ON TOTAL $H$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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Abstract

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least to one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the $H$-irregular total $k$-labeling of a graph $G$ admitting

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an $H$-covering if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$ is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling.

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

Keywords: $H$-covering, $H$-irregular labeling, total $H$-irregularity strength, copies of graphs, union of graphs.

2010 Mathematics Subject Classification: 05C78, 05C70.

1. Introduction

Consider a simple and finite graph $G$ with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a total labeling. For a total $k$-labeling $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated total vertex-weight of a vertex $x$ is

$$wt_{\psi}(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

and the associated total edge-weight of an edge $xy$ is

$$wt_{\psi}(xy) = \psi(x) + \psi(xy) + \psi(y).$$

A total $k$-labeling $\psi$ is defined to be an edge irregular total $k$-labeling of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ there is $wt_{\psi}(xy) \neq wt_{\psi}(x'y')$ and to be a vertex irregular total $k$-labeling of $G$ if for every two distinct vertices $x$ and $y$ of $G$ there is $wt_{\psi}(x) \neq wt_{\psi}(y)$. This concept was given by Bača, Jendrol’, Miller and Ryan in [8].

The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of the graph $G$, $\text{tes}(G)$. Analogously, we define the total vertex irregularity strength of $G$, $\text{tvs}(G)$, as the minimum $k$ for which there exists a vertex irregular total $k$-labeling of $G$.

The following lower bound on the total edge irregularity strength of a graph $G$ is given in [8].

$$\text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\},$$

where $\Delta(G)$ is the maximum degree of $G$. This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$. 

Ivančo and Jendrol’ [12] posed a conjecture that for an arbitrary graph $G$ different from $K_5$ with maximum degree $\Delta(G)$, $\text{tes}(G) = \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{(\Delta(G) + 1)/2} \right\rceil \right\}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2$ in [10].

The next theorem gives a lower bound for the total $H$-irregularity strength.

The bounds for the total vertex irregularity strength are given in [8] as follows.

\begin{equation}
\left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1,
\end{equation}

where $\delta(G)$ is the minimum degree of $G$.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for $r$-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

\begin{equation}
\text{tvs}(G) \leq 3 \left\lceil \frac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \frac{3|V(G)|}{\delta(G)} + 1 + 4.
\end{equation}

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq (|V(G)|)^{0.5} \ln |V(G)|$, then

\begin{equation}
\text{tvs}(G) \leq \frac{(2+o(1))|V(G)|}{\delta(G)} + 4.
\end{equation}

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering.

Let $G$ be a graph admitting an $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\psi$, we define the associated $H$-weight as

$$
wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).
$$

A total $k$-labeling $\psi$ is called to be an $H$-irregular total $k$-labeling if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. This definition was introduced by Ashraf, Baća, Lásicsáková and Semaničová-Feňovčíková [7]. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total $H$-irregularity strength.
Theorem 1 [7]. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G, H) \geq \left[ 1 + \frac{t-1}{|V(H)| + |E(H)|} \right].$$

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. Let $r, s$, $2 \leq s \leq r$, be positive integers. Then

$$\text{ths}(P_r, P_s) = \left\lceil \frac{r+s-1}{2s-1} \right\rceil.$$

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

Corollary 3. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \geq \left[ 1 + \frac{mt-1}{|V(H)| + |E(H)|} \right].$$

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. Let $G$ be a graph having an $H$-irregular total $\text{ths}(G, H)$-labeling $f$. Let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{w_{t}^\max(H) - w_{t}^\min(H) + 1}{|V(H)| + |E(H)|} \right\rceil,$$

where $w_{t}^\max(H)$ and $w_{t}^\min(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$.

Proof. Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$. The smallest
weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $\text{wt}_{f}^{\min}(H)$. Evidently

$$\text{wt}_{f}^{\min}(H) \geq |V(H)| + |E(H)|.$$  

(5)

Analogously, the largest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $\text{wt}_{f}^{\max}(H)$. It holds that

$$\text{wt}_{f}^{\max}(H) \geq \text{wt}_{f}^{\min}(H) + t - 1$$

and

$$\text{wt}_{f}^{\max}(H) \leq (|V(H)| + |E(H)|)k.$$  

(7)

Thus $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ and

$$\{\text{wt}_{f}(H^i_j) : j = 1, 2, \ldots, t\} \subset \{\text{wt}_{f}^{\min}(H), \text{wt}_{f}^{\min}(H) + 1, \ldots, \text{wt}_{f}^{\max}(H)\}.$$

(8)

By the symbol $x_i$, $i = 1, 2, \ldots, m$, we denote an element (a vertex or an edge) in the $i$th copy of $G$, denoted by $G_i$, corresponding to the element $x$ in $G$, i.e., $x \in V(G) \cup E(G)$. Analogously, let $H^i_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, be the subgraph in the $i$th copy of $G$ corresponding to the subgraph $H^j$ in $G$.

Let us define the total labeling $g$ of $mG$ in the following way. For $i = 1, 2, \ldots, m$ let

$$g(x_i) = f(x) + (i - 1) \left\lceil \frac{\text{wt}_{f}^{\max}(H) - \text{wt}_{f}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$  

Evidently, all the labels are at most

$$k + (m - 1) \left\lceil \frac{\text{wt}_{f}^{\max}(H) - \text{wt}_{f}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$  

For the weight of every subgraph $H^i_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, isomorphic to the graph $H$ under the labeling $g$ we have

$$\text{wt}_{g}(H^i_j) = \sum_{v \in V(H^i_j)} g(v) + \sum_{e \in E(H^i_j)} g(e)$$

$$= \sum_{v \in V(H^i_j)} \left( f(v) + (i - 1) \left\lceil \frac{\text{wt}_{f}^{\max}(H) - \text{wt}_{f}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right)$$

$$+ \sum_{e \in E(H^i_j)} \left( f(e) + (i - 1) \left\lceil \frac{\text{wt}_{f}^{\max}(H) - \text{wt}_{f}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right).$$
\[\begin{align*}
&= \sum_{v \in V(H^j)} f(v) + \sum_{e \in E(H^j)} f(e) + |V(H)|((i-1) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + (wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1) \\
&= wt_f(H^j) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.
\end{align*}\]

This means that in the given copy of \(G\) the \(H\)-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to \(H\) under the total labeling \(g\) in the \(i\)th copy of \(G\), \(i = 1, 2, \ldots, m\), denoted by \(wt_g^{\text{max}}(H : H \subset G_i)\), is at most

\[wt_g^{\text{max}}(H : H \subset G_i) \leq wt_f^{\text{max}}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil\]

and the smallest weight of a subgraph isomorphic to \(H\) under the total labeling \(g\) in the \((i+1)\)th copy of \(G\), \(i = 1, 2, \ldots, m - 1\), denoted by \(wt_g^{\text{min}}(H : H \subset G_{i+1})\), is at least

\[wt_g^{\text{min}}(H : H \subset G_{i+1}) \geq wt_f^{\text{min}}(H) + (|V(H)| + |E(H)|)(i) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil\]

After some manipulation we get

\[wt_g^{\text{min}}(H : H \subset G_{i+1}) \geq wt_f^{\text{min}}(H) + (|V(H)| + |E(H)|)(i) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil\]

As

\[\left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \geq \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}\]

we obtain

\[wt_g^{\text{min}}(H : H \subset G_{i+1}) \geq wt_f^{\text{min}}(H) + (|V(H)| + |E(H)|)(i) \left\lceil \frac{wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + (wt_{f}^{\text{max}}(H) - wt_{f}^{\text{min}}(H) + 1)\]
\[
= \ wt_f^{\text{max}}(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{\ wt_f^{\text{max}}(H) - \ wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right] + 1
\]
\[
\geq \ wt_g^{\text{max}}(H : H \subset G_i) + 1 > \ wt_g^{\text{max}}(H : H \subset G_i).
\]

Thus in all components the \(H\)-weights are distinct. This concludes the proof. \(\blacksquare\)

We obtain the following corollary.

**Corollary 5.** Let \(G\) be a graph admitting an \(H\)-irregular total \(\text{ths}(G, H)\)-labeling \(f\). Let \(m\) be a positive integer. Then

\[
\text{ths}(mG, H) \leq m \text{ths}(G, H).
\]

**Proof.** Let \(f\) be a \(\text{ths}(G, H)\)-labeling of a graph \(G\) and let \(\text{ths}(G, H) = k\). As \(\ wt_f^{\text{min}}(H) \geq |V(H)| + |E(H)|\) and \(\ wt_f^{\text{max}}(H) \leq (|V(H)| + |E(H)|)k\) we get

\[
\left[ \frac{\ wt_f^{\text{max}}(H) - \ wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right] \leq \left[ \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right]
\]
\[
= \left[ k - 1 + \frac{1}{|V(H)| + |E(H)|} \right] = k.
\]

Hence, by Theorem 4,

\[
\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left[ \frac{\ wt_f^{\text{max}}(H) - \ wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right] \leq k + (m - 1)k = mk.
\]

Let \(\{H^1, H^2, \ldots, H^t\}\) be the set of all subgraphs of \(G\) isomorphic to \(H\). Let \(f\) be an \(H\)-irregular total \(k\)-labeling of a graph \(G\) with \(\text{ths}(G, H) = k\) such that

\[
\{ \ wt_f(H^j) : j = 1, 2, \ldots, t \}
\]
\[
= \{ \ wt_f^{\text{min}}(H), \ wt_f^{\text{min}}(H) + 1, \ldots, \ wt_f^{\text{min}}(H) + t - 1 \}.
\]

(9)

Evidently, if the fraction

\[
\frac{\ wt_f^{\text{max}}(H) - \ wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} = \frac{t}{|V(H)| + |E(H)|}
\]

is an integer then the weights of all \(H\)-weights in \(mG\) under the total labeling \(g\) of \(mG\) defined in the proof of Theorem 4 constitute the set

\[
\{ \ wt_g^{\text{min}}(H), \ wt_g^{\text{min}}(H) + 1, \ldots, \ wt_g^{\text{min}}(H) + mt - 1 \}.
\]

In particular, this implies that the upper bound for \(\text{ths}(mG, H)\) given in Theorem 4 is tight if \(G\) is a graph that satisfies the conditions mentioned above.
**Theorem 6.** Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total $\text{ths}(G,H)$-labeling of $G$ such that 

$$\{wt_f(H^j) : j = 1, 2, \ldots, t\} = \{wt_f^\min(H), wt_f^\min(H) + 1, \ldots, wt_f^\min(H) + t - 1\}.$$ 

If the fraction $\frac{t}{|V(H)| + |E(H)|}$ is an integer then 

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|}.$$ 

Moreover, if $\text{ths}(G, H) = \left[1 + \frac{t}{|V(H)| + |E(H)|}\right] = 1 + \frac{t}{|V(H)| + |E(H)|}$ then 

$$\text{ths}(mG, H) = \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|} = 1 + \frac{ml}{|V(H)| + |E(H)|}.$$ 

Theorem 2 gives the exact value for the total $P_s$-irregularity strength for a path $P_r$. Moreover, the $P_s$-irregular total $|((s+r-1)/(2s-1))|$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_s$-weights consists of $t$ consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that 

$$\text{ths}(mP_r, P_s) = \text{ths}(P_r, P_s) + (m - 1)\frac{r-s+1}{2s-1} = \left[r-s+1 \atop 2s-1\right] + (m - 1)\frac{r-s+1}{2s-1}$$ 

$$= \left[r-s+1 + 2s-1 \atop 2s-1\right] + (m - 1)\frac{r-s+1}{2s-1}$$ 

$$= \left(r-s+1 \atop 2s-1\right) + 1 + (m - 1)\frac{r-s+1}{2s-1} = m\frac{r-s+1}{2s-1} + 1.$$ 

Thus we obtain the following result.

**Corollary 7.** Let $m, r, s, m \geq 1, 2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then 

$$\text{ths}(mP_r, P_s) = \frac{m(r-s+1)}{2s-1} + 1.$$ 

If $H$ is isomorphic to $K_2$ then $\text{ths}(G, K_2) = \text{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

**Corollary 8.** Let $m$ be a positive integer. Then 

$$\left\lfloor \frac{m|E(G)|+2}{3} \right\rfloor \leq \text{ths}(mG, K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m - 1)\left\lfloor \frac{\min wt_{\text{max}} - \min wt_{\text{min}}+1}{3} \right\rfloor,$$ 

where $\min wt_{\text{max}}$ and $\min wt_{\text{min}}$ are the largest and smallest edge weights under a total $\text{tes}(G)$-labeling $f$ of $G$. 


3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total \( H \)-irregularity strength of two graphs \( G_1 \) and \( G_2 \) admitting an \( H \)-covering. From Theorem 1 we immediately obtain

**Corollary 9.** Let \( G_i, \ i = 1, 2, \) be a graph admitting an \( H \)-covering given by \( t_i \) subgraphs isomorphic to \( H \). Then

\[
\text{ths}(G_1 \cup G_2, H) \geq 1 + \left\lceil \frac{t_1+t_2-1}{|V(H)|+|E(H)|} \right\rceil.
\]

The next theorem gives an upper bound for \( \text{ths}(G_1 \cup G_2, H) \).

**Theorem 10.** Let \( G_i, \ i = 1, 2, \) be a graph having an \( H \)-irregular total \( \text{ths}(G_i, H) \)-labeling \( f_i \). Then

\[
\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \frac{w_{f_i}^{\max}(H) - w_{f_i}^{\min}(H) + 1}{|V(H)|+|E(H)|} \right\rceil \right\}, \\
\max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \frac{w_{f_i}^{\max}(H) - w_{f_i}^{\min}(H) + 1}{|V(H)|+|E(H)|} \right\rceil \right\} \right\},
\]

where \( w_{f_i}^{\max}(H) \) and \( w_{f_i}^{\min}(H) \) are the largest and smallest weights of a subgraph \( H \) under a total \( \text{ths}(G, H) \)-labeling \( f_i \) of \( G_i \).

**Proof.** Let \( G_i, \ i = 1, 2, \) be a graph that admits an \( H \)-covering given by \( t_i \) subgraphs isomorphic to \( H \). We denote these subgraphs as \( H^1_i, H^2_i, \ldots, H^{t_i}_i \).

Assume that \( f_i \) is an \( H \)-irregular total \( k_i \)-labeling of a graph \( G_i \) with \( \text{ths}(G_i, H) = k_i \). The smallest weight of a subgraph \( H \) under the total \( k_i \)-labeling \( f_i \) is denoted by the symbol \( w_{f_i}^{\min}(H) \). Evidently

\[
(10) \quad w_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|.
\]

Analogously, the largest weight of a subgraph \( H \) under the total \( k_i \)-labeling \( f_i \) is denoted by the symbol \( w_{f_i}^{\max}(H) \). It holds that

\[
(11) \quad w_{f_i}^{\max}(H) \geq w_{f_i}^{\min}(H) + t_i - 1
\]

and

\[
(12) \quad w_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i.
\]

Thus \( f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, k_i\} \) and

\[
(13) \quad \{w_{f_i}(H^j_i) : j = 1, 2, \ldots, t_i\} \subset \{w_{f_i}^{\min}(H), w_{f_i}^{\min}(H) + 1, \ldots, w_{f_i}^{\max}(H)\}.
\]
Let us define the total labeling $g$ of $G_1 \cup G_2$ in the following way.

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in V(G_1) \cup E(G_1), \\ f_2(x) + \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} & \text{if } x \in V(G_2) \cup E(G_2). \end{cases}$$

Evidently, all the labels are not greater than

$$\max \left\{ k_1, k_2 + \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\}.$$

For the weight of the subgraph $H_j^1$, $j = 1, 2, \ldots, t_1$, isomorphic to the graph $H$ under the labeling $g$ we get

$$wt_g(H_j^1) = \sum_{v \in V(H_j^1)} g(v) + \sum_{e \in E(H_j^1)} g(e) = \sum_{v \in V(H_j^1)} f_1(v) + \sum_{e \in E(H_j^1)} f_1(e) = wt_{f_1}(H_j^1).$$

For the weight of the subgraph $H_j^2$, $j = 1, 2, \ldots, t_2$, isomorphic to the graph $H$ under the labeling $g$ we get

$$wt_g(H_j^2) = \sum_{v \in V(H_j^2)} g(v) + \sum_{e \in E(H_j^2)} g(e) = \sum_{v \in V(H_j^2)} f_2(v) + \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} + \sum_{e \in E(H_j^2)} f_2(e) + \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|}$$

$$= \sum_{v \in V(H_j^2)} f_2(v) + \sum_{e \in E(H_j^2)} f_2(e) + |V(H)| \left[ \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right]$$

$$+ |E(H)| \left[ \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right]$$

$$= wt_{f_2}(H_j^2) + (|V(H)| + |E(H)|) \left[ \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right].$$

According to (13) we get that the largest weight of a subgraph $H$ under the total labeling $g$ in $G_1$, denoted by $wt_g^{\max}(H : H \subset G_1)$, is at most

$$wt_g^{\max}(H : H \subset G_1) = wt_{f_1}^{\max}(H)$$

and the smallest weight of a subgraph $H$ under the total labeling $g$ in $G_2$, denoted by $wt_g^{\min}(H : H \subset G_2)$, is at least

$$wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left[ \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right].$$
Note, that when writing $H_i$ we only consider subgraphs of $G_i$ isomorphic to $H$. As
\[
\left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \geq \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|}
\]
we get
\[
wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil
\]
\[
\geq wt_{f_2}^{\min}(H) + (wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1) = wt_{f_1}^{\max}(H) + 1
\]
\[
> wt_{f_1}^{\max}(H) = wt_g^{\max}(H : H \subset G_1).
\]
Thus all the $H$-weights under the labeling $g$ in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling $h$ of $G_1 \cup G_2$ such that
\[
h(x) = f_2(x) \quad \text{if } x \in V(G_2) \cup E(G_2),
\]
\[
h(x) = f_1(x) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \quad \text{if } x \in V(G_1) \cup E(G_1).
\]
Using similar arguments we can also show that under the total labeling $h$ the $H$-weights in $G_1 \cup G_2$ are distinct.

Thus $g$ and $h$ are $H$-irregular total labelings of $G$. Immediately from this fact we get
\[
ths(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ ths(G_2, H), ths(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}, \right. \\
\left. \max \left\{ ths(G_1, H), ths(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\} \right\}.
\]

Ramdani, Salman, Assiyatum, Semaníčová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

**Theorem 11** [18]. The total edge irregularity strength of the disjoint union of graphs $G_1, G_2, \ldots, G_m$, $m \geq 2$, is
\[
tes \left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.
\]
If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that

$$\text{ths}(G_1 \cup G_2, K_2) = \text{tes}(G_1 \cup G_2)$$

$$\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lceil \frac{3\text{tes}(G_2) - 2}{3} \right\rceil \right\}, \right.$$

$$\max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lceil \frac{3\text{tes}(G_1) - 2}{3} \right\rceil \right\} \right\}$$

$$= \text{tes}(G_1) + \text{tes}(G_2)$$

which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total $\text{ths}(G, H)$-labeling $f$ and $m$ is a positive integer then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left[ \frac{\text{wt}_{\text{max}}(H) - \text{wt}_{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right],$$

where $\text{wt}_{\text{max}}(H)$ and $\text{wt}_{\text{min}}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

Acknowledgments

The research for this article was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-15-0116, by VEGA 1/0233/18, by the Spanish Research Council under project MTM2014-60127-P and symbolically by the Catalan Research Council under grant 2014SGR1147.

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Received 12 April 2017
Revised 5 February 2018
Accepted 22 February 2018