ON TOTAL $H$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

Faraha Ashraf
Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan
e-mail: faraha27@gmail.com

Susana Clara López
Dept. Matemàtiques, Universitat Politècnica de Catalunya
C/Esteve Terrades 5, 08860 Castelldefels, Spain
e-mail: susana.clara.lopez@upc.edu

Francesc Antoni Muntaner-Batle, Akito Oshima
Graph Theory and Applications Research Group, School of Electrical Engineering
and Computer Science, Faculty of Engineering and Built Environment
The University of Newcastle, Newcastle, Australia
e-mail: fambles@yahoo.es
akitoism@yahoo.co.jp

AND

Martin Bača¹, Andrea Semaničová-Feňovčíková
Department of Applied Mathematics and Informatics
Technical University, Košice, Slovakia
e-mail: martin.baca@tuke.sk
andrea.fenovcikova@tuke.sk

Abstract

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least to one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the $H$-irregular total $k$-labeling of a graph $G$ admitting

¹Corresponding author.
an \( H \)-covering if all subgraphs of \( G \) isomorphic to \( H \) have distinct weights. The \textit{total \( H \)-irregularity strength} of a graph \( G \) is the smallest integer \( k \) such that \( G \) has an \( H \)-irregular total \( k \)-labeling.

In this paper, we estimate lower and upper bounds on the total \( H \)-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

\textbf{Keywords:} \( H \)-covering, \( H \)-irregular labeling, total \( H \)-irregularity strength, copies of graphs, union of graphs.

\textbf{2010 Mathematics Subject Classification:} 05C78, 05C70.

1. Introduction

Consider a simple and finite graph \( G \) with vertex set \( V(G) \) and edge set \( E(G) \). By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is \( V(G) \cup E(G) \) then we call the labeling a \textit{total labeling}. For a total \( k \)-labeling \( \psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) the associated total vertex-weight of a vertex \( x \) is

\[ wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy) \]

and the associated total edge-weight of an edge \( xy \) is

\[ wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y). \]

A total \( k \)-labeling \( \psi \) is defined to be an \textit{edge irregular total \( k \)-labeling} of the graph \( G \) if for every two different edges \( xy \) and \( x'y' \) of \( G \) there is \( wt_\psi(xy) \neq wt_\psi(x'y') \) and to be a \textit{vertex irregular total \( k \)-labeling} of \( G \) if for every two distinct vertices \( x \) and \( y \) of \( G \) there is \( wt_\psi(x) \neq wt_\psi(y) \). This concept was given by Bača, Jendrol’, Miller and Ryan in [8].

The minimum \( k \) for which the graph \( G \) has an edge irregular total \( k \)-labeling is called the \textit{total edge irregularity strength} of the graph \( G \), \( \text{tes}(G) \). Analogously, we define the \textit{total vertex irregularity strength} of \( G \), \( \text{tvs}(G) \), as the minimum \( k \) for which there exists a vertex irregular total \( k \)-labeling of \( G \).

The following lower bound on the total edge irregularity strength of a graph \( G \) is given in [8].

\[
\text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\},
\]

where \( \Delta(G) \) is the maximum degree of \( G \). This lower bound is tight for paths, cycles and complete bipartite graphs of the form \( K_{1,n} \).
Ivančo and Jendröl [12] posed a conjecture that for an arbitrary graph $G$ different from $K_5$ with maximum degree $\Delta(G)$, $\text{tes}(G) = \max \left\{ \lceil \left( |E(G)| + 2 \right) / 3 \rceil , \left( \lceil \Delta(G) + 1 \rceil / 2 \right) \right\}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)| + 2) / 3 \leq \lceil \Delta(G) + 1 \rceil / 2$ in [10].

The next theorem gives a lower bound for the total $H$-irregularity strength.

The bounds for the total vertex irregularity strength are given in [8] as follows.

\begin{equation}
\left[ \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right] \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1,
\end{equation}

where $\delta(G)$ is the minimum degree of $G$.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)| / \delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)| / r + 3$ for $r$-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

\begin{equation}
\text{tvs}(G) \leq 3 \left[ \frac{|V(G)|}{\delta(G)} \right] + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4.
\end{equation}

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq |V(G)|^{0.5} \ln |V(G)|$, then

\begin{equation}
\text{tvs}(G) \leq \frac{(2+o(1))|V(G)|}{\delta(G)} + 4.
\end{equation}

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i, i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering.

Let $G$ be a graph admitting an $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\psi$, we define the associated $H$-weight as

$$wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).$$

A total $k$-labeling $\psi$ is called to be an $H$-irregular total $k$-labeling of the graph $G$ if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. This definition was introduced by Ashraf, Bača, Lásicsáková and Semaničová-Feňovčíková [7]. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total $H$-irregularity strength.
Theorem 1 [7]. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G, H) \geq \left[ 1 + \frac{t-1}{|V(H)|+|E(H)|} \right].$$

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. Let $r, s$, $2 \leq s \leq r$, be positive integers. Then

$$\text{ths}(P_r, P_s) = \left\lceil \frac{s+r-1}{2s-1} \right\rceil.$$

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

Corollary 3. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \geq \left[ 1 + \frac{mt-1}{|V(H)|+|E(H)|} \right].$$

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. Let $G$ be a graph having an $H$-irregular total $\text{ths}(G, H)$-labeling $f$. Let $m$ be a positive integer. Then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{\text{wt}_{\text{max}}(H) - \text{wt}_{\text{min}}(H) + 1}{|V(H)|+|E(H)|} \right\rceil,$$

where $\text{wt}_{\text{max}}(H)$ and $\text{wt}_{\text{min}}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$.

Proof. Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$. The smallest
weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $\text{wt}_{f}^{\text{min}}(H)$. Evidently

$$\text{wt}_{f}^{\text{min}}(H) \geq |V(H)| + |E(H)|.$$  

(5)

Analogously, the largest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $\text{wt}_{f}^{\text{max}}(H)$. It holds that

$$\text{wt}_{f}^{\text{max}}(H) \geq \text{wt}_{f}^{\text{min}}(H) + t - 1$$

and

$$\text{wt}_{f}^{\text{max}}(H) \leq (|V(H)| + |E(H)|)k.$$  

(7)

Thus $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ and

$$\{\text{wt}_{f}(H^j) : j = 1, 2, \ldots, t\} \subset \{\text{wt}_{f}^{\text{min}}(H), \text{wt}_{f}^{\text{min}}(H) + 1, \ldots, \text{wt}_{f}^{\text{max}}(H)\}.$$

(8)

By the symbol $x_i$, $i = 1, 2, \ldots, m$, we denote an element (a vertex or an edge) in the $i^{\text{th}}$ copy of $G$, denoted by $G_i$, corresponding to the element $x$ in $G$, i.e., $x \in V(G) \cup E(G)$. Analogously, let $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, be the subgraph in the $i^{\text{th}}$ copy of $G$ corresponding to the subgraph $H^j$ in $G$.

Let us define the total labeling $g$ of $mG$ in the following way. For $i = 1, 2, \ldots, m$ let

$$g(x_i) = f(x) + (i - 1) \left[ \frac{\text{wt}_{f}^{\text{max}}(H) - \text{wt}_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right].$$

Evidently, all the labels are at most

$$k + (m - 1) \left[ \frac{\text{wt}_{f}^{\text{max}}(H) - \text{wt}_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right].$$

For the weight of every subgraph $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, isomorphic to the graph $H$ under the labeling $g$ we have

$$\text{wt}_{g}(H^j_i) = \sum_{v \in V(H^j_i)} g(v) + \sum_{e \in E(H^j_i)} g(e)$$

$$= \sum_{v \in V(H^j)} \left( f(v) + (i - 1) \left[ \frac{\text{wt}_{f}^{\text{max}}(H) - \text{wt}_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right] \right)$$

$$+ \sum_{e \in E(H^j)} \left( f(e) + (i - 1) \left[ \frac{\text{wt}_{f}^{\text{max}}(H) - \text{wt}_{f}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right] \right).$$
After some manipulation we get

\[ \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) + |V(H)| \{ wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1 \} \]

This means that in the given copy of \( G \) the \( H \)-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \( i \)th copy of \( G \), \( i = 1, 2, \ldots, m \), denoted by \( wt_{g}^\text{max}(H : H \subset G_{i}) \), is at most

\[ wt_{g}^\text{max}(H : H \subset G_{i}) \leq wt_{f}^\text{max}(H) + (|V(H)| + |E(H)|)(i-1) \left[ \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \]

and the smallest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \((i+1)\)th copy of \( G \), \( i = 1, 2, \ldots, m - 1 \), denoted by \( wt_{g}^\text{min}(H : H \subset G_{i+1}) \), is at least

\[ wt_{g}^\text{min}(H : H \subset G_{i+1}) \geq wt_{f}^\text{min}(H) + (|V(H)| + |E(H)|)i \left[ \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \]

After some manipulation we get

\[ wt_{g}^\text{min}(H : H \subset G_{i+1}) \geq wt_{f}^\text{min}(H) + (|V(H)| + |E(H)|)i \left[ \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \]

As

\[ \left[ \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \geq \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \]

we obtain

\[ wt_{g}^\text{min}(H : H \subset G_{i+1}) \geq wt_{f}^\text{min}(H) \]

\[ + (|V(H)| + |E(H)|)(i - 1) \left[ \frac{wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right] \]

\[ + (wt_{f}^\text{max}(H) - wt_{f}^\text{min}(H) + 1) \]
On Total \(H\)-Irregularity Strength of the Disjoint Union ... 187

\[ \text{wt}^f_{\max}(H) + (|V(H)| + |E(H)|)(i - 1) \left( \frac{\text{wt}^\text{max}(H) - \text{wt}^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right) + 1 \]

\[ \geq \text{wt}^g_{\max}(H : H \subset G_i) + 1 > \text{wt}^g_{\max}(H : H \subset G_i). \]

Thus in all components the \(H\)-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

**Corollary 5.** Let \(G\) be a graph admitting an \(H\)-irregular total \(\text{ths}(G,H)\)-labeling \(f\). Let \(m\) be a positive integer. Then

\[ \text{ths}(mG,H) \leq m \text{ths}(G,H). \]

**Proof.** Let \(f\) be a \(\text{ths}(G,H)\)-labeling of a graph \(G\) and let \(\text{ths}(G,H) = k\). As \(\text{wt}^\text{min}_f(H) \geq |V(H)| + |E(H)|\) and \(\text{wt}^\text{max}_f(H) \leq (|V(H)| + |E(H)|)k\) we get

\[ \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \leq \left\lfloor \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rfloor = k - 1 + \frac{1}{|V(H)| + |E(H)|} = k. \]

Hence, by Theorem 4,

\[ \text{ths}(mG,H) \leq \text{ths}(G,H) + (m - 1) \left\lfloor \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \leq k + (m - 1)k = mk. \]

Let \(\{H^1, H^2, \ldots, H^t\}\) be the set of all subgraphs of \(G\) isomorphic to \(H\). Let \(f\) be an \(H\)-irregular total \(k\)-labeling of a graph \(G\) with \(\text{ths}(G,H) = k\) such that

\[ \{\text{wt}_f(H^j) : j = 1, 2, \ldots, t\} = \{\text{wt}^\text{min}_f(H), \text{wt}^\text{min}_f(H) + 1, \ldots, \text{wt}^\text{min}_f(H) + t - 1\}. \]

Evidently, if the fraction

\[ \frac{\text{wt}^\text{max}_f(H) - \text{wt}^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \]

is an integer then the weights of all \(H\)-weights in \(mG\) under the total labeling \(g\) of \(mG\) defined in the proof of Theorem 4 constitute the set

\[ \{\text{wt}^\text{min}_f(H), \text{wt}^\text{min}_f(H) + 1, \ldots, \text{wt}^\text{min}_f(H) + mt - 1\}. \]

In particular, this implies that the upper bound for \(\text{ths}(mG,H)\) given in Theorem 4 is tight if \(G\) is a graph that satisfies the conditions mentioned above.
**Theorem 6.** Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total $\text{ths}(G,H)$-labeling of $G$ such that 
\[ \{\text{wt}_{f}(H^j) : j = 1,2,\ldots,t\} = \{\text{wt}^{\min}_{f}(H),\text{wt}^{\min}_{f}(H) + 1,\ldots,\text{wt}^{\min}_{f}(H) + t - 1\} .\]

If the fraction $\frac{t}{|V(H)|+|E(H)|}$ is an integer then 
\[ \text{ths}(mG,H) \leq \text{ths}(G,H) + \frac{(m-1)t}{|V(H)|+|E(H)|} .\]

Moreover, if $\text{ths}(G,H) = \left[ 1 + \frac{t}{|V(H)|+|E(H)|} \right] = 1 + \frac{t}{|V(H)|+|E(H)|}$ then 
\[ \text{ths}(mG,H) = \text{ths}(G,H) + \frac{(m-1)t}{|V(H)|+|E(H)|} = 1 + \frac{ml}{|V(H)|+|E(H)|} .\]

Theorem 2 gives the exact value for the total $P_r$-irregularity strength for a path $P_r$. Moreover, the $P_r$-irregular total $((s+r-1)/(2s-1))$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_r$-weights consists of $t$ consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that 
\[
\text{ths}(mP_r,P_s) = \text{ths}(P_r,P_s) + (m - 1) \frac{r-s+1}{2s-1} = \left[ \frac{s+r-1}{2s-1} \right] + (m - 1) \frac{r-s+1}{2s-1} 
= \left[ \frac{r-s+1+2s-1-1}{2s-1} \right] + (m - 1) \frac{r-s+1}{2s-1} 
= \left[ \frac{r-s+1}{2s-1} + 1 - \frac{1}{2s-1} \right] + (m - 1) \frac{r-s+1}{2s-1} 
= \frac{r-s+1}{2s-1} + 1 + (m - 1) \frac{r-s+1}{2s-1} = m \frac{r-s+1}{2s-1} + 1 .
\]

Thus we obtain the following result.

**Corollary 7.** Let $m,r,s$, $m \geq 1$, $2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then 
\[ \text{ths}(mP_r,P_s) = \frac{m(r-s+1)}{(2s-1)} + 1 .\]

If $H$ is isomorphic to $K_2$ then $\text{ths}(G,K_2) = \text{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

**Corollary 8.** Let $m$ be a positive integer. Then 
\[ \left[ \frac{m|E(G)|+2}{3} \right] \leq \text{ths}(mG,K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m - 1) \left[ \frac{\text{wt}_{f}^{\max} - \text{wt}_{f}^{\min} + 1}{3} \right] ,\]

where $\text{wt}_{f}^{\max}$ and $\text{wt}_{f}^{\min}$ are the largest and smallest edge weights under a total $\text{tes}(G)$-labeling $f$ of $G$. 
3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total $H$-irregularity strength of two graphs $G_1$ and $G_2$ admitting an $H$-covering. From Theorem 1 we immediately obtain

**Corollary 9.** Let $G_i$, $i = 1, 2$, be a graph admitting an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G_1 \cup G_2, H) \geq 1 + \left\lfloor \frac{t_1 + t_2 - 1}{|V(H)| + |E(H)|} \right\rfloor.$$  

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

**Theorem 10.** Let $G_i$, $i = 1, 2$, be a graph having an $H$-irregular total $\text{ths}(G_i, H)$-labeling $f_i$. Then

$$\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lfloor \frac{w_{f_2}^{\max}(H) - w_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right\}, \right.$$

$$\left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lfloor \frac{w_{f_2}^{\max}(H) - w_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right\} \right\},$$

where $w_{f_i}^{\max}(H)$ and $w_{f_i}^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f_i$ of $G_i$.

**Proof.** Let $G_i$, $i = 1, 2$, be a graph that admits an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. We denote these subgraphs as $H_1^i, H_2^i, \ldots, H_t^i$. Assume that $f_i$ is an $H$-irregular total $k_i$-labeling of a graph $G_i$ with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $w_{f_i}^{\min}(H)$. Evidently

$$w_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|.$$  

(10)

Analogously, the largest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $w_{f_i}^{\max}(H)$. It holds that

$$w_{f_i}^{\max}(H) \geq w_{f_i}^{\min}(H) + t_i - 1$$

(11)

and

$$w_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i.$$  

(12)

Thus $f_i : V(G_i) \cup E(G_i) \rightarrow \{1, 2, \ldots, k_i\}$ and

$$\{w_{f_i}(H_j^i) : j = 1, 2, \ldots, t_i\} \subset \{w_{f_i}^{\min}(H), w_{f_i}^{\min}(H) + 1, \ldots, w_{f_i}^{\max}(H)\}.$$  

(13)
Let us define the total labeling $g$ of $G_1 \cup G_2$ in the following way.
\[
g(x) = \begin{cases} 
  f_1(x) & \text{if } x \in V(G_1) \cup E(G_1), \\
  f_2(x) + \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor & \text{if } x \in V(G_2) \cup E(G_2).
\end{cases}
\]

Evidently, all the labels are not greater than
\[
\max \left\{ k_1, k_2 + \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right\}.
\]

For the weight of the subgraph $H^j_1$, $j = 1, 2, \ldots, t_1$, isomorphic to the graph $H$ under the labeling $g$ we get
\[
\text{wt}_g(H^j_1) = \sum_{v \in V(H^j_1)} g(v) + \sum_{e \in E(H^j_1)} g(e) = \sum_{v \in V(H^j_1)} f_1(v) + \sum_{e \in E(H^j_1)} f_1(e) = \text{wt}_{f_1}(H^j_1).
\]

For the weight of the subgraph $H^j_2$, $j = 1, 2, \ldots, t_2$, isomorphic to the graph $H$ under the labeling $g$ we get
\[
\text{wt}_g(H^j_2) = \sum_{v \in V(H^j_2)} g(v) + \sum_{e \in E(H^j_2)} g(e)
= \sum_{v \in V(H^j_2)} \left( f_2(v) + \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right)
+ \sum_{e \in E(H^j_2)} \left( f_2(e) + \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right)
= \sum_{v \in V(H^j_2)} f_2(v) + \sum_{e \in E(H^j_2)} f_2(e) + |V(H)| \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor
+ |E(H)| \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor
= \text{wt}_{f_2}(H^j_2) + (|V(H)| + |E(H)|) \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\]

According to (13) we get that the largest weight of a subgraph $H$ under the total labeling $g$ in $G_1$, denoted by $\text{wt}_g^{\text{max}}(H : H \subset G_1)$, is at most
\[
\text{wt}_g^{\text{max}}(H : H \subset G_1) = \text{wt}_{f_1}^{\text{max}}(H)
\]
and the smallest weight of a subgraph $H$ under the total labeling $g$ in $G_2$, denoted by $\text{wt}_g^{\text{min}}(H : H \subset G_2)$, is at least
\[
\text{wt}_g^{\text{min}}(H : H \subset G_2) \geq \text{wt}_{f_2}^{\text{min}}(H) + (|V(H)| + |E(H)|) \left\lfloor \frac{\text{wt}_{f_1}^{\text{max}}(H) - \text{wt}_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\]
Note, that when writing $H$, we only consider subgraphs of $G_i$ isomorphic to $H$. As

$$\left[ \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H)+1}{|V(H)| + |E(H)|} \right] \leq \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H)+1}{|V(H)| + |E(H)|}$$

we get

$$wt_{g}^{min}(H) : H \subset G_2) \geq wt_{f_2}^{min}(H) + (|V(H)| + |E(H)|)\frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H)+1}{|V(H)| + |E(H)|}$$

$$\geq wt_{f_2}^{min}(H) + (wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H)+1) = wt_{f_1}^{max}(H)+1$$

$$> wt_{f_1}^{max}(H) = wt_{g}^{max}(H : H \subset G_1).$$

Thus all the $H$-weights in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling $h$ of $G_1 \cup G_2$ such that

$$h(x) = f_2(x) \quad \text{if } x \in V(G_2) \cup E(G_2),$$

$$h(x) = f_1(x) + \left[ \frac{wt_{f_2}^{max}(H) - wt_{f_1}^{min}(H)+1}{|V(H)| + |E(H)|} \right] \quad \text{if } x \in V(G_1) \cup E(G_1).$$

Using similar arguments we can also show that under the total labeling $h$ the $H$-weights in $G_1 \cup G_2$ are distinct.

Thus $g$ and $h$ are $H$-irregular total labelings of $G$. Immediately from this fact we get

$$\text{ths}(G_1 \cup G_2, H) \leq \min\left\{ \max\left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left[ \frac{wt_{f_2}^{max}(H) - wt_{f_1}^{min}(H)+1}{|V(H)| + |E(H)|} \right] \right\}, \right.$$}

$$\left. \max\left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left[ \frac{wt_{f_1}^{max}(H) - wt_{f_2}^{min}(H)+1}{|V(H)| + |E(H)|} \right] \right\} \right\}. \quad \blacksquare$$

Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

**Theorem 11** [18]. The total edge irregularity strength of the disjoint union of graphs $G_1, G_2, \ldots, G_m, m \geq 2$, is

$$\text{tes}\left( \bigcup_{i=1}^{m} G_i \right) \leq \sum_{i=1}^{m} \text{tes}(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$
If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that

$$\text{ths}(G_1 \cup G_2, K_2) = \text{tes}(G_1 \cup G_2)$$

$$\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lceil \frac{3\text{tes}(G_2) - 2}{3} \right\rceil \right\}, \right.$$

$$\max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lceil \frac{3\text{tes}(G_1) - 2}{3} \right\rceil \right\} \right\}$$

$$= \text{tes}(G_1) + \text{tes}(G_2)$$

which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total $\text{ths}(G, H)$-labeling $f$ and $m$ is a positive integer then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lfloor \frac{\text{wt}_{\text{max}}^f(H) - \text{wt}_{\text{min}}^f(H) + 1}{|V(H)| + |E(H)|} \right\rfloor,$$

where $\text{wt}_{\text{max}}^f(H)$ and $\text{wt}_{\text{min}}^f(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

Acknowledgments

The research for this article was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-15-0116, by VEGA 1/0233/18, by the Spanish Research Council under project MTM2014-60127-P and symbolically by the Catalan Research Council under grant 2014SGR1147.

References


On Total $H$-Irregularity Strength of the Disjoint Union...


Received 12 April 2017
Revised 5 February 2018
Accepted 22 February 2018