). A series of relative researches have shown that such enumeration problem is much more complicated. Historically, the problems involving the number of Hamiltonian cycles were mainly centered in cubic graphs and 4-regular graphs. Even so, few of researches gave exact values for such counting problems. A classic result of Smith [13] says that every edge of a cubic graph is contained in a set of even number Hamiltonian cycles. Thus, every 3-regular Hamiltonian graph contains a second (or more) Hamiltonian cycle. Thomason [12] extended Smith's result to all r -regular graphs where r is odd, and further obtained lower bounds for the number of Hamiltonian cycles in 4-regular graphs. For such counting problem on generalized Petersen graphs, one important result was also obtained by Thomason [11] in 1982. He proved that $P(6k + 3, 2)$ has exactly 3 Hamiltonian cycles for $k \geq 0$. Furthermore, Schwenk [10] gave the exact number of Hamiltonian cycles in $P(N, 2)$. Meanwhile, other scholars also considered the problem of finding the number of Hamiltonian cycles in a random graph (Cooper and Frieze [6]). In this paper, we show the following result.

Theorem 1. *Let $P(N, k)$ be a generalized Petersen graph. Then*

$$\Psi(P(N, 3)) \geq N \cdot \alpha_N,$$

where α_N satisfies that for any $\varepsilon > 0$, there exists a positive integer M such that when $N > M$,

$$\left((1 - \varepsilon) \frac{(1 - r^3)}{6r^3 + 5r^2 + 3} \right) \left(\frac{1}{r} \right)^{N+2} < \alpha_N < \left((1 + \varepsilon) \frac{(1 - r^3)}{6r^3 + 5r^2 + 3} \right) \left(\frac{1}{r} \right)^{N+2},$$

where $\frac{1}{r} = \max \left\{ \left| \frac{1}{r_j} \right| : j = 1, 2, \dots, 6 \right\}$ and each r_j is a root of equation $x^6 + x^5 + x^3 - 1 = 0$, $r \approx 0.782$.

2. PROOF OF THEOREM 1

Let $V = \{u_i : i = 1, 2, \dots, N\} \cup \{v_i : i = 1, 2, \dots, N\}$ be the vertex set of $P(N, 3)$ such that $G[u_1, u_2, \dots, u_N] = u_1 u_2 \cdots u_N u_1$ is the Outer rim. Then we further let $\gcd(a, b)$ denote the greatest common divisor of two positive integers a and b . Observe that the Inner rims induced by $G[v_1, v_2, \dots, v_N]$ consist of three cycles of length $\frac{N}{3}$ if $\gcd(N, 3) \neq 1$, and otherwise the Inner rim is exactly a cycle of length N . Edge set $\{u_i v_i : i = 1, 2, \dots, N\}$ is called *Spokes*.

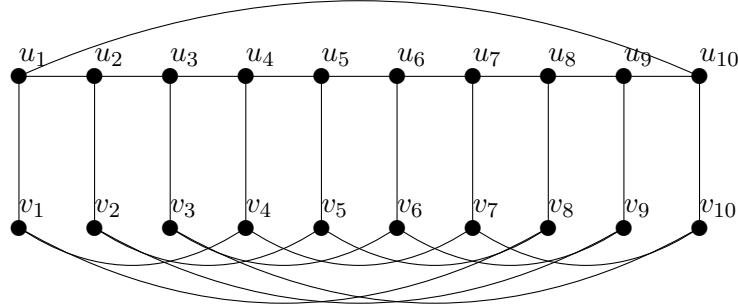


Figure 1. A drawing of $P(10, 3)$.

In order to count the number of Hamiltonian cycles of $P(N, 3)$, we need to study the construction of Hamiltonian cycles of $P(N, 3)$. Before the next discussion, we first introduce some new definitions. Let C be a Hamiltonian cycle of $P(N, 3)$. Then C intersects the Outer rim at some paths. And these paths can be called *Outer paths*. Similarly, the *Inner paths* are defined as the paths produced by the intersection of C and Inner rims. Hence a Hamiltonian cycle may be seen as a combination of the following paths:

Outer paths, Spokes, Inner paths, . . . , Inner paths, Spokes.

That is, each Hamiltonian cycle determines the number of Outer paths and Inner paths. Therefore, it also determines a combination type of Outer paths, Spokes and Inner paths.

We consider an Outer path of a Hamiltonian cycle as below. Assume that $P_m = u_{i+1}u_{i+2} \cdots u_{i+m}$ is an Outer path. Then $u_{i+1}v_{i+1}$ and $u_{i+m}v_{i+m}$ are the two Spokes connecting P_m in this Hamiltonian cycle. Moreover, none of the Spokes $u_{i+2}v_{i+2}, u_{i+3}v_{i+3}, \dots, u_{i+m-1}v_{i+m-1}$ appear on this Hamiltonian cycle. Simultaneously, $u_{i+1}v_{i+1}$ and $u_{i+m}v_{i+m}$ also connect Inner paths. Without loss of generality, we assume that $u_{i+1}v_{i+1}$ connects the Inner path P_{m1} and $u_{i+m}v_{i+m}$ connects the Inner path P_{m2} . According to the expanding direction of P_{m1} and P_{m2} , the Outer path can be divided into four types as follows.

- (i) If P_{m1} joins v_{i+1} to v_{i-2} and P_{m2} joins v_{i+m} to v_{i+m+3} , then we say that the Outer path P_m is with open leads, denoted by OP_m .
- (ii) If P_{m1} joins v_{i+1} to v_{i+4} and P_{m2} joins v_{i+m} to v_{i+m-3} , then we say that the Outer path P_m is with crossed leads, denoted by CP_m .
- (iii) If P_{m1} joins v_{i+1} to v_{i-2} and P_{m2} joins v_{i+m} to v_{i+m-3} , then we say that the Outer path P_m is with left leads, denoted by LP_m .
- (iv) If P_{m1} joins v_{i+1} to v_{i+4} and P_{m2} joins v_{i+m} to v_{i+m+3} , then we say that the Outer path P_m is with right leads, denoted by RP_m .

The above four types of Outer paths are shown in Figure 2 as $m = 3$, while it may not hold at the same time for other m .

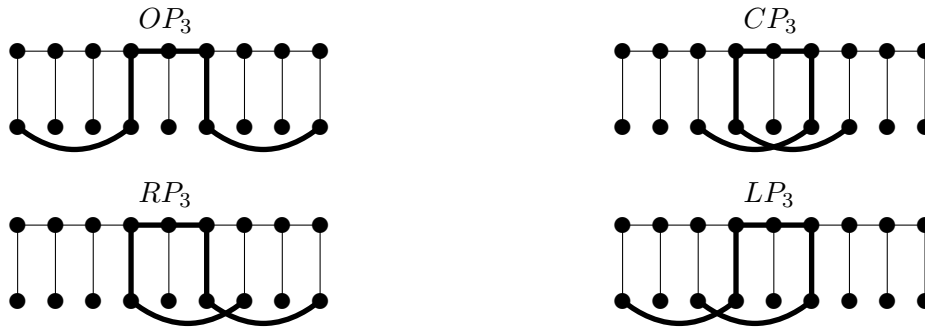


Figure 2. Four types of Outer paths.

Lemma 2. Let $P_m (m \geq 2)$ be an Outer path of a Hamiltonian cycle C in $P(N, 3)$. Then the existence of four types of P_m are as follows.

- (i) OP_m, CP_m, LP_m and RP_m all exist for $m = 2, 3$,
- (ii) only OP_4 exists for $m = 4$,
- (iii) only CP_m exists for $m \geq 5$ and $m \equiv 0$ or $2 \pmod{3}$,
- (iv) none of OP_m, CP_m, LP_m, RP_m exists for $m \geq 5$ and $m \equiv 1 \pmod{3}$.

Proof. It is easy to show the case (i). To prove (ii), suppose that $P_4 = u_{i+1}u_{i+2}u_{i+3}u_{i+4}$. Obviously, $u_{i+1}v_{i+1}$ and $u_{i+4}v_{i+4}$ are the two Spokes connecting P_4 . Then v_{i+1} can only be joined to v_{i-2} by the Inner path. And v_{i+4} can only be joined to v_{i+7} by the Inner path. Otherwise, it forms a cycle $u_{i+1}u_{i+2}u_{i+3}u_{i+4}v_{i+4}v_{i+1}u_{i+1}$. Therefore, (ii) holds. For $m \geq 5$, let $P_m = u_{i+1}u_{i+2} \cdots u_{i+m}$. Then $u_{i+1}v_{i+1}$ and $u_{i+m}v_{i+m}$ are Spokes connecting P_m . Since the Hamiltonian cycle C does not pass the edge $u_{i+4}v_{i+4}$, v_{i+4} is joined to v_{i+1} by an Inner path. Likewise, v_{i+m} joins with v_{i+m-3} . Hence CP_m appears only in two cases as below.

If $m = 3k + 2$, then let $P_m = u_{i+1}u_{i+2} \cdots u_{i+3k+2}$ and P_m is a CP_{3k+2} . We may extend CP_{3k+2} to

$$v_{i+3k+4}v_{i+3k+1} \cdots v_{i+1}u_{i+1}u_{i+2} \cdots u_{i+3k+2}v_{i+3k+2}v_{i+3k-1} \cdots v_{i+2}v_{i-1}.$$

If $m = 3k$, then let $P_m = u_{i+1}u_{i+2} \cdots u_{i+3k}$ and P_m is a CP_{3k} . We may extend CP_{3k} to

$$v_{i+3k+1}v_{i+3k-1} \cdots v_{i+4}v_{i+1}u_{i+1}u_{i+2} \cdots u_{i+3k}v_{i+3k}v_{i+3k-3} \cdots v_{i+3}v_i.$$

For $m = 3k + 1$, since v_{i+1} and v_{i+m} join only with v_{i+4} and v_{i+m-3} , respectively, it forms a cycle $u_{i+3k+1} \cdots u_{i+2}u_{i+1}v_{i+1}v_{i+4} \cdots v_{i+3k-2}v_{i+3k+1}u_{i+3k+1}$, a contradiction. Therefore, CP_{3k+1} will not exist. ■

Obviously, if the type of an Outer path in a Hamiltonian cycle is confirmed, then its corresponding Inner path is also determined. By Lemma 2, the type of P_m is determined if the value m ($m \geq 4$) is given. Namely, if the length of P_m in a Hamiltonian cycle is given, then the corresponding Hamiltonian cycle is also determined.

Let $P_m = u_{i+1}u_{i+2} \cdots u_{i+m}$ and $P_n = u_{j+1}u_{j+2} \cdots u_{j+n}$ be two Outer paths. Then we say that P_m and P_n are adjacent, if $u_{i+m+1} = u_{j+1}$. Here, we need to notice that not any two types of Outer paths can be adjacent. Hence, we further say that the types of P_m and P_n are compatible, if P_m and P_n are adjacent.

Lemma 3. OP_3 is only compatible with CP_m in $P(N, 3)$. Furthermore, CP_m is compatible with OP_3 , RP_2 or LP_2 in $P(N, 3)$.

Proof. We first consider the compatibility of OP_3 . Let $OP_3 = u_{i+1}u_{i+2}u_{i+3}$ and P_m be the adjacent Outer path of P_3 . Without loss of generality, we assume that P_m contains vertex u_{i+4} . As a result, $P_m = u_{i+4}u_{i+5} \cdots u_{i+3+m}$. Then u_{i+4} only joins with $\{u_{i+5}, v_{i+4}\}$, and v_{i+4} only joins with $\{v_{i+7}, u_{i+4}\}$. Moreover, v_{i+2} needs to join with v_{i+5} and v_{i+3} joins with v_{i+6} . It implies that P_m is the type of CP_m when $m \leq 3$. Furthermore, Lemma 2 shows that $m \neq 4$. Thus we only consider the remaining case for $m \geq 5$. Since v_{i+m} does not joins with u_{i+m} , v_{i+m} needs to join with v_{i+3+m} . Hence, P_m is the type of CP_m .

Now we study the compatibility of CP_m . As $m \equiv 2 \pmod{3}$, suppose that $P_m = u_{i+1}u_{i+2} \cdots u_{i+3k+1}u_{i+3k+2}$ and the extension path of P_m is denoted as follows: $v_{i+3k+4}v_{i+3k+1} \cdots v_{i+1}u_{i+1}u_{i+2} \cdots u_{i+3k+2}v_{i+3k+2}v_{i+3k-1} \cdots v_{i+2}v_{i-1}$. Let P_n denote the adjacent Outer path of P_m such that P_n contains u_{i+3k+3} . Since v_{i+3k+3} and v_{i+3k+5} join only with $\{v_{i+3k}, u_{i+3k+3}\}$ and $\{v_{i+3k+8}, u_{i+3k+5}\}$, respectively, P_n is precisely the type of OP_3 when $n \geq 3$. If $n = 2$, then v_{i+3k+3} joins with v_{i+3k} , and v_{i+3k+4} joins with v_{i+3k+1} . Hence P_n is the type of LP_2 . The case for P_n containing u_i can be solved analogously. For the same reason as above, Lemma 3 is also right for $m \equiv 0 \pmod{3}$. ■

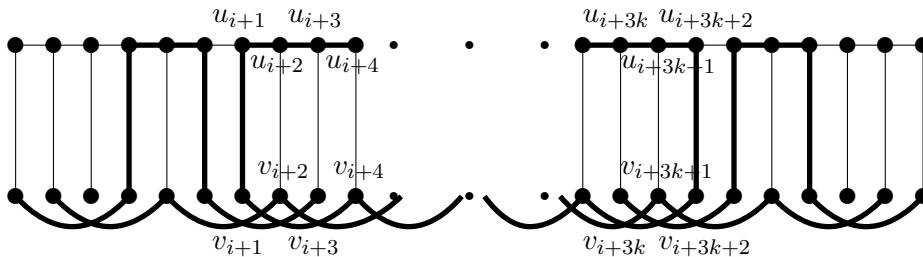


Figure 3. OP_3 and CP_m are pairwise compatible.

Lemma 3 implies that $P(N, 3)$ contains a Hamilton cycle such that its Outer paths consist of $RP_2, CP_{n_1}, OP_3, CP_{n_2}, OP_3, \dots, CP_{n_k}, LP_2$ (shown in Figure 4).

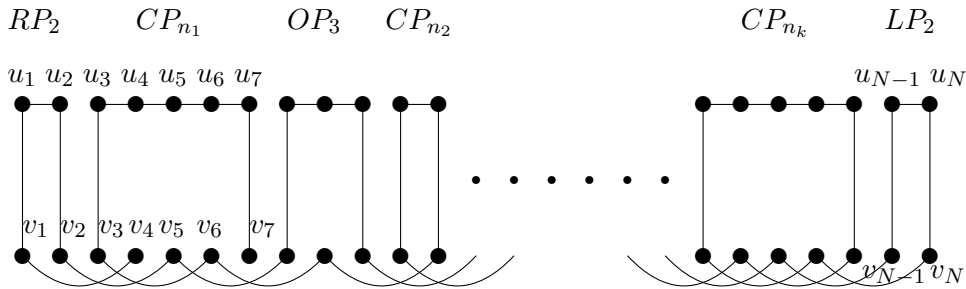


Figure 4. A type of Hamilton cycle of $P(N, 3)$.

Let α_N be the number of Hamilton cycles on the above type with $RP_2 = u_1u_2$. Then the number of Hamiltonian cycles corresponding to $P(N, 3)$ is $N \cdot \alpha_N$.

When $RP_2 = u_1u_2$, each Hamiltonian cycle in Figure 4 decides a partition of Outer path of an Outer cycle uniquely. Conversely, each partition of Outer paths also determines a Hamiltonian cycle uniquely. Hence the number of Hamiltonian cycles equals to the number of partitions of Outer paths. Observe that each partition of Outer paths is corresponded to an ordered partition of $N - 4$ such that its parts are $n_1, 3, n_2, 3, \dots, n_{k-1}, 3, n_k$, where $n_i \equiv 0$ or $2 \pmod 3$ and $n_i \geq 2$, $i = 1, 2, \dots, k$. Thus each Hamiltonian cycle can be regarded as an ordered partition, whose parts from left to right are: $2, n_1, 3, n_2, 3, \dots, n_{k-1}, 3, n_k, 2$. Then the number of partitions satisfying the above condition is α_N , where α_N is the number of solutions in the following equation

$$(1) \quad 2 + n_1 + 3 + n_2 + 3 + \dots + n_{k-1} + 3 + n_k + 2 = N.$$

Set $x_i = n_i + 3, i = 1, 2, \dots, k$. Then equation (1) can be translated into

$$(2) \quad x_1 + x_2 + \dots + x_k = N - 1,$$

where $x_i \equiv 0$ or $2 \pmod 3$ and $x_i \geq 5, i = 1, 2, \dots, k$. Here we consider the case for $N \geq 6$. The following lemma gives a recursive relation for α_N .

Lemma 4. *The number of partitions of $N - 1$ satisfies the condition of equation (2) having the following recursive relation*

$$(3) \quad \alpha_N = \alpha_{N-5} + \alpha_{N-6} + \alpha_{N-3},$$

where $\alpha_6 = \alpha_7 = \alpha_9 = \alpha_{10} = \alpha_{11} = 1, \alpha_8 = 0$.

Proof. According to the first part of the partition of $N - 1$, α_N can be divided into three cases. When the first part $x_1 = 5$ in equation (2), it has α_{N-5} distinct partitions. We continue to consider the second case for the first part $x_1 = 6$ in equation (2). Then it has α_{N-6} distinct partitions. Finally, we consider the third case for the first part $x_1 > 6$ in equation (2). Let $y_1 = x_1 - 3$, $y_i = x_i$, $i = 2, \dots, k$. Then equation (2) is translated into

$$y_1 + y_2 + \dots + y_k = N - 4,$$

where $y_i \equiv 0$ or $2 \pmod{3}$, $y_i \geq 5$. Thus it has α_{N-3} distinct partitions. ■

By the recursive relation and the initial value of α_N , we have

$$\alpha_N = \sum_{i=1}^6 \left(\frac{1}{r_i}\right)^{N+2} \frac{(1 - r_i^3)}{6r_i^3 + 5r_i^2 + 3},$$

where $r_j (j = 1, 2, \dots, 6)$ is the root of the following equation

$$x^6 + x^5 + x^3 - 1 = 0.$$

The asymptotic value of each r_j is, respectively, $r_1 \approx 0.782$, $r_2 \approx -1.538$, $r_{3,4} \approx 0.400 \pm 0.963i$, $r_{5,6} \approx -0.521 \pm 0.702i$, where $i = \sqrt{-1}$.

Since $\frac{1}{r_1} = \max \left\{ \left| \frac{1}{r_j} \right| : j = 1, 2, \dots, 6 \right\}$, for any $\varepsilon > 0$, there exists a positive integer M such that when $N > M$,

$$\left((1 - \varepsilon) \frac{(1 - r_1^3)}{6r_1^3 + 5r_1^2 + 3} \right) \left(\frac{1}{r_1} \right)^{N+2} < \alpha_N < \left((1 + \varepsilon) \frac{(1 - r_1^3)}{6r_1^3 + 5r_1^2 + 3} \right) \left(\frac{1}{r_1} \right)^{N+2}.$$

Since $\frac{1}{r_1} > 1$, α_N increases exponentially. Hence the lower bound of $\Psi(P(N, 3))$ is also exponential.

Suppose that C is a Hamiltonian cycle of $P(N, 3)$, then $P(N, 3) - E(C)$ is a 1-factor of $P(N, 3)$. This yields the following corollary.

Theorem 5. *The number of 1-factors of a generalized Petersen graph $P(N, 3)$ is exponential.*

REFERENCES

- [1] B. Alspach, *The classification of Hamiltonian generalized Petersen graphs*, J. Combin. Theory Ser. B **34** (1983) 293–312.
doi:10.1016/0095-8956(83)90042-4
- [2] K. Bannai, *Hamiltonian cycles in generalized Petersen graphs*, J. Combin. Theory Ser. B **24** (1978) 181–188.
doi:10.1016/0095-8956(78)90019-9

- [3] J.A. Bondy, *Variations on the Hamiltonian theme*, Canad. Math. Bull. **15** (1972) 57–62.
doi:10.4153/CMB-1972-012-3
- [4] J.A. Bondy and U.S.R. Murty, *Graph Theory* (Springer, 2008).
- [5] F. Castagna and G. Prins, *Every generalized Petersen graph has a Tait coloring*, Pacific J. Math. **40** (1972) 53–58.
doi:10.2140/pjm.1972.40.53
- [6] C. Cooper and A.M. Frieze, *On the number of Hamiltonian cycles in a random graph*, J. Graph Theory **13** (1989) 719–735.
doi:10.1002/jgt.3190130608
- [7] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (Freeman, San Francisco, 1979).
- [8] C.H. Papadimitriou, *Computational Complexity* (Addison-Wesley, Reading, MA, 1994).
- [9] G.N. Robertson, *Graphs under Girth, Valency, and Connectivity Constraints*, PhD. Thesis (University of Waterloo, Ontario, Canada, 1968).
- [10] A. Schwenk, *Enumeration of Hamiltonian cycles in certain generalized Petersen graphs*, J. Combin. Theory Ser. B **47** (1989) 53–59.
doi:10.1016/0095-8956(89)90064-6
- [11] A. Thomason, *Cubic graphs with three Hamiltonian cycles are not always uniquely edge colorable*, J. Graph Theory **6** (1982) 219–221.
doi:10.1002/jgt.3190060218
- [12] A.G. Thomason, *Hamiltonian cycles and uniquely edge colourable graphs*, Ann. Discrete Math. **3** (1978) 259–268.
doi:10.1016/S0167-5060(08)70511-9
- [13] W.T. Tutte, *On Hamiltonian circuits*, J. Lond. Math. Soc. (2) **21** (1946) 98–101.
doi:10.1112/jlms/s1-21.2.98
- [14] M.E. Watkins, *A theorem on tait colorings with an application to the generalized Petersen graphs*, J. Combin. Theory **6** (1969) 152–164.
doi:10.1016/S0021-9800(69)80116-X

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