ON TOTAL $H$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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Abstract

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the $H$-irregular total $k$-labeling of a graph $G$ admitting

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an H-covering if all subgraphs of G isomorphic to H have distinct weights. The total H-irregularity strength of a graph G is the smallest integer k such that G has an H-irregular total k-labeling.

In this paper, we estimate lower and upper bounds on the total H-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

Keywords: H-covering, H-irregular labeling, total H-irregularity strength, copies of graphs, union of graphs.

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1. Introduction

Consider a simple and finite graph G with vertex set V(G) and edge set E(G). By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is V(G) ∪ E(G) then we call the labeling a total labeling. For a total k-labeling ψ : V(G) ∪ E(G) → {1, 2, . . . , k} the associated total vertex-weight of a vertex x is

wt_ψ(x) = ψ(x) + ∑_{xy ∈ E(G)} ψ(xy)

and the associated total edge-weight of an edge xy is

wt_ψ(xy) = ψ(x) + ψ(xy) + ψ(y).

A total k-labeling ψ is defined to be an edge irregular total k-labeling of the graph G if for every two different edges xy and x'y' of G there is wt_ψ(xy) ≠ wt_ψ(x'y') and to be a vertex irregular total k-labeling of G if for every two distinct vertices x and y of G there is wt_ψ(x) ≠ wt_ψ(y). This concept was given by Bača, Jendrol’, Miller and Ryan in [8].

The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of the graph G, tes(G). Analogously, we define the total vertex irregularity strength of G, tvs(G), as the minimum k for which there exists a vertex irregular total k-labeling of G.

The following lower bound on the total edge irregularity strength of a graph G is given in [8].

\[ \text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \]

where Δ(G) is the maximum degree of G. This lower bound is tight for paths, cycles and complete bipartite graphs of the form K_{1,n}. 

Ivančo and Jendrol’ [12] posed a conjecture that for an arbitrary graph $G$ different from $K_5$ with maximum degree $\Delta(G)$, $\text{tes}(G) = \max \{ \lceil |E(G)| + 2 \rceil / 3, \lceil (\Delta(G) + 1) / 2 \rceil \}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $|E(G)| + 2 / 3 \leq (\Delta(G) + 1) / 2$ in [10].

The next theorem gives a lower bound for the total $H$-irregularity strength.

(2) \[ \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1, \]
where $\delta(G)$ is the minimum degree of $G$.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for $r$-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

(3) \[ \text{tvs}(G) \leq 3 \left\lceil \frac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4. \]

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq (|V(G)|)^{0.5} \ln |V(G)|$, then

(4) \[ \text{tvs}(G) \leq \frac{(2 + o(1))|V(G)|}{\delta(G)} + 4. \]

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering.

Let $G$ be a graph admitting an $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\psi$, we define the associated $H$-weight as

\[ wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e). \]

A total $k$-labeling $\psi$ is called to be an $H$-irregular total $k$-labeling of the graph $G$ if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. This definition was introduced by Ashraf, Bača, Lásicsáková and Semaničová-Feňovčíková [7]. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total $H$-irregularity strength.
Theorem 1 [7]. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G, H) \geq 1 + \left\lceil \frac{t-1}{|V(H)|+|E(H)|} \right\rceil.$$ 

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. Let $r, s, 2 \leq s \leq r$, be positive integers. Then

$$\text{ths}(P_r, P_s) = \left\lceil \frac{s+r-1}{2s-1} \right\rceil.$$ 

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

Corollary 3. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \geq 1 + \left\lceil \frac{mt-1}{|V(H)|+|E(H)|} \right\rceil.$$ 

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. Let $G$ be a graph having an $H$-irregular total $\text{ths}(G, H)$-labeling $f$. Let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{\text{wt}_f^{\text{max}}(H) - \text{wt}_f^{\text{min}}(H) + 1}{|V(H)|+|E(H)|} \right\rceil,$$

where $\text{wt}_f^{\text{max}}(H)$ and $\text{wt}_f^{\text{min}}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$.

Proof. Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$. The smallest
weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^\text{min}(H)$. Evidently

$$ wt_f^\text{min}(H) \geq |V(H)| + |E(H)|. $$

(5)

Analogously, the largest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^\text{max}(H)$. It holds that

$$ wt_f^\text{max}(H) \geq wt_f^\text{min}(H) + t - 1 $$

and

$$ wt_f^\text{max}(H) \leq (|V(H)| + |E(H)|)k. $$

(7)

Thus $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ and

$$ \{wt_f(H^j_i) : j = 1, 2, \ldots, t\} \subset \{wt_f^\text{min}(H), wt_f^\text{min}(H) + 1, \ldots, wt_f^\text{max}(H)\}. $$

(8)

By the symbol $x_i$, $i = 1, 2, \ldots, m$, we denote an element (a vertex or an edge) in the $i$th copy of $G$, denoted by $G_i$, corresponding to the element $x$ in $G$, i.e., $x \in V(G) \cup E(G)$. Analogously, let $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, be the subgraph in the $i$th copy of $G$ corresponding to the subgraph $H^j$ in $G$.

Let us define the total labeling $g$ of $mG$ in the following way. For $i = 1, 2, \ldots, m$ let

$$ g(x_i) = f(x) + (i - 1) \left\lceil \frac{wt_f^\text{max}(H)-wt_f^\text{min}(H)+1}{|V(H)|+|E(H)|} \right\rceil. $$

Evidently, all the labels are at most

$$ k + (m - 1) \left\lceil \frac{wt_f^\text{max}(H)-wt_f^\text{min}(H)+1}{|V(H)|+|E(H)|} \right\rceil. $$

For the weight of every subgraph $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, isomorphic to the graph $H$ under the labeling $g$ we have

$$ wt_g(H^j_i) = \sum_{v \in V(H^j_i)} g(v) + \sum_{e \in E(H^j_i)} g(e) $$

$$ = \sum_{v \in V(H^j_i)} \left( f(v) + (i - 1) \left\lceil \frac{wt_f^\text{max}(H)-wt_f^\text{min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right) $$

$$ + \sum_{e \in E(H^j_i)} \left( f(e) + (i - 1) \left\lceil \frac{wt_f^\text{max}(H)-wt_f^\text{min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right). $$
After some manipulation we get

\[
wt_f^g(H : H \subseteq G_i) = wt_f^g(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]

This means that in the given copy of \( G \) the \( H \)-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \( i \)th copy of \( G \), \( i = 1, 2, \ldots, m \), denoted by \( wt_g^\text{max}(H : H \subseteq G_i) \), is at most

\[
wt_g^\text{max}(H : H \subseteq G_i) \leq wt_f^\text{max}(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]

and the smallest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \((i + 1)\)th copy of \( G \), \( i = 1, 2, \ldots, m - 1 \), denoted by \( wt_g^\text{min}(H : H \subseteq G_{i+1}) \), is at least

\[
wt_g^\text{min}(H : H \subseteq G_{i+1}) \geq wt_f^\text{min}(H) + (|V(H)| + |E(H)|)i \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]

After some manipulation we get

\[
wt_g^\text{min}(H : H \subseteq G_{i+1}) \geq wt_f^\text{min}(H) + (|V(H)| + |E(H)|)i \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right],
\]

As

\[
\left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \geq \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|},
\]

we obtain

\[
wt_g^\text{min}(H : H \subseteq G_{i+1}) \geq wt_f^\text{min}(H)
\]

\[
+ (|V(H)| + |E(H)|)(i - 1) \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right],
\]

\[
+ (wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1)
\]

\[
= wt_f^\text{max}(H) + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]
On Total $H$-Irregularity Strength of the Disjoint Union ...

$$= wt_f^{\text{max}}(H) + (|V(H)| + |E(H)|)(i - 1) \left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + 1$$

$$\geq wt_g^{\text{max}}(H : H \subset G_i) + 1 > wt_g^{\text{max}}(H : H \subset G_i).$$

Thus in all components the $H$-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

**Corollary 5.** Let $G$ be a graph admitting an $H$-irregular total ths($G,H$)-labeling $f$. Let $m$ be a positive integer. Then

$$\text{ths}(mG,H) \leq m \text{ths}(G,H).$$

**Proof.** Let $f$ be a ths($G,H$)-labeling of a graph $G$ and let ths($G,H$) = $k$. As $wt_f^{\text{min}}(H) \geq |V(H)| + |E(H)|$ and $wt_f^{\text{max}}(H) \leq (|V(H)| + |E(H)|)k$ we get

$$\left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq \left\lceil \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rceil = \left\lfloor k - 1 + \frac{1}{|V(H)| + |E(H)|} \right\rfloor = k.$$

Hence, by Theorem 4,

$$\text{ths}(mG,H) \leq \text{ths}(G,H) + (m - 1) \left\lceil \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq k + (m - 1)k = mk.$$

Let $\{H^1, H^2, \ldots, H^t\}$ be the set of all subgraphs of $G$ isomorphic to $H$. Let $f$ be an $H$-irregular total $k$-labeling of a graph $G$ with ths($G,H$) = $k$ such that

$$\{ wt_f(H^j) : j = 1, 2, \ldots, t \}$$

$$= \{ wt_f^{\text{min}}(H), wt_f^{\text{min}}(H) + 1, \ldots, wt_f^{\text{min}}(H) + t - 1 \}.$$ (9)

Evidently, if the fraction

$$\frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}$$

is an integer then the weights of all $H$-weights in $mG$ under the total labeling $g$ of $mG$ defined in the proof of Theorem 4 constitute the set

$$\{ wt_f^{\text{min}}(H), wt_f^{\text{min}}(H) + 1, \ldots, wt_f^{\text{min}}(H) + mt - 1 \}.$$

In particular, this implies that the upper bound for ths($mG,H$) given in Theorem 4 is tight if $G$ is a graph that satisfies the conditions mentioned above.
Theorem 6. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total $\text{ths}(G, H)$-labeling of $G$ such that 
\[ \{\text{wt}_f(H^j) : j = 1, 2, \ldots, t\} = \{\text{wt}_{f}^{\min}(H), \text{wt}_{f}^{\min}(H) + 1, \ldots, \text{wt}_{f}^{\min}(H) + t - 1\}. \]

If the fraction \( \frac{t}{|V(H)| + |E(H)|} \) is an integer then

\[ \text{ths}(mG, H) \leq \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|}. \]

Moreover, if \( \text{ths}(G, H) = \left[ 1 + \frac{t}{|V(H)| + |E(H)|}\right] = 1 + \frac{(m-1)t}{|V(H)| + |E(H)|} \) then

\[ \text{ths}(mG, H) = \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|} = 1 + \frac{ml}{|V(H)| + |E(H)|}. \]

Theorem 2 gives the exact value for the total $P_r$-irregularity strength for a path $P_r$. Moreover, the $P_r$-irregular total $\{(s + r - 1)/(2s - 1)\}$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_r$-weights consists of $t$ consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_r)| = s$ and $|E(P_r)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that

\[ \text{ths}(mP_r, P_s) = \text{ths}(P_r, P_s) + (m - 1)\frac{r - s + 1}{2s - 1} = \left[\frac{s + r - 1}{2s - 1}\right] + (m - 1)\frac{r - s + 1}{2s - 1} \]
\[ = \left[\frac{s + r - 1 + 2s - 1 - 1}{2s - 1}\right] + (m - 1)\frac{r - s + 1}{2s - 1} \]
\[ = \left[\frac{r - s + 1}{2s - 1} + 1 - \frac{1}{2s - 1}\right] + (m - 1)\frac{r - s + 1}{2s - 1} \]
\[ = \frac{r - s + 1}{2s - 1} + 1 + (m - 1)\frac{r - s + 1}{2s - 1} = m\frac{r - s + 1}{2s - 1} + 1. \]

Thus we obtain the following result.

Corollary 7. Let $m, r, s, m \geq 1, 2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then

\[ \text{ths}(mP_r, P_s) = \frac{m(r - s + 1)}{(2s - 1)} + 1. \]

If $H$ is isomorphic to $K_2$ then $\text{ths}(G, K_2) = \text{tes}(G)$. Immediately from Theorem 4 the following corollary follows.

Corollary 8. Let $m$ be a positive integer. Then

\[ \left\lceil \frac{m|E(G)| + 2}{3} \right\rceil \leq \text{ths}(mG, K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m - 1)\left\lceil \frac{\text{wt}_{f}^{\min} - \text{wt}_{f}^{\max} + 1}{3}\right\rceil, \]

where $\text{wt}_{f}^{\max}$ and $\text{wt}_{f}^{\min}$ are the largest and smallest edge weights under a total $\text{tes}(G)$-labeling $f$ of $G$. 
3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total $H$-irregularity strength of two graphs $G_1$ and $G_2$ admitting an $H$-covering. From Theorem 1 we immediately obtain

**Corollary 9.** Let $G_i$, $i = 1, 2$, be a graph admitting an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G_1 \cup G_2, H) \geq \left[ 1 + \frac{t_1 + t_2 - 1}{|V(H)| + |E(H)|} \right].$$

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

**Theorem 10.** Let $G_i$, $i = 1, 2$, be a graph having an $H$-irregular total $\text{ths}(G_i, H)$-labeling $f_i$. Then

$$\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left[ \frac{\text{wt}_{f_2}^{\max}(H) - \text{wt}_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \right\}, \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left[ \frac{\text{wt}_{f_1}^{\max}(H) - \text{wt}_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \right\} \right\},$$

where $\text{wt}_{f_i}^{\max}(H)$ and $\text{wt}_{f_i}^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G_i, H)$-labeling $f_i$ of $G_i$.

**Proof.** Let $G_i$, $i = 1, 2$, be a graph that admits an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. We denote these subgraphs as $H_i^1, H_i^2, \ldots, H_i^{t_i}$. Assume that $f_i$ is an $H$-irregular total $k_i$-labeling of a graph $G_i$ with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $\text{wt}_{f_i}^{\min}(H)$. Evidently

$$\text{wt}_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|. \tag{10}$$

Analogously, the largest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $\text{wt}_{f_i}^{\max}(H)$. It holds that

$$\text{wt}_{f_i}^{\max}(H) \geq \text{wt}_{f_i}^{\min}(H) + t_i - 1 \tag{11}$$

and

$$\text{wt}_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i. \tag{12}$$

Thus $f_i : V(G_i) \cup E(G_i) \rightarrow \{1, 2, \ldots, k_i\}$ and

$$\{\text{wt}_{f_i}(H_i^j) : j = 1, 2, \ldots, t_i\} \subset \{\text{wt}_{f_i}^{\min}(H), \text{wt}_{f_i}^{\min}(H) + 1, \ldots, \text{wt}_{f_i}^{\max}(H)\}. \tag{13}$$
Let us define the total labeling \( g \) of \( G_1 \cup G_2 \) in the following way.

\[
g(x) = \begin{cases} 
  f_1(x) & \text{if } x \in V(G_1) \cup E(G_1), \\
  f_2(x) + \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] & \text{if } x \in V(G_2) \cup E(G_2).
\end{cases}
\]

Evidently, all the labels are not greater than

\[
\max \left\{ k_1, k_2 + \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \right\}.
\]

For the weight of the subgraph \( H^j_1 \), \( j = 1, 2, \ldots, t_1 \), isomorphic to the graph \( H \) under the labeling \( g \) we get

\[
wt_g(H^j_1) = \sum_{v \in V(H^j_1)} g(v) + \sum_{e \in E(H^j_1)} g(e) = \sum_{v \in V(H^j_1)} f_1(v) + \sum_{e \in E(H^j_1)} f_2(e) = wt_{f_1}(H^j_1).
\]

For the weight of the subgraph \( H^j_2 \), \( j = 1, 2, \ldots, t_2 \), isomorphic to the graph \( H \) under the labeling \( g \) we get

\[
wt_g(H^j_2) = \sum_{v \in V(H^j_2)} g(v) + \sum_{e \in E(H^j_2)} g(e) \\
= \sum_{v \in V(H^j_2)} \left( f_2(v) + \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \right) \\
+ \sum_{e \in E(H^j_2)} \left( f_2(e) + \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \right) \\
= \sum_{v \in V(H^j_2)} f_2(v) + \sum_{e \in E(H^j_2)} f_2(e) + |V(H)| \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \\
+ |E(H)| \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right] \\
= wt_{f_2}(H^j_2) + (|V(H)| + |E(H)|) \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]

According to (13) we get that the largest weight of a subgraph \( H \) under the total labeling \( g \) in \( G_1 \), denoted by \( wt_g^{\max}(H : H \subset G_1) \), is at most

\[
wt_g^{\max}(H : H \subset G_1) = wt_{f_1}^{\max}(H)
\]

and the smallest weight of a subgraph \( H \) under the total labeling \( g \) in \( G_2 \), denoted by \( wt_g^{\min}(H : H \subset G_2) \), is at least

\[
wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left[ \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right].
\]
Note, that when writing \( H \), we only consider subgraphs of \( G_i \) isomorphic to \( H \).

As

\[
\frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \geq \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}
\]

we get

\[
wt_g^{\text{min}}(H : H \subset G_2) \geq wt_{f_2}^{\text{min}}(H) + (|V(H)| + |E(H)|) \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}
\]

\[
\geq wt_{f_2}^{\text{min}}(H) + (wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1) = wt_{f_1}^{\max}(H) + 1
\]

\[
> wt_{f_1}^{\max}(H) = wt_g^{\max}(H : H \subset G_1).
\]

Thus all the \( H \)-weights under the labeling \( g \) in \( G_1 \cup G_2 \) are distinct.

Analogously we can define the total labeling \( h \) of \( G_1 \cup G_2 \) such that

\[
h(x) = \begin{cases} f_2(x) & \text{if } x \in V(G_2) \cup E(G_2), \\ f_1(x) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil & \text{if } x \in V(G_1) \cup E(G_1). \end{cases}
\]

Using similar arguments we can also show that under the total labeling \( h \) the \( H \)-weights in \( G_1 \cup G_2 \) are distinct.

Thus \( g \) and \( h \) are \( H \)-irregular total labelings of \( G \). Immediately from this fact we get

\[
\text{ths}(G_1 \cup G_2, H) 
\leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}, \\
\max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\} \right\}.
\]

Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

**Theorem 11** [18]. The total edge irregularity strength of the disjoint union of graphs \( G_1, G_2, \ldots, G_m, m \geq 2 \), is

\[
\text{tes} \left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} \text{tes}(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.
\]
If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that
\[
\text{ths}(G_1 \cup G_2, K_2) = \text{tes}(G_1 \cup G_2)
\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lfloor \frac{3 \text{tes}(G_2) - 2}{3} \right\rfloor \right\},
\max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lfloor \frac{3 \text{tes}(G_1) - 2}{3} \right\rfloor \right\} \right\}
= \text{tes}(G_1) + \text{tes}(G_2)
\]
which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total $\text{ths}(G, H)$-labeling $f$ and $m$ is a positive integer then
\[
\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lfloor \frac{\text{wt}^\text{max}(H) - \text{wt}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor,
\]
where $\text{wt}^\text{max}(H)$ and $\text{wt}^\text{min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

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