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NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES[1]

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Abstract

For a given graph $G = (V(G), E(G))$, a proper total coloring $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is neighbor sum distinguishing if $f(u) \neq f(v)$ for each edge $uv \in E(G)$, where $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. The smallest integer k in such a coloring of G is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak first introduced this coloring and conjectured that $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$ for any graph with maximum degree $\Delta(G)$. In this paper, by using the discharging method, we prove that for any planar graph G without 5-cycles, $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 10\}$. The bound $\Delta(G) + 2$ is sharp. Furthermore, we get the exact value of $\chi''_{\Sigma}(G)$ if $\Delta(G) \geq 9$.

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If G is a planar graph embedded in the plane, we use $F(G)$ to denote its face set. A vertex v is a *t-vertex*, *t^--vertex*, *t^+-vertex* if $d_G(v) = t$, $d_G(v) \leq t$, $d_G(v) \geq t$ in G, respectively. A *t-face* is defined similarly. An l-face $v_1v_2 \cdots v_l$ is a $(b_1, b_2, \ldots, b_l)$-*face*, where v_i is a b_i-vertex, for $i = 1, 2, \ldots, l$. Let $d^t_G(v)$ denote the number of t-vertices adjacent to v in G. Let $n^d_G(v)$ denote the number of d-faces incident with v in G. A configuration F is *reducible* to G, if it cannot be a configuration of G.

Given a graph G, set $n_i(G) = |\{v \in V(G) : d_G(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph G' is *smaller* than G if one of the following holds:

(1) $|E(G')| < |E(G)|$,

(2) $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_1(G))$ with respect to the standard lexicographic order, where $t = \max\{\Delta(G), \Delta(G')\}$.

A graph is *minimum* for a property if no smaller graph satisfies it.

Given a graph G and a positive integer k, a *proper total k-coloring* of G is a mapping ϕ: $V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements x, $y \in V(G) \cup E(G)$. Let $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. If $f(u) \neq f(v)$ for each edge $uv \in E(G)$, then ϕ is a *neighbor sum distinguishing total k-coloring*, or *k-tnsd-coloring* for simplicity. The smallest number k is the *neighbor sum distinguishing total chromatic number* of G, denoted by $\chi''_\Sigma(G)$. For k-tnsd-coloring, Pilśniak and Woźniak gave the following conjecture.

Conjecture 1 [11]**.** *For any graph G, $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

Pilśniak and Woźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong *et al.* [3] showed that Conjecture 1 holds for some sparse graphs. Yao *et al.* [21, 22] considered tnsd-coloring of degenerate graphs. Li *et al.* [9] proved that Conjecture 1 holds for K_4-minor free graphs. Song *et al.* [15] determined $\chi''_\Sigma(G)$ for K_4-minor free graph G with $\Delta(G) \geq 5$. For planar graph, it was proved that this conjecture holds with $\Delta(G) \geq 13$ by Li *et al.* [7] and $\Delta(G) \geq 11$ by Qu *et al.* [12]. For planar graph, it was proved that $\chi''_\Sigma(G) \leq \Delta(G) + 2$ holds with $\Delta(G) \geq 14$ by Cheng *et al.* [2], $\Delta(G) \geq 12$ by Song *et al.* [14] and $\Delta(G) \geq 11$ by Yang *et al.* [20]. The bound $\Delta(G) + 2$ is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].

Recently, Ge *et al.* [6] got the following result.

Theorem 2 [6]**.** *Let G be a planar graph without 5-cycles. Then*
$$\chi''_{\Sigma}(G) \leq \max\{\ \Delta(G)+3, 10\}.$$

In this paper, we prove the following results.

Theorem 3. *Let G be a planar graph without 5-cycles. Then*
$$\chi''_{\Sigma}(G) \leq \max\{\Delta(G)+2, 10\}.$$

Theorem 4. *Let G be a planar graph without 5-cycles and without adjacent $\Delta(G)$-vertices. Then $\chi''_{\Sigma}(G) \leq \max\{\Delta(G)+1, 10\}$.*

Clearly, $\chi''_{\Sigma}(G) \geq \Delta(G)+1$ for any graph G. If G has adjacent $\Delta(G)$-vertices, then $\chi''_{\Sigma}(G) \geq \Delta(G)+2$. Thus we get the following corollary.

Corollary 5. *Let G be a planar graph without 5-cycles and $\Delta(G) \geq 9$. If G has no adjacent $\Delta(G)$-vertices, then $\chi''_{\Sigma}(G) = \Delta(G)+1$, otherwise $\chi''_{\Sigma}(G) = \Delta(G)+2$.*

2. The Proof of Theorem 3

We will prove it by contradiction. Let G be a minimum counterexample to Theorem 3 which is embedded in the plane. Set $k = \max\{\Delta(G)+2, 10\}$. By the choice of G, any planar graph G' without 5-cycles which is smaller than G has a k-tnsd-coloring ϕ'. In the following, we will choose some G' and extend the coloring ϕ' of G' to a desired coloring ϕ of G to get a contradiction. Unless otherwise stated, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$, set $\phi(x) = \phi'(x)$.

In the following proof, we will omit the coloring of all 3^--vertices. Since they have at most 9 forbidden colors and $k \geq 10$, they can be colored easily.

In Figure 1, we draw a vertex x in black if it has no other neighbors than the ones already depicted, and a vertex x in white if it might have more neighbors than the ones shown in the figure.

Claim 1. *These configurations of F_1, F_2, F_3 and F_4 in Figure 1 are reducible.*

Proof. (1) Suppose to the contrary that G contains configuration F_1. We obtain a smaller graph G' by splitting v_i into u_i, v_i for $i = 1, 2$ (see F'_1 in Figure 1). Thus G' is a planar graph without 5-cycles which is smaller than G. Hence G' admits a k-tnsd-coloring ϕ'. We can stick u_i, v_i together properly for $i = 1, 2$ (if necessary, exchange the colors of uu_1 and uu_2), and then recolor u_i, v_i, thus we can obtain a k-tnsd-coloring ϕ of G, a contradiction.

(2) Suppose to the contrary that G contains configuration F_2. We obtain a smaller graph G' by splitting v_i into u_i, v_i for $i = 1, 2$ (see F'_2 in Figure 1) without producing 5-cycles. Thus G' has a k-tnsd-coloring ϕ'.

(i) If $\phi'(wu_1) \neq \phi'(uu_2)$ or $\phi'(wu_1) = \phi'(uu_2) \notin \{\phi'(vv_1), \phi'(vv_2)\}$, then we can stick u_i, v_i together for $i = 1, 2$ (if necessary, exchange the colors of vv_1 and vv_2).

(ii) If $\phi'(wu_1) = \phi'(uu_2) \in \{\phi'(vv_1), \phi'(vv_2)\}$, without loss of generality, suppose that $\phi'(uu_2) = \phi'(vv_1)$. Exchange the colors of vv_1 (uu_2) and uv. Therefore, we can stick u_i, v_i together for $i = 1, 2$. Thus, by recoloring, we can obtain a k-tnsd-coloring ϕ of G, a contradiction.

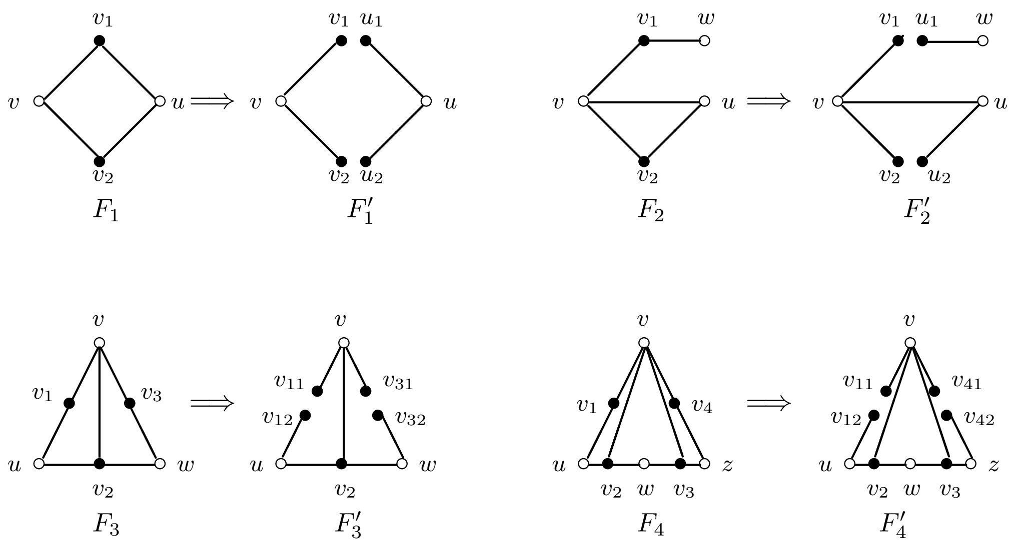

Figure 1. Illustration of Claim 1.

(3) Suppose to the contrary that G contains configuration F_3. We obtain a smaller graph G' by splitting v_i into v_{i1}, v_{i2} for $i = 1, 3$ (see F_3' in Figure 1) without producing 5-cycles. Thus G' has a k-tnsd-coloring ϕ'.

(i) If $\phi'(uv_{12}) \neq \phi'(wv_{32})$ or $\phi'(uv_{12}) = \phi'(wv_{32}) \notin \{\phi'(vv_{11}), \phi'(vv_{31})\}$, then we can stick v_{i1}, v_{i2} together for $i = 1, 3$ (if necessary, exchange the colors of vv_{11} and vv_{31}).

(ii) If $\phi'(uv_{12}) = \phi'(wv_{32}) \in \{\phi'(vv_{11}), \phi'(vv_{31})\}$, without loss of generality, suppose that $\phi'(uv_{12}) = \phi'(vv_{11})$. Then we exchange the colors of uv_{12} and uv_2. Therefore, we can stick v_{i1}, v_{i2} together for $i = 1, 3$. Thus, by recoloring, we can obtain a k-tnsd-coloring ϕ of G, a contradiction.

(4) Suppose to the contrary that G contains configuration F_4. We obtain a smaller graph G' by splitting v_i into v_{i1}, v_{i2} for $i = 1, 4$ (see F_4' in Figure 1) without producing 5-cycles. Thus G' admits a k-tnsd-coloring ϕ'.

(i) If $\phi'(uv_{12}) \neq \phi'(zv_{42})$ or $\phi'(uv_{12}) = \phi'(zv_{42}) \notin \{\phi'(vv_{11}), \phi'(vv_{41})\}$, then we can stick v_{i1}, v_{i2} together for $i = 1, 4$ (if necessary, exchange the colors of vv_{11} and vv_{41}).

(ii) If $\phi'(uv_{12}) = \phi'(zv_{42}) \in \{\phi'(vv_{11}), \phi'(vv_{41})\}$, without loss of generality, suppose that $\phi'(uv_{12}) = \phi'(zv_{42}) = \phi'(vv_{11})$. Since $\phi'(wv_2) \neq \phi'(wv_3)$, suppose that $\phi'(wv_2) \neq \phi'(uv_{12})$. We exchange the colors of uv_{12} and uv_2. Therefore, we can stick v_{i1}, v_{i2} together for $i = 1, 4$. Thus, by recoloring, we can obtain a k-tnsd-coloring ϕ of G, a contradiction. ■

It is easy to see that the following claim given in [16] also holds with the graph G in our proof.

Claim 2 [16]. *In the graph G, the following results holds.*

(1) *Each t^--vertex is not adjacent to any $(7-t)^-$-vertex, where $t = 4, 5$.*

(2) *For each vertex $v \in V(G)$, if $d_G^1(v) \geq 1$, then $d_G^2(v) = 0$; if $d_G^1(v) \geq 2$, then $d_G^3(v) = 0$.*

(3) *If $d_G(v) = 5$, then $d_G^3(v) \leq 1$.*

(4) *If $d_G(v) = 6$, then $d_G^{3^-}(v) \leq 2$. Furthermore, if $d_G^{2^-}(v) \geq 1$, then $d_G^{3^-}(v) \leq 1$.*

(5) *If $d_G(v) = 7$, then $d_G^{2^-}(v) \leq 2$. Furthermore, if $d_G^{2^-}(v) \geq 1$, then $d_G^{3^-}(v) \leq 2$.*

(6) *If $d_G(v) = l$ $(l \geq 8)$, then $d_G^1(v) < \lceil \frac{l}{2} \rceil$.*

(7) *If $d_G(v) = l$ $(l \geq 8)$ and $d_G^2(v) \geq 1$, then $d_G^2(v) + d_G^3(v) \leq l - 1$.*

(8) *Each 3-face in G is a $(2, 6^+, 6^+)$-face, a $(3, 5^+, 5^+)$-face or a $(4^+, 4^+, 5^+)$-face.*

Claim 3. *Each 4-face in G is a $(2, 6^+, 3^+, 6^+)$-face, a $(3, 6^+, 3, 6^+)$-face, a $(3, 5^+, 4^+, 5^+)$-face or a $(4^+, 4^+, 4^+, 4^+)$-face.*

Proof. Let $T = v_1v_2v_3v_4v_1$ be a 4-face of G, and assume that $d_G(v_1) \leq d_G(v_i)$, where $i = 2, 3, 4$. If $d_G(v_1) = 2$, by Claim 2(1), $d_G(v_2) \geq 6, d_G(v_4) \geq 6$. By Claim 1, F_1 is reducible, thus T is a $(2, 6^+, 3^+, 6^+)$-face. If $d_G(v_1) = d_G(v_3) = 3$, by Claim 2(1) and Claim 2(3), $d_G(v_2) \geq 6$ and $d_G(v_4) \geq 6$, thus T is a $(3, 6^+, 3, 6^+)$-face. If $d_G(v_1) = 3$ and $d_G(v_3) \geq 4$, by Claim 2(1), $d_G(v_2) \geq 5$ and $d_G(v_4) \geq 5$, thus T is a $(3, 5^+, 4^+, 5^+)$-face. If $d_G(v_1) \geq 4$ and $d_G(v_3) \geq 4$, by Claim 2(1), $d_G(v_2) \geq 4$ and $d_G(v_4) \geq 4$, thus T is a $(4^+, 4^+, 4^+, 4^+)$-face. ■

Let H be the graph obtained from G by removing all 1-vertices. By Claims 1–3, we have the following facts.

Fact 1. For the graph H, we have $\delta(H) \geq 2$; $d_H(v) = d_G(v)$, for $2 \leq d_G(v) \leq 5$. If $d_G(v) \geq 6$, then $d_H(v) \geq 5$.

Fact 2.

(1) In the graph H, each 3^--vertex is not adjacent to any 4^--vertex.

(2) If $d_H(v) = 5$, then $d_H^2(v) = 0$ and $d_H^3(v) \leq 1$.

(3) If $d_H(v) = 6$, then $d_H^2(v) \leq 1$; furthermore, if $d_H^2(v) = 1$, then $d_H^3(v) = 0$; if $d_H^2(v) = 0$, then $d_H^3(v) \leq 2$.

(4) If $d_H(v) = 7$, then $d_H^2(v) \leq 2$; furthermore, if $d_H^2(v) = 2$, then $d_H^3(v) = 0$; if $d_H^2(v) = 1$, then $d_H^3(v) \leq 1$.

(5) If $d_H(v) = l$ ($l \geq 8$), then $d_H^2(v) \leq l - 1$.

Fact 3.

(1) Each 3-face in H is a $(2, 6^+, 6^+)$-face, a $(3, 5^+, 5^+)$-face or a $(4^+, 4^+, 5^+)$-face.

(2) Each 4-face in H is a $(2, 6^+, 3^+, 6^+)$-face, a $(3, 6^+, 3, 6^+)$-face, a $(3, 5^+, 4^+, 5^+)$-face or a $(4^+, 4^+, 4^+, 4^+)$-face.

A $(2, 6^+, 6^+)$-face or a $(3, 5^+, 5^+)$-face is called a *bad* 3-face. A $(4^+, 5^+, 5^+)$-face is called a *normal* 3-face. A $(2, 6^+, 3, 6^+)$-face or a $(3, 6^+, 3, 6^+)$-face is called a *bad* 4-face, and other 4-face is a *normal* 4-face. We use $n_i'(v)$, $n_i''(v)$ to denote the number of bad i-faces and the number of normal i-faces incident with v in H, respectively, $i = 3, 4$.

Since G has no 5-cycles, we have the following fact.

Fact 4. These configurations are reducible to H:

(1) a 5-face,

(2) a 3-face adjacent to two 3-faces,

(3) a 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

Fact 5. If $d_H(v) = l$ and $n_H^3(v) > 0$, then $n_H^3(v) + n_H^4(v) \leq l - 2$.

By Euler's formula, we have

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.$$

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: $w(v) = 2d_H(v) - 6$ for each $v \in V(H)$; $w(f) = d_H(f) - 6$ for each $f \in F(H)$. Next, we will design some discharging rules. Let w' be the new charge after the discharging process. It suffices to show that $w'(x) \geq 0$ for each $x \in V(H) \cup F(H)$, which leads to a contradiction.

In the following, a k-face means a k-face in H, the discharging rules are defined as follows.

R1 Every 2-vertex v in H takes 1 from each neighbor.

R2 Every 4-vertex v in H gives 1 to each incident 3-face, gives $\frac{1}{2}$ to each incident 4-face.

R3 Every 5^+-vertex v in H gives $\frac{3}{2}$ to each incident bad 3-face, gives 1 to each incident normal 3-face.

R4 Every 5^+-vertex v in H gives 1 to each incident bad 4-face, gives $\frac{3}{4}$ to each incident normal 4-face.

We will verify the new charge of each $x \in V(H) \cup F(H)$. In the following, we use $d(v)$, $d_i(v)$, $n_i(v)$ and $d(f)$ to denote $d_H(v)$, $d_H^i(v)$, $n_H^i(v)$ and $d_H(f)$, respectively. We first consider the new charge of each $f \in F(H)$.

• $d(f) = 3$. If f is a bad 3-face, by R3, $w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0$; otherwise, by R2 and R3, $w'(f) = 3 - 6 + 1 \cdot 3 = 0$.

• $d(f) = 4$. If f is a bad 4-face, by R4, $w'(f) = 4 - 6 + 1 \cdot 2 = 0$. If f is a $(2, 6^+, 4^+, 6^+)$-face or a $(3, 5^+, 4^+, 5^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{3}{4} \cdot 2 + \frac{1}{2} = 0$. If f is a $(4^+, 4^+, 4^+, 4^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0$.

• $d(f) = t$ $(t \geq 6)$. $w'(f) = w(f) = t - 6 \geq 0$.

Next we will consider the new charge of each $v \in V(H)$.

• $d(v) = 2$. By R1, $w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0$.

• $d(v) = 3$. No rule applies to v, $w'(v) = 2 \cdot 3 - 6 = 0$.

• $d(v) = 4$. By Fact 2(1), $d_2(v) = d_3(v) = 0$. If $n_3(v) = 0$, by R2, $w'(v) = 2 \cdot 4 - 6 - \frac{1}{2} \cdot n_4(v) \geq 2 - \frac{1}{2} \cdot 4 = 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 2$. By R2, $w'(v) = 2 \cdot 4 - 6 - 1 \cdot n_3(v) - \frac{1}{2} \cdot n_4(v) \geq 2 - 1 \cdot 2 = 0$.

• $d(v) = 5$. By Fact 2(2), $d_2(v) = 0$, $d_3(v) \leq 1$, so we have $n_3'(v) \leq 2$ and $n_4'(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{4} \cdot n_4''(v) \geq 4 - \frac{3}{4} \cdot 5 = \frac{1}{4} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4''(v) \leq 3$. By R3 and R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_3''(v) - \frac{3}{4} \cdot n_4''(v) \geq 4 - \frac{3}{2} \cdot 2 - 1 = 0$.

• $d(v) = 6$. By Fact 2(3), $d_2(v) \leq 1$.

(a) $d_2(v) = 1$. By Fact 2(3), $d_3(v) = 0$, so we have $n_3'(v) \leq 1$ and $n_4'(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{4} \cdot n_4''(v) \geq 6 - 1 - \frac{3}{4} \cdot 6 = \frac{1}{2} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4''(v) \leq 4$. By R1, R3 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_3''(v) - \frac{3}{4} \cdot n_4''(v) \geq 6 - 1 - \frac{3}{2} \cdot 1 - 1 \cdot 3 = \frac{1}{2} > 0$.

(b) $d_2(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) \geq 2 \cdot 6 - 6 - 1 \cdot n_4(v) \geq 6 - 1 \cdot 6 = 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 4$. By R3 and R4, $w'(v) \geq 2 \cdot 6 - 6 - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 6 - \frac{3}{2} \cdot 4 = 0$.

• $d(v) = 7$. By Fact 2(4), $d_2(v) \leq 2$.

(a) $d_2(v) = 2$. By Fact 2(4), $d_3(v) = 0$. By Claim 1, F_1 and F_2 are reducible, so we have $n_3'(v) = n_4'(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - \frac{3}{4} \cdot n_4''(v) \geq 8 - 2 - \frac{3}{4} \cdot 7 = \frac{3}{4} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 5$. Noting that $n_3'(v) = n_4'(v) = 0$, By R1, R3 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_3''(v) - \frac{3}{4} \cdot n_4''(v) \geq 8 - 2 - 1 \cdot 5 > 0$.

(b) $d_2(v) \leq 1$. If $n_3(v) = 0$, by R1 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_4(v) \geq 8 - 1 - 1 \cdot 7 = 0$. If $n_3(v) > 0$, by Fact 4 and Fact 5, $n_3(v) \leq 4$ and $n_3(v) + n_4(v) \leq 5$. By R1, R3 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 8 - 1 - \frac{3}{2} \cdot 4 - 1 = 0$.

• $d(v) = l$ $(l \geq 8)$, by Fact 2(5), $d_2(v) \leq l - 1$.

(a) $d_2(v) = l-1$. By Claim 1, F_1 and F_2 are reducible, so we have $n_3(v) = 0$ and $n_4(v) \leq 2$. By R1 and R4 ,$w'(v) \geq 2l-6-d_2(v)-1\cdot n_4(v) \geq 2l-6-(l-1)-1\cdot 2 = l-7 > 0$.

(b) $d_2(v) = l-2$.

(b1) $n_3(v) = 0$. By Claim 1, F_1 is reducible, so we have $n_4(v) \leq 4$. By R1 and R4, $w'(v) \geq 2l-6-d_2(v)-1\cdot n_4(v) \geq 2l-6-(l-2)-4 = l-8 \geq 0$.

(b2) $n_3(v) > 0$. By Claim 1, F_1 and F_2 are reducible, and by Fact 4, we have $n_3(v) = 1$ and $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l-6-d_2(v)-\frac{3}{2}\cdot n_3(v) \geq 2l-6-(l-2)-\frac{3}{2} = l-\frac{11}{2} > 0$.

(c) $d_2(v) = l-3$.

(c1) $n_3(v) = 0$. By Claim 1, F_1 is reducible, so we have $n_4(v) \leq 6$.

If $n_4(v) = 6$, by Claim 1, F_3 is reducible, so we have $n'_4(v) = 0$. By R1 and R4, $w'(v) = 2l-6-d_2(v)-\frac{3}{4}\cdot n''_4(v) = 2l-6-(l-3)-\frac{3}{4}\cdot 6 = l-\frac{15}{2} > 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l-6-d_2(v)-1\cdot n_4(v) \geq 2l-6-(l-3)-1\cdot 5 = l-8 \geq 0$.

(c2) $n_3(v) > 0$. By Claim 1, F_2 is reducible, so we have $n_3(v) \leq 2$. By Claim 1, F_1 is reducible, and by Fact 4, we have $n_4(v) \leq 2$. By R1, R3 and R4, $w'(v) \geq 2l-6-d_2(v)-\frac{3}{2}\cdot n_3(v)-1\cdot n_4(v) \geq 2l-6-(l-3)-\frac{3}{2}\cdot 2-2 = l-8 \geq 0$.

(d) $d_2(v) = l-4$.

(d1) $n_3(v) = 0$. By Claim 1, F_1 is reducible, so we have $n_4(v) \leq 8$.

$n_4(v) = i$ $(i = 7, 8)$. By Claim 1, F_3 is reducible, so we have $n'_4(v) \leq 8-i$. By R1 and R4, $w'(v) = 2l-6-d_2(v)-1\cdot n'_4(v)-\frac{3}{4}\cdot n''_4(v) \geq 2l-6-(l-4)-1\cdot(8-i)-\frac{3}{4}\cdot(i-(8-i)) = l-4-\frac{i}{2} \geq 0$.

$n_4(v) \leq 6$. By R1 and R4, $w'(v) \geq 2l-6-d_2(v)-1\cdot n_4(v) \geq 2l-6-(l-4)-1\cdot 6 = l-8 \geq 0$.

(d2) $n_3(v) > 0$. By Claim 1, F_2 is reducible, so each 2-neighbor of v is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have $n_3(v) \leq 2$.

$n_3(v) = i$ $(i = 1, 2)$. By Claim 1, F_1 and F_2 are reducible, and note that each 3-face is not adjacent to a 4-face, we have $n_4(v) \leq 6-2i$. By R1, R3 and R4, $w'(v) \geq 2l-6-d_2(v)-\frac{3}{2}\cdot n_3(v)-1\cdot n_4(v) \geq 2l-6-(l-4)-\frac{3}{2}\cdot i-1\cdot(6-2i) = l-8+\frac{i}{2} > 0$.

(e) $d_2(v) = l-5$.

(e1) $n_3(v) = 0$. If $n_4(v) \leq l-1$, by R1 and R4, $w'(v) \geq 2l-6-d_2(v)-1\cdot n_4(v) \geq 2l-6-(l-5)-1\cdot(l-1) = 0$. Now suppose that $n_4(v) = l$. By Claim 1, F_1 is reducible, so we have $d_2(v) \leq \lfloor \frac{l}{2} \rfloor$. Noting that $d_2(v) = l-5$, we have $8 \leq l \leq 10$. By Claim 1, F_1, F_3 and F_4 are reducible, so we have $n'_4(v) \leq 4$. By R1 and R4, $w'(v) = 2l-6-d_2(v)-1\cdot n'_4(v)-\frac{3}{4}\cdot n''_4(v) \geq 2l-6-(l-5)-1\cdot 4-\frac{3}{4}\cdot(l-4) = \frac{l}{4}-2 \geq 0$.

(e2) $n_3(v) > 0$. By Claim 1, F_2 is reducible, and by Fact 4, we have $n_3(v) \leq 3$.

$n_3(v) = 3$. By Claim 1, F_1 is reducible, and by Fact 4, we have $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l-5) - \frac{3}{2} \cdot 3 = l - \frac{11}{2} > 0$.

$n_3(v) = i$ ($i = 1, 2$). By Claim 1, F_1 is reducible, and by Fact 4, we have $n_4(v) \leq 8 - 2i$. By Claim 1, F_3 is reducible. So if $n_4(v) = 8 - 2i$, we have $n'_4(v) = 0$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - \frac{3}{4} \cdot n''_4(v) \geq 2l - 6 - (l-5) - \frac{3}{2} \cdot i - \frac{3}{4} \cdot (8 - 2i) = l - 7 > 0$. If $n_4(v) \leq 7 - 2i$, by R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l-5) - \frac{3}{2} \cdot i - 1 \cdot (7 - 2i) = l + \frac{i}{2} - 8 > 0$.

(f) $d_2(v) \leq l - 6$. Set $t = \left\lceil \frac{2(l - d_2(v) - 1)}{3} \right\rceil$. By Claim 1, F_2 is reducible, and by Fact 4, we have $n_3(v) \leq t$, $n_4(v) \leq l$ and if $n_3(v) > 0$, then $n_3(v) + n_4(v) \leq l - 2$.

(f1) $n_3(v) = 0$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - l \geq 2l - 6 - (l-6) - l = 0$.

(f2)$n_3(v) > 0$, by R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - n_4(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - (l - 2 - n_3(v)) \geq l - 4 - d_2(v) - \frac{1}{2} \cdot t = l - 4 - d_2(v) - \frac{1}{2} \left\lceil \frac{2(l - d_2(v) - 1)}{3} \right\rceil \geq 0$.

Now we get that for each $x \in V(H) \cup F(H)$, $w'(x) \geq 0$, which is a contradiction. This completes the proof of Theorem 3.

3. The Proof of Theorem 4

The proof of Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let G be a minimum counterexample to Theorem 4 which is embedded in the plane. Set $k = \max\{\Delta(G) + 1, 10\}$. By the choice of G, any planar graph G' without 5-cycles and without adjacent $\Delta(G)$-vertices which is smaller than G has a k-tnsd-coloring ϕ'. Similarly, we will choose some G' and extend the coloring ϕ' of G' to a desired coloring ϕ of G to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

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