

SPECTRAL CONDITIONS FOR GRAPHS TO BE k -HAMILTONIAN OR k -PATH-COVERABLE

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Abstract

A graph G is k -Hamiltonian if for all $X \subset V(G)$ with $|X| \leq k$, the subgraph induced by $V(G) \setminus X$ is Hamiltonian. A graph G is k -path-coverable if $V(G)$ can be covered by k or fewer vertex disjoint paths. In this paper, by making use of the vertex degree sequence and an appropriate closure concept (due to Bondy and Chvátal), we present sufficient spectral conditions of a connected graph with fixed minimum degree and large order to be k -Hamiltonian or k -path-coverable.

Keywords: spectral radius, minimum degree, k -Hamiltonian, k -path-coverable.

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1. INTRODUCTION

Throughout this paper we consider only connected simple graphs. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. Generally, we say G is of order n and size m . Let d_v be the degree of a vertex v in G . The minimum degree of G is denoted by $\delta(G)$. Let K_n, \overline{K}_n denote the complete graph, the empty graph on n vertices, respectively. For vertex disjoint graphs G and H , $G \cup H$ and $G \vee H$ denote the union and join of G and H , respectively.

The *adjacency matrix* of a simple graph G of order n is $A(G) = (a_{ij})_{n \times n}$, whose entries satisfy $a_{ij} = 1$ if vertices i and j are adjacent in G , and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G is $P_G(x) = \det(xI - A(G))$, and the *eigenvalues* of G are the zeros of $P_G(x)$ (with multiplicities). Since $A(G)$ is

real and symmetric, the eigenvalues of G are real. The largest eigenvalue of G is called the *spectral radius* of G and is denoted by $\lambda(G)$.

The spectral radius of a graph is an important invariant in this subject. Brualdi and Solheid [3] proposed the problem concerning the spectral radius of graphs: *Given a set \mathcal{G} of graphs, find an upper bound for the spectral radius in this set and identify the graph(s) for which the maximal spectral radius is attained.* For various classes of graphs, the above problem is well studied in the literature (see for planar graphs [34], graphs with given matching number [8], edge chromatic number [9], diameter [13], and domination number [32], etc.). For more details, see also the book by Stevanović [33]. Needless to add, the eigenvalues of graphs, apart from theoretical interest in mathematics and computer science, are important in variety of applications of graph theory (see, for example, [4, 15, 20–22]). It is worth mentioning that, besides adjacency matrix, some other matrices like Laplacian and signless Laplacian, or distance matrix are included in investigations (see, for example, [36, 37] and references therein).

Analogous to Brualdi-Solheid problem, Nikiforov [27] proposed the following related Turán-type problem, and it has been studied extensively in the literature.

Problem 1. For a given graph F , what is the maximum spectral radius of a graph G on n vertices without a subgraph isomorphic to F ?

The study of Problem 1 is largely due to Nikiforov. Fiedler and Nikiforov in [12] obtained the following tight sufficient conditions for graphs to be Hamiltonian or traceable.

Theorem 1 [12]. *Let G be a graph of order n .*

- (1) *If $\lambda(G) > n - 2$, then G is Hamiltonian unless $G = K_1 \vee (K_{n-2} \cup K_1)$.*
- (2) *If $\lambda(G) \geq n - 2$, then G is traceable unless $G = K_{n-1} \cup K_1$.*

In general sense, the problem of deciding whether a given graph is Hamiltonian or traceable is NP-complete. One possible way to tackle this problem is to identify sufficient conditions guaranteeing the existence of a hamilton path (cycle) or a path partitioning. One may refer to the survey paper of Li [18] for details. Inspired by Theorem 1, many researchers are devoted to study the relation between the spectral radius and the Hamiltonian problems see [23, 24, 26, 30, 31, 35, 38–41] for more details. For other related topics in this area, see [10, 11, 25, 42].

When the minimum degree is involved, Li and Ning [17] obtained.

Theorem 2 [17]. *Let $t \geq 1$, and G be a graph of order n with minimum degree $\delta(G) \geq t$.*

- (1) *If $n \geq \max\{6t + 10, (t^2 + 7t + 8)/2\}$ and*

$$\lambda(G) \geq \lambda(K_t \vee (K_{n-2t-1} \cup (t+1)K_1)),$$

then G is traceable, unless $G = K_t \vee (K_{n-2t-1} \cup (t+1)K_1)$.

(2) If $n \geq \max\{6t+5, (t^2+6t+4)/2\}$ and

$$\lambda(G) \geq \lambda(K_t \vee (K_{n-2t} \cup tK_1)),$$

then G is Hamiltonian, unless $G = K_t \vee (K_{n-2t} \cup tK_1)$.

Theorem 2 is generalized by Nikiforov as follows.

Theorem 3 [28]. Let $t \geq 1$ and G be a graph of order n .

(1) If $n \geq t^3 + t + 4$, $\delta(G) \geq t$ and

$$\lambda(G) \geq n - t - 1,$$

then G is Hamiltonian unless $G = K_1 \vee (K_{n-t-1} \cup K_t)$ or $G = K_t \vee (K_{n-2t} \cup tK_1)$.

(2) If $n \geq t^3 + t^2 + 2t + 5$, $\delta(G) \geq t$ and

$$\lambda(G) \geq n - t - 2,$$

then G is traceable unless $G = K_{t+1} \cup K_{n-t-1}$ or $G = K_t \vee (K_{n-2t-1} \cup (t+1)K_1)$.

In what follows, we put our focus on k -Hamiltonian and k -path-coverable graphs. A graph G is k -Hamiltonian [6, 16] if for all $X \subset V(G)$ with $|X| \leq k$, the subgraph induced by $V(G) \setminus X$ is Hamiltonian. Thus 0-Hamiltonian is the same as Hamiltonian. A graph G is k -path-coverable if $V(G)$ can be covered by k or fewer vertex-disjoint paths. In particular, 1-path-coverable is the same as traceable. The disjoint path cover problem is strongly related to the well-known Hamiltonian problem (one may refer to [18] for a survey), which is among the most fundamental ones in graph theory, and which attracts much attention in theoretical computer science.

In [6], it is obtained that for a graph G , if $\delta(G) \geq \frac{n+k}{2}$, then G is k -Hamiltonian. In [7], the authors obtained sufficient conditions for a general graph without any minimum degree restriction to be k -Hamiltonian and k -path coverable. In this paper, by utilizing the degree sequences and the closure concepts, we aim to generalize Theorem 3 to the k -Hamiltonian and k -path-coverable. Our results can be considered as the spectral counterpart for the above Dirac-type condition.

For convenience, we denote

$$A(n, k, \delta) := K_{k+1} \vee (K_{\delta-k} \cup K_{n-\delta-1}),$$

$$B(n, k, \delta) := K_\delta \vee (K_{n-2\delta-k} \cup \overline{K_{\delta+k}}).$$

The main results of this paper read as follows.

Theorem 4. *Let $k \geq 1$ and $\delta \geq k + 2$. If G is a connected graph on $n \geq \max\{2\delta^2 - 2k\delta + 2\delta - k + 2, (\delta - k)(k^2 + 2k + 5) + 1\}$ vertices and minimum degree $\delta(G) \geq \delta$ such that*

$$\lambda(G) \geq n - \delta + k - 1,$$

then G is k -Hamiltonian unless $G = A(n, k, \delta)$.

Since $\lambda(A(n, k, \delta))$ contains $K_{n-\delta+k}$ as a subgraph, $\lambda(A(n, k, \delta)) \geq n - \delta + k - 1$, we immediately have.

Corollary 5. *Let $k \geq 1$ and $\delta \geq k + 2$. If G is a connected graph on $n \geq \max\{2\delta^2 - 2k\delta + 2\delta - k + 2, (\delta - k)(k^2 + 2k + 5) + 1\}$ vertices and minimum degree $\delta(G) \geq \delta$ such that*

$$\lambda(G) \geq \lambda(A(n, k, \delta)),$$

then G is k -Hamiltonian unless $G = A(n, k, \delta)$.

Theorem 6. *Let $k \geq 1$ and $\delta \geq 2$. If G is a connected graph on $n \geq \max\{\delta^2(\delta + k) + \delta + k + 5, 5k + 6\delta + 6\}$ vertices and minimum degree $\delta(G) \geq \delta$ such that*

$$\lambda(G) \geq n - \delta - k - 1,$$

then G is k -path-coverable unless $G = B(n, k, \delta)$.

Since $\lambda(B(n, k, \delta)) \geq n - \delta - k - 1$, we immediately have.

Corollary 7. *Let $k \geq 1$ and $\delta \geq 2$. If G is a connected graph on $n \geq \max\{\delta^2(\delta + k) + \delta + k + 5, 5k + 6\delta + 6\}$ vertices and minimum degree $\delta(G) \geq \delta$ such that*

$$\lambda(G) \geq \lambda(B(n, k, \delta)),$$

then G is k -path-coverable unless $G = B(n, k, \delta)$.

In the remainder of this section we give some further notation. Given a graph G , and a nonnegative integer k , a property P is said to be k -stable [2] if whenever $G + uv$ has property P and

$$d_u + d_v \geq k,$$

where $uv \notin E(G)$, then G itself has property P . It is well known that the Hamiltonicity and traceability are n -stable and $(n - 1)$ -stable, respectively. Among all graphs H of order n such that G is a spanning subgraph of H and

$$d_u + d_v < k$$

for all $uv \notin E(H)$, there is a unique smallest one, we shall call this graph the k -closure of G , denoted $cl_k(G)$. Obviously, $cl_k(G)$ can be obtained from G by recursively joining two nonadjacent vertices such that their degree sum is at

least k . This concept plays a prominent role in many problems of structural graph theory.

An integer sequence $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ is called *graphical* if there exists a graph G having π as its vertex degree sequence; in that case, G is called a *realization* of π . If P is a graph property, such as Hamiltonicity or k -connectedness, we call a graphical sequence π *forcibly P* if every realization of π has property P . A survey in this area can be found in [1].

2. PRELIMINARIES

We first give some lemmas that will be used later.

Lemma 8 [5]. *Let G be a connected graph and $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a graphical sequence. Suppose $n \geq 3$, and $0 \leq k \leq n - 3$. If*

$$d_i \leq i + k \Rightarrow d_{n-i-k} \geq n - i,$$

for $1 \leq i < \frac{1}{2}(n - k)$, then π is forcibly k -Hamiltonian.

Lemma 9 [1, 2]. *Let G be a connected graph and $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a graphical sequence. Suppose $k \geq 1$. If*

$$d_{i+k} \leq i \Rightarrow d_{n-i} \geq n - i - k,$$

for $1 \leq i < \frac{1}{2}(n - k)$, then π is forcibly k -path-coverable.

Lemma 10 [2]. *The property that “ G is k -Hamiltonian” is $(n + k)$ -stable.*

Lemma 11 [2]. *The property that “ G is k -path-coverable” is $(n - k)$ -stable.*

Lemma 12 [2]. *Let P be a property of graph G of order $n \geq 4$. If P is k -stable and the k -closure of G $cl_k(G)$ has property P , then G itself has property P .*

Lemma 13 [6]. *Let G be a graph with $n \geq 3$ vertices and let $0 \leq k \leq n - 3$. If for every pair of non-adjacent vertices u and v of G ,*

$$d_u + d_v \geq n + k,$$

then G is k -Hamiltonian.

Lemma 14 [19]. *Let G be a connected graph of order $n \geq 2k$, where $k \geq 1$. If for every pair of nonadjacent vertices u and v*

$$d_u + d_v \geq n - k - 1,$$

then G can be partitioned into k vertex-disjoint path unless G is of the form $\overline{K_{\frac{n+k+1}{2}}} \vee L_{\frac{n-k-1}{2}}$, where $L_{\frac{n-k-1}{2}}$ is any graph of order $\frac{n-k-1}{2}$, $n - k - 1$ is even, $k \geq 1$.

Lemma 15 [14, 29]. *If G is a graph of order n , size m , and minimum degree δ , then*

$$\lambda(G) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{8m - 4n\delta + (\delta + 1)^2} \right).$$

Lemma 16 [14]. *If $2m \leq n(n - 1)$, then the function*

$$f(x) = \frac{x - 1}{2} + \sqrt{2m - nx + \frac{(x + 1)^2}{4}}$$

is decreasing in x for $x \leq n - 1$.

3. MAIN RESULTS

3.1. k -Hamiltonicity and spectral radius

Observe first that if G has order n and is k -Hamiltonian, then $\delta(G) \geq k + 2$. Otherwise, if there exists a vertex $v \in V(G)$ with $d_G(v) \leq k + 1$, then we can delete at most k neighbours of v to obtain a subgraph of G which is not Hamiltonian.

Next theorem is crucial for proving Theorem 4. It is required that the K_{k+1} part of $A(n, k, \delta)$ has at least two vertices, so we assume $k \geq 1$ in Theorem 17.

Theorem 17. *Let $k \geq 1$ and $\delta \geq k + 2$. If G is a connected graph of order $n \geq (\delta - k)(k^2 + 2k + 5) + 1$ and minimum degree δ and if G is a subgraph of $A(n, k, \delta)$, then*

$$\lambda(G) < n - \delta + k - 1,$$

unless $G = A(n, k, \delta)$.

Proof. Let $\lambda := \lambda(G)$ for short, and let $x = (x_1, x_2, \dots, x_n)^T$ be the unit positive eigenvector (also known as the Perron vector) corresponding to λ . Then we have

$$\lambda = x^T A(G)x = \langle A(G)x, x \rangle = 2 \sum_{ij \in E(G)} x_i x_j.$$

Let G be a proper subgraph of $A(n, k, \delta)$. Without loss of generality, let G be a proper subgraph obtained from $A(n, k, \delta)$ by deleting from it just one edge, say uv . Denote by X for the set of vertices of $A(n, k, \delta)$ of degree δ , by Y the set of their neighbors which is not in X , and by Z the set of the remaining $n - \delta - 1$ vertices. Since G is a connected graph with minimum degree δ , G must contain all the edges incident with X . Therefore $\{u, v\} \subset Y \cup Z$, and this implies three possible cases: (a) $\{u, v\} \subset Y$; (b) $u \in Y, v \in Z$; (c) $\{u, v\} \subset Z$. We denote the corresponding graph in each of these three cases by G_1, G_2 and G_3 , respectively. We shall show that $\lambda(G_1) \leq \lambda(G_2) \leq \lambda(G_3)$.

Case (a). $\{u, v\} \subset Y$ and $G = G_1$. We will show that $\lambda(G_1) \leq \lambda(G_2)$. If $w \in Z$, we remove the edge uw and add the edge uv to obtain a graph G' (as is G_2 in case (b)). If $x_w \leq x_v = x_u$, then $x^T A(G')x - x^T A(G_1)x = 2x_u(x_v - x_w) \geq 0$, and thus $\lambda(G') \geq \lambda(G_1)$ by the Rayleigh principle. If $x_w > x_v$, we construct the vector x' of G' from x by swapping the entries x_v and x_w . Since $G_1 \setminus \{v, w\}$ and $G' \setminus \{v, w\}$ are isomorphic, we have

$$\begin{aligned} & x'^T A(G')x' - x^T A(G_1)x \\ &= 2x_w \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_u + x_v \right) \\ &+ 2x_v \left(\sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i \right) \\ &- 2x_v \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_w \right) \\ &- 2x_w \left(\sum_{i \in Y \setminus \{u, v\}} x_i + \sum_{i \in Z \setminus \{w\}} x_i + x_u \right) = 2(x_w - x_v) \sum_{i \in X} x_i > 0, \end{aligned}$$

so by the Rayleigh principle, we have $\lambda(G') > \lambda(G_1)$.

Case (b). Now $u \in Y, v \in Z$, and $G = G_2$. We will show that $\lambda(G_2) \leq \lambda(G_3)$. If $w \in Z$, we remove the edge vw and add the edge uv to obtain one graph G'' (as is G_3 in case (c)). If $x_u \geq x_w$, then $x^T A(G'')x - x^T A(G_2)x = 2x_v(x_u - x_w) \geq 0$, and thus $\lambda(G'') \geq \lambda(G_2)$ by the Rayleigh principle. If $x_u < x_w$, we construct the vector x' from x by swapping the entries x_w and x_u . Since $G_2 \setminus \{u, w\}$ and $G'' \setminus \{u, w\}$ are isomorphic, we have

$$\begin{aligned} & x'^T A(G'')x' - x^T A(G_2)x \\ &= 2x_w \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_u + x_v \right) \\ &+ 2x_u \left(\sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i \right) \\ &- 2x_u \left(\sum_{i \in X} x_i + \sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_w \right) \\ &- 2x_w \left(\sum_{i \in Y \setminus \{u\}} x_i + \sum_{i \in Z \setminus \{v, w\}} x_i + x_v \right) = 2(x_w - x_u) \sum_{i \in X} x_i > 0, \end{aligned}$$

again by the Rayleigh principle, we have $\lambda(G'') > \lambda(G_2)$.

Case (c). Now $\{u, v\} \subset Z$ and $G = G_3$. From symmetry, we obtain $x_i = x_j := x$ for any $i, j \in X$; $x_i = x_j := y$ for any $i, j \in Y$; and $x_i = x_j := z$ for any $i, j \in Z \setminus \{u, v\}$. As the vertices u and v are symmetric, we may write $x_u = x_v := t$. Since $\lambda x_i = \sum_{j \in E(G)} x_j$, we have

$$\begin{aligned}\lambda x &= (\delta - k - 1)x + (k + 1)y, \\ \lambda y &= (\delta - k)x + ky + (n - \delta - 3)z + 2t, \\ \lambda z &= (k + 1)y + (n - \delta - 4)z + 2t, \\ \lambda t &= (k + 1)y + (n - \delta - 3)z.\end{aligned}$$

From above, we find that

$$\begin{aligned}(1) \quad x &= \left(\frac{k + 1}{\lambda - \delta + k + 1} \right) y, \\ z &= \left(1 - \frac{(\delta - k)(k + 1)}{(\lambda - \delta + k + 1)(\lambda + 1)} \right) y, \\ (2) \quad t &= \left(\frac{\lambda + 1}{\lambda + 2} \right) \left(1 - \frac{(\delta - k)(k + 1)}{(\lambda - \delta + k + 1)(\lambda + 1)} \right) y.\end{aligned}$$

Further, note that if we remove all edges incident with X and add the edge uv to G , we obtain the graph $\overline{K_{\delta-k}} \cup K_{n-\delta+k}$. Let x'' be the restriction of x to $K_{n-\delta+k}$. Then we find that

$$\begin{aligned}\langle A(K_{n-\delta+k})x'', x'' \rangle &= \langle A(G)x, x \rangle + 2t^2 - 2(k + 1)(\delta - k)xy - (\delta - k - 1)(\delta - k)x^2 \\ &= \lambda + 2t^2 - 2(k + 1)(\delta - k)xy - (\delta - k - 1)(\delta - k)x^2.\end{aligned}$$

We have that

$$\langle A(K_{n-\delta+k})x'', x'' \rangle < \lambda(K_{n-\delta+k}) = n - \delta + k - 1,$$

that is,

$$(3) \quad \lambda + 2t^2 - 2(k + 1)(\delta - k)xy - (\delta - k - 1)(\delta - k)x^2 < n - \delta + k - 1.$$

Assume on the contrary that $\lambda \geq n - \delta + k - 1$, then from (3) we have

$$2(k + 1)(\delta - k)xy + (\delta - k - 1)(\delta - k)x^2 > 2t^2.$$

And from (1) and (2) it follows that

$$2 \frac{(k + 1)^2(\delta - k)}{\lambda - \delta + k + 1} y^2 + \frac{(\delta - k - 1)(\delta - k)(k + 1)^2}{(\lambda - \delta + k + 1)^2} y^2$$

$$> 2 \left(1 - \frac{1}{\lambda + 2}\right)^2 \left(1 - \frac{(\delta - k)(k + 1)}{(\lambda - \delta + k + 1)(\lambda + 1)}\right)^2 y^2.$$

Cancelling y^2 and applying Bernoulli inequality (which states that $(1 + x)^n > 1 + nx$ for any nonzero $x > -1$ and $n > 1$) to the right side, we obtain

$$\begin{aligned} & 2 \frac{(k + 1)^2(\delta - k)}{\lambda - \delta + k + 1} + \frac{(\delta - k - 1)(\delta - k)(k + 1)^2}{(\lambda - \delta + k + 1)^2} \\ & > 2 \left(1 - \frac{2}{\lambda + 2}\right) \left(1 - \frac{2(\delta - k)(k + 1)}{(\lambda - \delta + k + 1)(\lambda + 1)}\right) \\ & > 2 \left(1 - \frac{2}{\lambda + 2} - \frac{2(\delta - k)(k + 1)}{(\lambda - \delta + k + 1)(\lambda + 1)}\right), \end{aligned}$$

which in turn yields

$$\begin{aligned} & 2(k + 1)^2(\delta - k) + \frac{(\delta - k - 1)(\delta - k)(k + 1)^2}{\lambda - \delta + k + 1} \\ (4) \quad & > 2 \left(\lambda - \delta + k + 1 - \frac{2(\lambda - \delta + k + 1)}{\lambda + 2} - \frac{2(\delta - k)(k + 1)}{\lambda + 1} \right) \\ & > 2((\lambda - \delta + k + 1) - 2 - 2). \end{aligned}$$

The last inequality holds since $\lambda - \delta + k + 1 < \lambda + 2$, and $(\delta - k)(k + 1) < \lambda + 1$ when $n > (\delta - k)(k + 2) := G_1(\delta, k)$.

Furthermore, since $\lambda \geq n - \delta + k - 1$, when $n \geq (\delta - k)(k^2 + 2k + 5) + 1 := G_2(\delta, k)$, we have

$$\begin{aligned} \lambda - \delta + k + 1 & \geq n - 2\delta + 2k \\ & \geq (\delta - k)(k^2 + 2k + 5) + 1 - 2\delta + 2k \\ & = (\delta - k)(k^2 + 2k + 3) + 1. \end{aligned}$$

Bearing this in mind, from (4), we have

$$\begin{aligned} & (\delta - k - 1)(\delta - k)(k + 1)^2 \\ & \geq 2(\lambda - \delta + k + 1) [(\lambda - \delta + k + 1) - (k + 1)^2(\delta - k) - 4] \\ & \geq 2 [(\delta - k)(k^2 + 2k + 3) + 1] [((\delta - k)(k^2 + 2k + 3) + 1) - (k + 1)^2(\delta - k) - 4] \\ & = 2 [(\delta - k)(k^2 + 2k + 3) + 1] [2(\delta - k) - 3] \\ & \geq 2 [(\delta - k)(k^2 + 2k + 3) + 1] (\delta - k - 1), \end{aligned}$$

this is obviously a contradiction when $n \geq \max\{G_1(\delta, k), G_2(\delta, k)\} = G_2(\delta, k)$. This completes the proof of Theorem 17. \blacksquare

Now we are ready to prove our main result.

Proof of Theorem 4. Suppose that $\lambda(G) \geq n - \delta + k - 1$ and G is not k -Hamiltonian, in order to prove the result, we need only to prove that $G = A(n, k, \delta)$. From Lemma 10, we consider the closure $H := cl_{n+k}(G)$, by Lemma 12, H is not k -Hamiltonian, and $\delta(H) \geq \delta(G) \geq \delta$ and $\lambda(H) \geq \lambda(G) \geq n - \delta + k - 1$. From Theorem 17, we need only to prove that $H = A(n, k, \delta)$.

Obviously, for two vertices $i, j \in V(H)$ such that i is not adjacent to j , we have $d_i + d_j \leq n + k - 1$.

According to the assumptions of the theorem, together with Lemmas 15, 16, we have

$$n - \delta + k - 1 \leq \lambda(H) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - n\delta)} \right),$$

and consequently

$$(5) \quad 2m \geq n^2 + (2k - 2\delta - 1)n + (\delta - k)(2\delta - k + 1) := h_1(n).$$

Since H is not k -Hamiltonian, from Lemma 8, there exists an integer $1 \leq i \leq \frac{n-k-1}{2}$ such that $d_i \leq i + k$ and $d_{n-i-k} \leq n - i - 1$. So we have

$$(6) \quad \begin{aligned} 2m &\leq i(i + k) + (n - 2i - k)(n - i - 1) + (i + k)(n - 1) \\ &= 3i^2 - (2n - 2k - 1)i + n(n - 1). \end{aligned}$$

Since $\delta \leq i + k$, we obtain $\delta - k \leq i \leq \frac{n-k-1}{2}$. We next prove that $i = \delta - k$. Suppose $i \geq \delta - k + 1$. Due to (6), let $f(x) = 3x^2 - (2n - 2k - 1)x$, where $\delta - k + 1 \leq x \leq \frac{1}{2}(n - k - 1)$. We have either

$$(7) \quad 2m \leq n^2 + (2k - 2\delta - 3)n + (3\delta - k + 4)(\delta - k + 1) := h_2(n),$$

or

$$(8) \quad 2m \leq \frac{3n^2}{4} + \left(\frac{k}{2} - 1 \right) n - \frac{k^2 - 1}{4} := h_3(n).$$

From (5) and (7), when $n > \frac{1}{2}(4 - 4k + 6\delta - \delta k + \delta^2) := F_1(\delta, k)$, we obtain

$$h_1(n) - h_2(n) = -4 + 4k + 2n - 6\delta + \delta k - \delta^2 > 0,$$

leading to a contradiction, hence inequality (7) cannot hold.

From (5) and (8), when $n > 7\delta - 3k := F_2(\delta, k)$, we obtain

$$\begin{aligned} h_1(n) - h_3(n) &= \frac{1}{4} [n^2 - (8\delta - 6k)n + 8\delta^2 - 12k\delta + 4\delta + 5k^2 - 4k - 1] \\ &= \frac{1}{4} [(n - 4\delta + 3k)^2 - 8\delta^2 + (12k\delta - 9k^2 + 4\delta) + (5k^2 - 4k - 1)] > 0, \end{aligned}$$

a contradiction. Hence inequality (8) cannot hold.

Therefore, either (7) or (8) leads to a contradiction, and we have $i = \delta - k$. Thus

$$d_1 = d_2 = \dots = d_{\delta-k} = \delta.$$

Next, we will show that $d_{\delta-k+1} \geq n - \delta + k - 1 + k\delta - \delta^2$ when $n > \delta - k + 1 - k\delta + \delta^2 := F_3(\delta, k)$. Suppose that $d_{\delta-k+1} < n - \delta + k - 1 + k\delta - \delta^2$. Then

$$\begin{aligned} 2m &< (\delta - k)\delta + (n - \delta + k - 1 + k\delta - \delta^2) \\ &\quad + (n - 2\delta + k - 1)(n - \delta + k - 1) + \delta(n - 1) \\ &= n^2 + (2k - 2\delta - 1)n + (\delta - k)(2\delta - k + 1) \leq 2m, \end{aligned}$$

a contradiction. So we get that $d_i \geq n - \delta + k - 1 + k\delta - \delta^2$ for any $i \in \{\delta - k + 1, \dots, n\}$.

Next, we claim that the vertex set $\{\delta - k + 1, \dots, n\}$ induces a complete graph. Suppose $i, j \in \{\delta - k + 1, \dots, n\}$ are not adjacent. If $n \geq 2\delta^2 - 2k\delta + 2\delta - k + 2 := F_4(\delta, k)$, we have

$$\begin{aligned} d_i + d_j &\geq 2n - 2\delta + 2k - 2 + 2k\delta - 2\delta^2 \\ &\geq n + (2\delta^2 - 2k\delta + 2\delta - k + 2) - 2\delta + 2k - 2 + 2k\delta - 2\delta^2 \\ &= n + k - 1 + (2\delta^2 - 2k\delta + 2\delta - k + 2) - 2\delta + k - 1 + 2k\delta - 2\delta^2 \\ &= n + k > n + k - 1, \end{aligned}$$

contradicting the definition of H . So our claim holds.

Let $X = \{1, 2, \dots, \delta - k\}$, and let Y be the set of vertices in $\{\delta - k + 1, \dots, n\}$ having neighbors in X . In fact, every vertex from Y is adjacent to every vertex in X . Suppose that is not the case, and let $w \in Y, u, v \in X$ be such that w is adjacent to u , but not to v . Since the vertices in X have degree δ , we have $d_w + d_v \geq (n - \delta + k) + \delta = n + k$, which contradicts the definition of the closure of G .

Next, let $\ell := |Y|$. As the degree of the vertices in X is δ , thus $k + 1 \leq \ell \leq \delta$.

If $\ell = k + 1$, then $H = K_{k+1} \vee (K_{\delta-k} \cup K_{n-\delta-1}) = A(n, k, \delta)$. If we delete k vertices in K_{k+1} , then we obtain a graph with one cut vertex, and thus H is not k -Hamiltonian.

If $\ell = \delta$, then $H = K_\delta \vee (K_{n-2\delta+k} \cup \overline{K_{\delta-k}})$. Since $\delta \geq k + 2$, $n > F_2(\delta, k)$, we have $n - 2\delta + k > 5\delta - 2k \geq 3k$. In this case $X = V(\overline{K_{\delta-k}})$, $Y = V(K_\delta)$, $Z = V(K_{n-2\delta+k})$. Now we delete a vertex subset K of cardinality k from H with x vertices from X , y vertices from Y , z vertices from Z , $x + y + z = k$. Then $0 \leq x, y \leq k$. Since $\delta - y = |Y| - y \geq |X| - x = \delta - k - x$, there always exists a hamilton cycle in the graph induced by $(X \cup Y) \setminus K$. As $Y \cup Z$ induces a complete graph, therefore in this case, H is k -Hamiltonian, contradicting the assumption of H .

If $k + 1 < \ell < \delta$, we will show that H is k -Hamiltonian, which contradicts the assumptions about H . Indeed, let F be the graph induced by the set $X \cup Y$. Since $K_\ell \vee \overline{K_{\delta-k}} \subset F$, and $\delta - k \geq 2$, we see that F is 2-connected. Further, if u, v are distinct nonadjacent vertices of F , with degrees d_u and d_v , they must belong to X , and thus $d_u = d_v = \delta$, that is to say,

$$d_u + d_v = 2\delta \geq \delta + \ell + 1 > (\delta - k + \ell) + k.$$

By Lemma 13, F is k -Hamiltonian. For H , if we delete k vertices, no matter where these vertices are from, note that $\ell \geq k + 2$, the resulting graph is still Hamiltonian, hence H is k -Hamiltonian, contradicting the assumption of H .

Finally, to make our proof valid, we need to add

$$\begin{aligned} n &\geq \max\{G_1(\delta, k), G_2(\delta, k), F_1(\delta, k), F_2(\delta, k), F_3(\delta, k), F_4(\delta, k)\} \\ &= \max\{G_2(\delta, k), F_4(\delta, k)\}. \end{aligned}$$

This completes the proof. ■

We should point out that Theorem 4 only deals with the problem for relatively large n , for general n , we believe that additional tools are needed.

3.2. k -path-coverable and spectral radius

The following result is crucial for the proof of Theorem 6. It is very similar to the proof of Theorem 17, and we present it here for completeness. We require the K_δ part of $B(n, k, \delta)$ has at least two vertices, so we assume $\delta \geq 2$.

Theorem 18. *Let $k \geq 1$, $\delta \geq 2$ and $n \geq \delta^2(\delta + k) + \delta + k + 5$. If G is a connected graph of order n and minimum degree δ and if G is a subgraph of $B(n, k, \delta)$, then*

$$\lambda(G) < n - \delta - k - 1,$$

unless $G = B(n, k, \delta)$.

Proof. Let $x = (x_1, \dots, x_n)^T$ be the Perron vector of G corresponding to $\lambda := \lambda(G)$. Assume that G is a proper subgraph of $B(n, k, \delta)$. To prove the result, we only need to consider the graph G obtained by deleting just one edge uv from $B(n, k, \delta)$. Let X be the set of vertices of $B(n, k, \delta)$ of degree δ , Y be the set of the vertices adjacent to X , and Z be the set of remaining $n - 2\delta - k$ vertices of $B(n, k, \delta)$. Obviously, G must contain all the edges between X and Y as its minimum degree is δ . So we have $\{u, v\} \subset Y \cup Z$, with three possible cases: (i) $\{u, v\} \subset Y$; (ii) $u \in Y, v \in Z$; (iii) $\{u, v\} \subset Z$. Similarly to the proof of Theorem 17, we only consider case (iii).

So assume that $\{u, v\} \subset Z$. From symmetry, we obtain $x_i = x_j := x$ for any $i, j \in X$; $x_i = x_j := y$ for any $i, j \in Y$; and $x_i = x_j := z$ for any $i, j \in Z \setminus \{u, v\}$.

As the vertices u and v are symmetric, we may write $x_u = x_v := t$. Since $\lambda x_i = \sum_{j \in E(G)} x_j$, we have

$$\begin{aligned}\lambda x &= \delta y; \\ \lambda y &= (\delta + k)x + (\delta - 1)y + (n - 2\delta - k - 2)z + 2t; \\ \lambda z &= \delta y + (n - 2\delta - k - 3)z + 2t; \\ \lambda t &= \delta y + (n - 2\delta - k - 2)z,\end{aligned}$$

and therefore

$$\begin{aligned}(9) \quad x &= \frac{\delta}{\lambda}y; \\ z &= \left(1 - \frac{\delta(\delta + k)}{\lambda(\lambda + 1)}\right)y; \\ (10) \quad t &= \frac{\lambda + 1}{\lambda + 2} \left(1 - \frac{\delta(\delta + k)}{\lambda(\lambda + 1)}\right)y.\end{aligned}$$

Note that if we remove all edges between X and Y and add the edge uv to G , we can obtain the graph $K_{n-\delta-k} \cup \overline{K_{\delta+k}}$. Let x_1 be the restriction of x to $K_{n-\delta-k}$. Obviously $\|x_1\| < \|x\| = 1$. Then

$$\begin{aligned}n - \delta - k - 1 &= \lambda(K_{n-\delta-k}) \geq x_1^T A(K_{n-\delta-k})x_1 \\ &= x^T A(G)x + 2t^2 - 2\delta(\delta + k)xy \\ &= \lambda + 2t^2 - 2\delta(\delta + k)xy,\end{aligned}$$

so we obtain

$$\lambda + 2t^2 - 2\delta(\delta + k)xy \leq n - \delta - k - 1.$$

Assume on the contrary that $\lambda \geq n - \delta - k - 1$. Then we have

$$\delta(\delta + k)xy \geq t^2.$$

Replacing x, t among this inequality by (9), (10), respectively, and then cancelling y^2 , we have

$$\begin{aligned}\frac{\delta^2(\delta + k)}{\lambda} &\geq \left(1 - \frac{1}{\lambda + 2}\right)^2 \left(1 - \frac{\delta(\delta + k)}{\lambda(\lambda + 1)}\right)^2 \\ &> \left(1 - \frac{2}{\lambda + 2}\right) \left(1 - \frac{2\delta(\delta + k)}{\lambda(\lambda + 1)}\right) \quad \text{by Bernoulli inequality} \\ &> 1 - \frac{2}{\lambda + 2} - \frac{2\delta(\delta + k)}{\lambda(\lambda + 1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\delta^2(\delta + k) &> \lambda - \frac{2\lambda}{\lambda + 2} - \frac{2\delta(\delta + k)}{\lambda + 1} \\ &> \lambda - 4 \geq n - \delta - k - 5,\end{aligned}$$

a contradiction if $n \geq \delta^2(\delta + k) + \delta + k + 5 := \widetilde{G}_1(\delta, k)$. \blacksquare

Now we will prove Theorem 6.

Proof of Theorem 6. Suppose that $\lambda(G) \geq n - \delta - k - 1$ and G is not k -path-coverable. To prove the result, we need only to prove that $G = B(n, k, \delta)$. From Lemma 11, we consider the closure $H := cl_{n-k}(G)$, in light of Lemma 12, H is not k -path-coverable. From Theorem 18, we need only to prove that $H = B(n, k, \delta)$.

Observe that $\delta(H) \geq \delta(G) \geq \delta$ and $\lambda(H) \geq \lambda(G) \geq n - \delta - k - 1$. For any two nonadjacent vertices $i, j \in V(H)$, we have $d_i + d_j \leq n - k - 1$.

According to the assumptions, by Lemmas 15 and 16, we have

$$n - \delta - k - 1 \leq \lambda(H) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - n\delta)} \right),$$

and therefore

$$(11) \quad 2m \geq n^2 - (2\delta + 2k + 1)n + 2\delta^2 + k^2 + 3k\delta + \delta + k := g_1(n).$$

Since H is not k -path-coverable, then from Lemma 9, there exists an integer i with $i < \frac{1}{2}(n - k)$, such that $d_{i+k} \leq i$ and $d_{n-i} \leq n - i - k - 1$ hold. We have

$$\begin{aligned}2m &\leq i(i + k) + (n - k - 2i)(n - i - k - 1) + i(n - 1) \\ &= (n - k)(n - k - 1) + 3i^2 - (2n - 4k - 1)i.\end{aligned}$$

Next, we show that $i = \delta$. Indeed, if $i \geq \delta + 1$, we consider $f(x) = 3x^2 - (2n - 4k - 1)x$, where $\delta + 1 \leq x \leq \frac{n-k-1}{2}$. Note that $f_{\max}(x) = \max \left\{ f(\delta + 1), f\left(\frac{n-k-1}{2}\right) \right\}$, $f(\delta + 1) = 3(\delta + 1)^2 - (2n - 4k - 1)(\delta + 1)$, and $f\left(\frac{n-k-1}{2}\right) = 3\left(\frac{n-k-1}{2}\right)^2 - (2n - 4k - 1)\left(\frac{n-k-1}{2}\right)$,

$$f\left(\frac{n-k-1}{2}\right) - f(\delta + 1) = -\frac{1}{4}(n - k - 2\delta - 3)(n - 5k - 6\delta - 5).$$

Hence, when $n > 5k + 6\delta + 5 := \widetilde{F}_1(\delta, k)$, $f_{\max}(x) = f(\delta + 1)$. Then

$$\begin{aligned}2m &\leq (n - k)(n - k - 1) + f(\delta + 1) \\ &\leq n^2 - (2\delta + 2k + 3)n + 3\delta^2 + k^2 + 4k\delta + 7\delta + 5k + 4 := g_2(n).\end{aligned}$$

If $n > \frac{1}{2}(\delta^2 + k\delta + 6\delta + 4k + 4) := \widetilde{F}_2(\delta, k)$, then

$$(12) \quad g_2(n) - g_1(n) = -2n + \delta^2 + k\delta + 6\delta + 4k + 4 < 0.$$

Therefore, (11) and (12) provide a contradiction. So $i = \delta$, and hence $d_1 = d_2 = \dots = d_{\delta+k} = \delta$.

Now, we will show that $d_{\delta+k+1} \geq n - k\delta - \delta^2 - \delta - k - 1$ if $n > k\delta + \delta^2 + \delta + k + 1 := \widetilde{F}_3(\delta, k)$. Indeed, if $d_{\delta+k+1} < n - k\delta - \delta^2 - \delta - k - 1$, then from the proof above, we have

$$\begin{aligned} 2m &< \delta(\delta + k) + (n - k\delta - \delta^2 - \delta - k - 1) \\ &+ (n - k - 2\delta - 1)(n - \delta - k - 1) + \delta(n - 1) \\ &= n^2 - (2\delta + 2k + 1)n + 2\delta^2 + k^2 + 3k\delta + \delta + k, \end{aligned}$$

a contradiction to (11). So we get that $d_i \geq n - k\delta - \delta^2 - \delta - k - 1$ for any $i \in \{\delta + k + 1, \dots, n\}$.

In what follows, we will show that the vertex set $\{\delta + k + 1, \dots, n\}$ induces a complete graph. Suppose $i, j \in \{\delta + k + 1, \dots, n\}$ are two nonadjacent vertices. If $n > 2\delta^2 + 2k\delta + 2\delta + k + 2 := \widetilde{F}_4(\delta, k)$, then

$$\begin{aligned} d_i + d_j &\geq 2n - 2\delta - 2k - 2\delta^2 - 2k\delta - 2 \\ &= (n - k) + (n - 2\delta - k - 2\delta^2 - 2k\delta - 2) > n - k, \end{aligned}$$

a contradiction to the definition of H . So the vertex set $\{\delta + k + 1, \dots, n\}$ induces a complete graph.

Suppose $X = \{1, 2, \dots, \delta + k\}$. Let Y be the set of vertices in $\{\delta + k + 1, \dots, n\}$ having neighbors in X . In fact, every vertex in Y is adjacent to every vertex in X . Indeed, suppose that this is not the case, and let $i \in Y$, $j, t \in X$ be such that i is adjacent to j , but not to t . Then we see that

$$d_i + d_t \geq (n - \delta - k) + \delta = n - k,$$

a contradiction.

Next, let $\ell := |Y|$. Since each vertex in X is of degree δ , we have $1 \leq \ell \leq \delta$.

If $\ell = \delta$, then $H = B(n, k, \delta)$, which is not k -path-coverable.

If $1 \leq \ell < \delta$, we will consider the subgraph H' induced by $X \cup Y$. Obviously $|H'| = \delta + k + \ell$, and we need to show that H' is $(k - 1)$ -path-coverable. If u and v are nonadjacent vertices of H' , then $u, v \in X$, and we easily see that $d_u = d_v = \delta$ in H' . Therefore, $d_u + d_v = 2\delta \geq \delta + \ell + 1 > |H'| - (k - 1) - 1$, and by Lemma 14, H' is $(k - 1)$ -path-coverable. Therefore, Z induces a complete subgraph and thus it is 1-path-coverable, so we have that H is k -path-coverable.

Finally, to make our proof valid, we need to add

$$n > \max\{\widetilde{G}_1(\delta, k), \widetilde{F}_1(\delta, k), \widetilde{F}_2(\delta, k), \widetilde{F}_3(\delta, k), \widetilde{F}_4(\delta, k)\} = \max\{\widetilde{G}_1(\delta, k), \widetilde{F}_1(\delta, k)\}.$$

This completes the proof. ■

Similar to Theorem 4, Theorem 6 also only deals with the problem for relatively large n , for general n , we believe that additional tools are needed.

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