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# 2-CONNECTED HAMILTONIAN CLAW-FREE GRAPHS INVOLVING DEGREE SUM OF ADJACENT VERTICES

Tao Tian

*School of Mathematics and Statistics*
*Beijing Institute of Technology*
*Beijing 102488, P.R. China*

*Faculty of EEMCS, University of Twente*
*P.O. Box 217, 7500 AE Enschede, The Netherlands*

**e-mail:** taotian0118@163.com

and

Liming Xiong

*School of Mathematics and Statistics*
*Beijing Key Laboratory on MCAACI*
*Beijing Institute of Technology*
*Beijing 102488, P.R. China*

**e-mail:** lmxiong@bit.edu.cn

**Abstract**

For a graph $H$, define $\bar{\sigma}_2(H) = \min\{d(u) + d(v) | \ uv \in E(H)\}$. Let $H$ be a 2-connected claw-free simple graph of order $n$ with $\delta(H) \geq 3$. In [J. Graph Theory 86 (2017) 193–212], Chen proved that if $\bar{\sigma}_2(H) \geq \frac{n}{2} - 1$ and $n$ is sufficiently large, then $H$ is Hamiltonian with two families of exceptions. In this paper, we refine the result. We focus on the condition $\bar{\sigma}_2(H) \geq \frac{2n}{5} - 1$, and characterize non-Hamiltonian 2-connected claw-free graphs $H$ of order $n$ sufficiently large with $\bar{\sigma}_2(H) \geq \frac{2n}{5} - 1$. As byproducts, we prove that there are exactly six graphs in the family of 2-edge-connected triangle-free graphs of order at most seven that have no spanning closed trail and give an improvement of a result of Veldman in [Discrete Math. 124 (1994) 229–239].

**Keywords:** Hamiltonian cycle, degree sum, dominating closed trail, closure.

**2010 Mathematics Subject Classification:** 05C45.

## 1. Introduction

### 1.1. Terminology and known results

For graph-theoretic notation not explained in this paper, we refer the reader to [1]. We consider only finite and loopless graphs in this paper. A graph is called *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. For a vertex $x$ of $G$, $N_G(x)$ is the neighborhood of $x$ in $G$, and $d_G(x)$ or $d(x)$ is the degree of $x$ in $G$. A graph $G$ is *Hamiltonian* if it has a Hamilton cycle, i.e., a spanning cycle. A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. Similarly, a graph is *triangle-free* if it has no $K_3$. As in [1], $\kappa'(G)$ denotes the edge-connectivity of $G$. Define $D_i(G) = \{v \in V(G) \,|\, d_G(v) = i\}$ and $D(G) = D_1(G) \cup D_2(G)$. An edge cut $X$ of $G$ is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph $G$ is *essentially $k$-edge-connected* if $G$ does not have an essential edge-cut $X$ with $|X| < k$. An edge $e = uv \in E(G)$ is called a *pendant edge* of $G$ if $\min\{d(u), d(v)\} = 1$. The circumference of $G$, denoted by $c(G)$, is the length of a longest cycle of $G$. Let $C_n$ denote a cycle of order $n$. The length of a path is the number of its edges. A path of length $k$ is called a *$k$-path*. A connected graph $Q$ is a *closed trail* if the degree of each vertex in $Q$ is even. A closed trail $Q$ is called a *spanning closed trail* (SCT) in $G$ if $V(G) = V(Q)$, and is called a *dominating closed trail* (DCT) if $E(G - V(Q)) = \emptyset$. A graph is *supereulerian* if it contains an SCT. The family of supereulerian graphs is denoted by $\mathcal{SL}$.

The line graph of a graph $G$, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ have a vertex in common. The following theorem shows the relationship between a graph and its line graph.

**Theorem 1** (Harary and Nash-Williams [9]). *The line graph $H = L(G)$ of a graph $G$ with at least three edges is Hamiltonian if and only if $G$ has a DCT.*

Ryjáček [14] introduced the line graph closure operation of a claw-free graph $G$, which becomes a very useful tool in investigating the Hamiltonicity in claw-free graphs. A vertex $v \in V(G)$ is *locally connected* if the neighborhood of $v$ induces a connected subgraph in $G$. Particularly, we say $v$ is *simplicial* if the subgraph induced by $N_G(v)$ is complete. For $v \in V(G)$, the graph $G'_v$ obtained from $G$ by adding the edges $\{uw \,|\, u, w \in N_G(v) \text{ and } uw \notin E(G)\}$ is called the *local completion* of $G$ at $v$. The *closure* of a claw-free graph $G$, denoted by $cl(G)$, is obtained from $G$ by recursively performing local completions at locally connected vertices with non-complete neighborhoods, as long as it is possible. The closure $cl(G)$ remains a claw-free graph and its connectivity is no less than the connectivity of $G$. A graph $G$ is said to be *closed* if $G = cl(G)$.

The following theorem translates claw-free graphs to line graphs when we consider the Hamiltonicity of claw-free graphs. Note that a graph $G$ is essentially $k$-edge-connected if and only if $L(G)$ is $k$-connected or complete.

**Theorem 2** (Ryjáček [14]). *Let $H$ be a claw-free graph and $cl(H)$ its closure. Then*

(i) *$cl(H)$ is well-defined, and $\kappa(cl(H)) \geq \kappa(H)$;*
(ii) *there is a triangle-free graph $G$ such that $cl(H) = L(G)$;*
(iii) *both graphs $H$ and $cl(H)$ have the same circumference.*

For a graph $H$, let $\Omega(H,t) = \{\delta(H), \sigma_2(H), \bar{\sigma}_2(H), \sigma_t(H), \delta_F(H)\}$, where

$\delta(H) = \min\{d(v) |\ v \in V(H)\}$ (Dirac-type);

$\sigma_2(H) = \min\{d(u) + d(v) |\ uv \notin E(H)\}$ (Ore-type);

$\bar{\sigma}_2(H) = \min\{d(u) + d(v) |\ uv \in E(H)\}$;

$\sigma_t(H) = \min\{\Sigma_{i=1}^{t} d(v_i) |\ \{v_1, v_2, \ldots, v_t\}$ is independent in $H$ $(t \geq 2)\}$;

$\delta_F(H) = \min\{\max\{d(u), d(v)\} |\ u, v \in V(H)$ with $\mathrm{dist}(u,v) = 2\}$ (Fan-type).

For the Hamiltonicity of a graph or a claw-free graph, a lot of research has focused on the conditions of the parameters in $\Omega(H,t)$ (see the surveys [6, 8]).

**Theorem 3** (Ore [13]). *If $H$ is a graph of order $n$ such that $\sigma_2(H) \geq n$, then $H$ is Hamiltonian.*

The following result extended the above result.

**Theorem 4** (Fan [5]). *If $H$ is a 2-connected graph of order $n \geq 3$ with $\delta_F(H) \geq \frac{n}{2}$, then $H$ is Hamiltonian.*

For claw-free graphs, Matthews and Sumner [12] proved the following.

**Theorem 5** (Matthews and Sumner [12]). *If $H$ is a 2-connected claw-free graph of order $n$ such that $\delta(H) \geq \frac{n-2}{3}$, then $H$ is Hamiltonian.*

In [17], Zhang improved the above result as follows.

**Theorem 6** (Zhang [17]). *If $H$ is a $k$-connected claw-free graph of order $n$ with $\sigma_{k+1}(H) \geq n - k$, then $H$ is Hamiltonian.*

The graphs $G_3$, $G_4^1$, $G_4^2$, $G_5^1, G_5^2, \ldots, G_5^7$ are shown in Figure 1 (where the circular and elliptical parts represent cliques of appropriate order containing at least one simplicial vertex). Let $\widetilde{\mathcal{G}}_3$ ($\widetilde{\mathcal{G}}_4$ or $\widetilde{\mathcal{G}}_5$) be the set of all spanning subgraphs of $G_3$ ($G_4^1$ and $G_4^2$ or $G_5^1, G_5^2, \ldots, G_5^7$, respectively).

In [11], Li *et al.* improved Theorem 5 and get the following result.

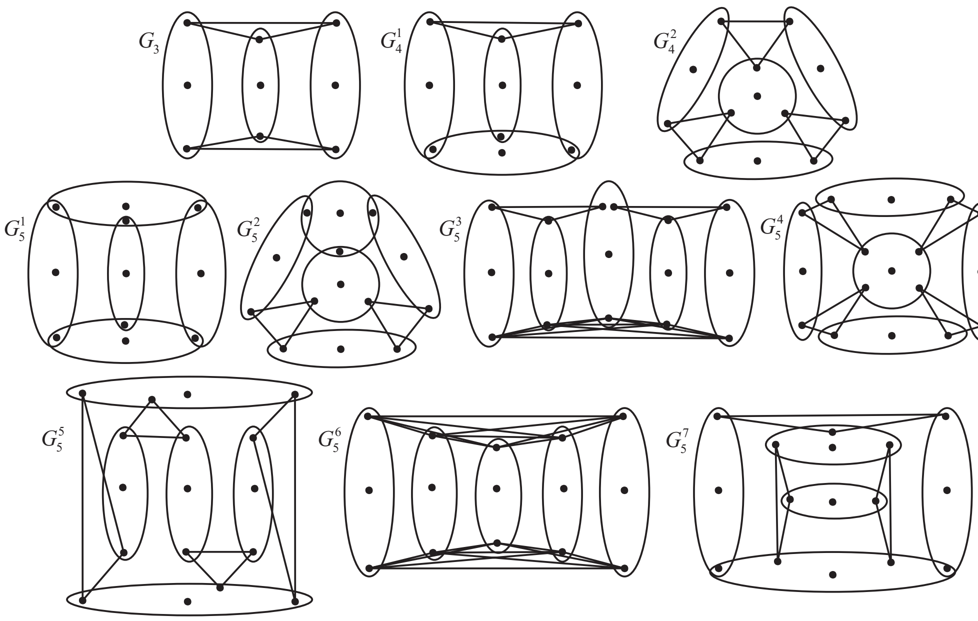


Figure 1. Ten classes of graphs which have no Hamiltonian cycles.

**Theorem 7** (Li *et al.* [11]). *If $H$ is a 2-connected claw-free graph of order $n$ with $\delta(H) \geq \frac{n+5}{5}$, then either $H$ is Hamiltonian or $H \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4$.*

In [7], Favaron *et al.* improved Theorem 7 and got the following two results.

**Theorem 8** (Favaron *et al.* [7]). *Let $H$ be a 2-connected claw-free graph with $n \geq 77$ vertices such that $\delta(H) \geq 14$ and $\sigma_6(H) > n + 19$. Then either $H$ is Hamiltonian or $H \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}_5$.*

**Theorem 9** (Favaron *et al.* [7]). *Let $H$ be a 2-connected claw-free graph of connectivity $\kappa(H) = 2$ with $n \geq 78$ vertices satisfying $\delta(H) > \frac{n+16}{6}$. Then either $H$ is Hamiltonian or $H \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}_5$.*

Degree conditions for Hamiltonicity in claw-free graphs were studied further in [10]; the authors gave a general algorithm that allows to generate all classes of exceptions, roughly speaking, a degree condition of form $\sigma_p(H) \geq n+\text{constant}(p)$ $\big($or, as a corollary, $\delta(H) \geq \frac{n+\text{constant}(p)}{p}\big)$, for arbitrary positive integer $p$. In [10], with the help of a computer, the computation was performed for $p = 8$, Kovářík *et al.* obtained a result for $\sigma_8(H) > n+39$ with an exception family that contains 318 infinite classes.

These conditions above are all related to degrees of non-adjacent vertices. However, the degree conditions of non-adjacent vertices exclude many Hamiltonian graphs. For example, let $H$ be the claw-free graph of order $n = s_1+s_2+s_3+$

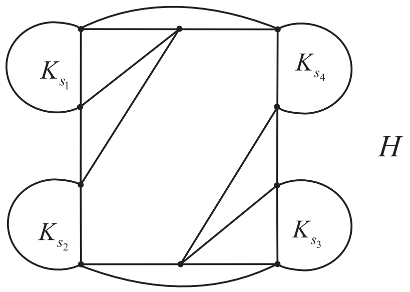


Figure 2. The Hamiltonian graph $H$.

$s_4+2$ ($\min\{s_1, s_2, s_3, s_4\} \geq \frac{n}{5}$, $n \geq 35$) depicted in Figure 2, where each cycle represents a complete graph $K_{s_i}$, and $\delta(H) = 4$, $\sigma_2(H) = 8$, $\frac{n}{5} - 1 \leq \delta_F(H) \leq \frac{n}{4} - 1$ and $\bar{\sigma}_2(H) \geq \frac{n}{5} + 5$. The graph $H$ is Hamiltonian but it does not satisfy any existing Dirac-type, Ore-type or Fan-type degree conditions for the existence of Hamiltonian cycles in graphs. Obviously, this family of graphs could be large.

Let $G$ be a connected multigraph. For $X \subseteq E(G)$, the *contraction* $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Even if $G$ is simple, $G/X$ may not be simple. If $\Gamma$ is a connected subgraph of $G$, then we write $G/\Gamma$ for $G/E(\Gamma)$ and use $v_\Gamma$ for the vertex in $G/\Gamma$ to which $\Gamma$ is contracted, and $v_\Gamma$ is called a *contracted vertex* if $\Gamma \neq K_1$.

For a $K_{2,3}$, let $D_2(K_{2,3}) = \{v_1, v_2, v_3\}$, $D_3(K_{2,3}) = \{u_1, u_2\}$. Let $\mathcal{K}_{2,3}(s_1, s_2, s_3, s, r)$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from a $K_{2,3}$ by replacing each $v_i \in D_2(K_{2,3})$ by a triangle-free graph of size $s_i$ and replacing the two vertices $u_1$ and $u_2$ by a triangle-free graph of size $s$ and $r$, respectively. Note that each graph in $\mathcal{K}_{2,3}(s_1, s_2, s_3, s, r)$ may be contractible to a $K_{2,3}$.

Let $\mathcal{Q}_{2,3}(s_1, s_2, s_3, s, r)$ be the set of 2-connected claw-free graphs $H$ whose Ryjáček's closure is the line graph of a graph $G$ in $\mathcal{K}_{2,3}(s_1, s_2, s_3, s, r)$, i.e., $cl(H) = L(G)$.

In [4], Chen proved that a claw-free graph $H$ of order $n$ with large $\overline{\sigma}_2(H)$ in terms of $n$ has a Hamiltonian cycle while its minimum degree or Ore-type degrees may be small and Fan-type degree may be less than $\frac{n}{2}$.

**Theorem 10** (Chen [4]). *Let $H$ be a 2-connected claw-free simple graph of order $n$ with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-4}{4}$ and $n$ is sufficiently large, then one of the following holds:*

(a) *$H$ is Hamiltonian;*

(b) *$H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, s, 0)$ and $\frac{2n-4}{4} \leq \overline{\sigma}_2(H) \leq \frac{2n-2}{4}$, where $\min\{s_1, s_2, s_3\} \geq \frac{n-6}{4}$, $s \geq \frac{n-10}{4}$ and $s_1 + s_2 + s_3 + s + 6 = n$; or*

(c) $H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, 0, 0)$ *and* $\frac{2n-4}{4} \leq \overline{\sigma}_2(H) \leq \frac{2n-6}{3}$, *where* $\min\{s_1, s_2, s_3\} \geq \frac{n-6}{4}$ *and* $s_1 + s_2 + s_3 + 6 = n$.

Our main goal of this paper is to give an extension of Theorem 10. As a byproduct, we can also get an extension (Corollary 17) of Theorem 7 while Theorem 10 is not an extension of Theorem 7. One of our results (Theorem 20) is also an improvement of the following result of Veldman.

**Theorem 11** (Veldman [16]). *Let $G$ be an essentially 2-edge-connected simple graph of order $n$ such that $\overline{\sigma}_2(G) > 2\left(\left\lfloor \frac{n}{7} \right\rfloor - 1\right)$. If $n$ is sufficiently large, then either $L(G)$ is Hamiltonian or $G$ is contractible to a $K_{2,3}$ such that all vertices of degree 2 in $K_{2,3}$ are nontrivial.*

### 1.2. Main results of this paper

In this paper, $K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}$ are depicted in Figure 3 and we get the following results.

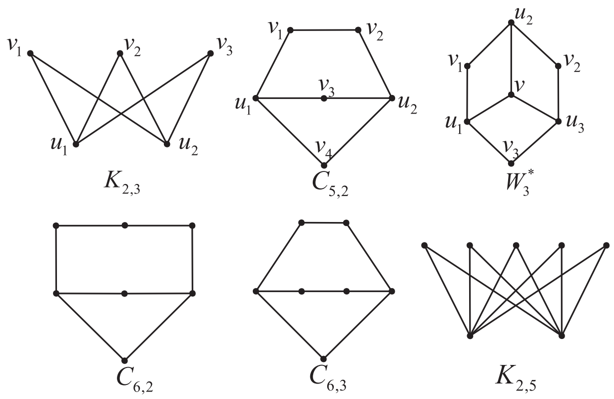


Figure 3. The graphs $K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}$ and $K_{2,5}$.

**Theorem 12.** *If $G$ is a 2-edge-connected triangle-free simple graph of order at most 7 which has no SCT, then $G \in \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\}$.*

Let $\mathcal{K}_{2,3}^{\triangle}$ be the set of graphs which are contractible to a $K_{2,3}$ by contracting exactly a triangle. The graphs in $\mathcal{K}_{2,3}^{\triangle}$ are depicted in Figure 4. (Obviously, none of the graphs in $\mathcal{K}_{2,3}^{\triangle}$ has an SCT.)

**Theorem 13.** *If $G$ is a 2-edge-connected simple graph of order at most 7 which has no SCT, then $G \in \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\} \cup \mathcal{K}_{2,3}^{\triangle}$.*

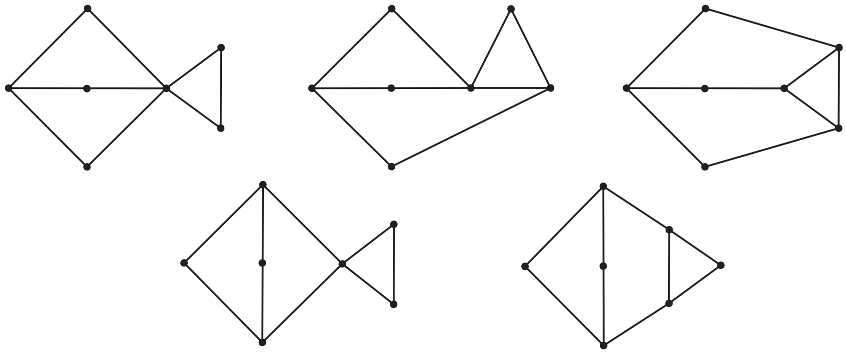

Figure 4. The five graphs in $\mathcal{K}_{2,3}^{\triangle}$.

Let $\mathcal{Q}_0(r,k)$ be the family of $k$-edge-connected triangle-free graphs of order at most $r$ and without an SCT. Then, by Theorem 12, $\mathcal{Q}_0(7,2) = \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\}$. In [4], Chen proved the following.

**Theorem 14** (Chen [4])**.** *Let $p > 0$ be a given integer and $\epsilon$ be a given number and $k \in \{2,3\}$. Let $H$ be a $k$-connected claw-free simple graph of order $n$ and $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n+\epsilon}{p}$ and $n$ is sufficiently large, then either $H$ is Hamiltonian or $cl(H) = L(G)$, where $G$ is an essentially $k$-edge-connected triangle-free graph that can be contracted to a graph in $\mathcal{Q}_0(5p-10,k)$ and $p \geq 3$.*

For a $C_{5,2}$, let $D_2(C_{5,2}) = \{v_1, v_2, v_3, v_4\}$ and $D_3(C_{5,2}) = \{u_1, u_2\}$. Let $\mathcal{C}_{5,2}(s_1, s_2, s_3, s_4, s)$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from a $C_{5,2}$ by replacing each $v_i \in D_2(C_{5,2})$ by a triangle-free graph of size $s_i$ and by replacing exactly one vertex $u_i \in D_3(C_{5,2})$ by a triangle-free graph of size $s$. Note that each graph in $\mathcal{C}_{5,2}(s_1, s_2, s_3, s_4, s)$ may be contractible to a $C_{5,2}$.

For a $W_3^*$, let $D_2(W_3^*) = \{v_1, v_2, v_3\}$ and $D_3(W_3^*) = \{u_1, u_2, u_3, v\}$, where $v$ is a vertex of degree three and $N_{W_3^*}(v) = \{u_1, u_2, u_3\}$. Let $\mathcal{W}_3^*(s_1, s_2, s_3, s, r)$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from a $W_3^*$ by replacing each $v_i \in D_2(W_3^*)$ by a triangle-free graph of size $s_i$ and replacing the vertex $v$ and one vertex $u_i \in D_3(W_3^*)$ by a triangle-free graph of size $s$ and $r$, respectively. Note that each graph in $\mathcal{W}_3^*(s_1, s_2, s_3, s, r)$ may be contractible to a $W_3^*$.

Let $\mathcal{Q}_{5,2}(s_1, s_2, s_3, s_4, s)$ and $\mathcal{Q}_3(s_1, s_2, s_3, s, r)$ be the set of 2-connected claw-free graphs $H$ whose Ryjáček's closure is the line graph of a graph $G$ in $\mathcal{C}_{5,2}(s_1, s_2, s_3, s_4, s)$ and $\mathcal{W}_3^*(s_1, s_2, s_3, s, r)$, respectively (i.e., $cl(H) = L(G)$).

In this paper, by using Theorem 14, we improve Theorem 10 and get the following result.

**Theorem 15.** *Let $H$ be a 2-connected claw-free simple graph of order $n$ with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$ and $n$ is sufficiently large, then one of the following holds:*

(a) *$H$ is Hamiltonian;*

(b) *$H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, 0, 0)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-6}{3}$, where $\min\{s_1, s_2, s_3\} \geq \frac{2n-15}{10}$, $s_1 + s_2 + s_3 + 6 = n$; or*

(c) *$H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, s, 0)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-2}{4}$, where $\min\{s_1, s_2, s_3\} \geq \frac{2n-15}{10}$, $s \geq \frac{2n-25}{10}$ and $s_1 + s_2 + s_3 + s + 6 = n$; or*

(d) *$H \in \mathcal{Q}_{5,2}(s_1, s_2, s_3, s_4, 0)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-6}{4}$, where $\min\{s_1, s_2, s_3, s_4\} \geq \frac{2n-15}{10}$ and $s_1 + s_2 + s_3 + s_4 + 7 = n$; or*

(e) *$H \in \mathcal{Q}_3(s_1, s_2, s_3, s, 0)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-8}{4}$, where $\min\{s_1, s_2, s_3\} \geq \frac{2n-15}{10}$, $s \geq \frac{2n-25}{10}$ and $s_1 + s_2 + s_3 + s + 9 = n$; or*

(f) *$H \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, s, r)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n+2}{5}$, where $\min\{s_1, s_2, s_3\} \geq \frac{2n-15}{10}$, $s, r \geq \frac{2n-25}{10}$ and $s_1 + s_2 + s_3 + s + r + 6 = n$; or*

(g) *$H \in \mathcal{Q}_{5,2}(s_1, s_2, s_3, s_4, s)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-2}{5}$, where $\min\{s_1, s_2, s_3, s_4\} \geq \frac{2n-15}{10}$, $s \geq \frac{2n-25}{10}$ and $s_1 + s_2 + s_3 + s_4 + s + 7 = n$; or*

(h) *$H \in \mathcal{Q}_3(s_1, s_2, s_3, s, r)$ and $\frac{2n-5}{5} \leq \overline{\sigma}_2(H) \leq \frac{2n-4}{5}$, where $\min\{s_1, s_2, s_3\} \geq \frac{2n-15}{10}$, $s, r \geq \frac{2n-25}{10}$ and $s_1 + s_2 + s_3 + s + r + 9 = n$.*

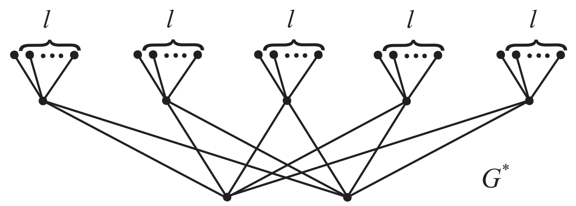


Figure 5. The graph $G^*$ with $\overline{\sigma}_2(L(G^*)) = \frac{2n-10}{5}$.

Let $G^*$ be the graph obtained from a $K_{2,5}$ by adding $l \geq 2$ pendant edges at each vertex of degree 2 in $K_{2,5}$, which is depicted in Figure 5.

**Remark 16.** Since $G^*$ has no DCT, by Theorem 1, $L(G^*)$ is non-Hamiltonian. The line graph $L(G^*)$ of order $n = 5l + 10$ ($n \geq 20$) is 2-connected with $\overline{\sigma}_2(L(G^*)) = 2l + 2 = \frac{2n-10}{5} < \frac{2n-5}{5}$, $\delta(L(G^*)) \geq 3$ and $L(G^*) \notin \mathcal{Q}_{2,3}(s_1, s_2, s_3, s, r) \cup \mathcal{Q}_{5,2}(s_1, s_2, s_3, s_4, s) \cup \mathcal{Q}_3(s_1, s_2, s_3, s, r)$. This example shows that the bound in Theorem 15 is asymptotically sharp.

Let $\widetilde{\mathcal{G}}'_5$ be the set of all spanning subgraphs of $G_5^1$ and $G_5^2$. By Theorem 15, we have the following result.

**Corollary 17.** *If $H$ is a 2-connected claw-free graph of order $n$ with $\delta(H) \geq \frac{2n-5}{10}$, then for $n$ sufficiently large, either $H$ is Hamiltonian or $H \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}'_5$.*

Obviously, Corollary 17 is not the best known result about $\delta(H)$, it follows from Theorems 8 and 9, and the results in [10]. Since Corollary 17 slightly improves Theorem 7 when $n$ is sufficiently large and $\widetilde{\mathcal{G}}'_5 \subset \widetilde{\mathcal{G}}_5$, it is still worth to be presented in this paper. The following result can be deduced from Theorem 15 immediately.

**Theorem 18.** *Let $H$ be a 2-connected claw-free simple graph of order $n$ with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$ and $n$ is sufficiently large, then either $H$ is Hamiltonian or $cl(H) = L(G)$, where $G$ is an essentially 2-edge-connected triangle-free graph which can be contracted to a $K_{2,3}$ or $W_3^*$ such that all vertices of degree 2 in $K_{2,3}$ or $W_3^*$ are nontrivial.*

By Theorem 18, we have the following result immediately.

**Corollary 19.** *Let $H$ be a 2-connected claw-free simple graph of order $n$. If $\delta(H) \geq \frac{2n-5}{10}$ and $n$ is sufficiently large, then either $H$ is Hamiltonian or $cl(H) = L(G)$, where $G$ is an essentially 2-edge-connected triangle-free graph which can be contracted to a $K_{2,3}$ or $W_3^*$ such that all vertices of degree 2 in $K_{2,3}$ or $W_3^*$ are nontrivial.*

As an application of Theorem 12, we get the following result, which is an extension of Theorem 11.

**Theorem 20.** *Let $G$ be an essentially 2-edge-connected graph of order $n$ such that $\overline{\sigma}_2(G) \geq 2\left(\left\lfloor \frac{n}{7} \right\rfloor - 1\right)$. If $n$ is sufficiently large, then either $L(G)$ is Hamiltonian or $G$ is contractible to a $K_{2,3}$ or $K_{2,5}$ or $W_3^*$ such that all vertices of degree 2 in $K_{2,3}$ or $K_{2,5}$ or $W_3^*$ are nontrivial.*

**Remark 21.** Although in [16] Veldman stated the following: there exist infinitely many essentially 2-edge-connected simple graphs with $\overline{\sigma}_2(G) = 2\left(\left\lfloor \frac{n}{7} \right\rfloor - 1\right)$ such that $L(G)$ is nonhamiltonian and $G$ is not contractible to $K_{2,3}$ and examples of such graphs can be found among the graphs contractible to $K_{2,5}$ or the 3-cubic minus a vertex ($W_3^*$). In Theorem 20, we give a more precise answer for the case $\overline{\sigma}_2(G) = 2\left(\left\lfloor \frac{n}{7} \right\rfloor - 1\right)$.

By Theorem 20, we have the following result immediately.

**Corollary 22.** *Let $G$ be an essentially 2-edge-connected simple graph of order $n$ such that $\delta(G) \geq \left\lfloor \frac{n}{7} \right\rfloor - 1$. If $n$ is sufficiently large, then either $L(G)$ is Hamiltonian or $G$ is contractible to a $K_{2,3}$ or $K_{2,5}$ or $W_3^*$ such that all vertices of degree 2 in $K_{2,3}$ or $K_{2,5}$ or $W_3^*$ are nontrivial.*

The remainder of this paper is organized as follows. In Section 2, we give a brief discussion of Catlin's reduction and present some auxiliary results, and also give proofs of Theorems 12 and 13. In Section 3, we give a brief discussion of the reduction of the core of essentially 2-edge-connected graphs and Veldman's reduction method, and we also present some useful results. In Section 4, proofs of Theorems 15, 18, 20 and Corollary 17 are given.

## 2. Preliminaries and Auxiliary Results

### 2.1. Catlin's reduction method

Let $O(G)$ be the set of vertices of odd degree in $G$. A graph $G$ is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph $\Gamma_R$ of $G$ with $O(\Gamma_R) = R$. The graph $K_1$ is regarded as a collapsible and supereulerian graph.

In [2], Catlin showed that every multigraph $G$ has a unique collection of maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_c$. The *reduction* of $G$ is $G' = G/\big(\bigcup_{i=1}^{c} \Gamma_i\big)$, the graph obtained from $G$ by contracting each $\Gamma_i$ into a single vertex $v_i$ ($1 \le i \le c$). For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma_0(v)$ such that $v$ is the contraction image of $\Gamma_0(v)$ and $\Gamma_0(v)$ is the *preimage* of $v$ and $v$ is a contracted vertex if $\Gamma_0(v) \neq K_1$. A graph $G$ is *reduced* if $G' = G$.

**Theorem 23** (Catlin *et al.* [2, 3])**.** *Let $G$ be a connected graph and let $G'$ be the reduction of $G$.*

(a) *$G$ is collapsible if and only if $G' = K_1$, and $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.*
(b) *$G$ has a DCT if and only if $G'$ has a DCT containing all the contracted vertices of $G'$.*
(c) *If $G$ is a reduced graph, then $G$ is simple and triangle-free with $\delta(G) \le 3$. For any subgraph $H$ of $G$, $H$ is reduced and either $H \in \{K_1, K_2, K_{2,t}\ (t \ge 2)\}$ or $|E(H)| \le 2|V(H)| - 5$.*

The following result is obvious.

**Lemma 24.** *Let $G$ be a graph and $F$ be a spanning subgraph of $G$. If $F$ has an SCT, then $G$ has an SCT.*

### 2.2. Proofs of Theorems 12 and 13

In this section, we shall present the proofs of Theorems 12 and 13. Let $G$ be a 2-connected simple graph and $C = v_0v_1v_2 \cdots v_{c(G)-1}v_0$ be a longest cycle of $G$, where the subscripts are taken modulo $c(G)$ in the following. Then any component of $G - C$ has at least two different neighbors on $C$. Denote by $d_C(v_i, v_j)$ the distance between $v_i, v_j \in V(C)$ $(v_i \neq v_j)$ on $C$. Obviously, $1 \le d_C(v_i, v_j) \le \left\lfloor \frac{|C|}{2} \right\rfloor$.

***Proof of Theorem* 12.** Suppose that $\kappa(G) = 1$. Let $B_1, B_2, \ldots, B_t$ $(t \geq 2)$ be the blocks of $G$. Since $G$ is triangle-free, $|V(B_i)| \geq 4$. Since $B_1$ and $B_t$ have at most one vertex in common, $7 \geq |V(G)| \geq |V(B_1)| + |V(B_t)| - 1 \geq 4 + 4 - 1 = 7$. Then $|V(G)| = 7$, and the equality holds only if $t = 2$ and $|V(B_1)| = |V(B_2)| = 4$. Since $G$ is triangle-free, $G[V(B_1)] = G[V(B_2)] = C_4$. Since $B_1$ and $B_2$ have a vertex in common, $G$ has an SCT, a contradiction. Hence $G$ is 2-connected.

Since $G \notin \mathcal{SL}$, $5 \leq |V(G)| \leq 7$. If $|V(G)| = 5$, then since $G$ is triangle-free and by $G \notin \mathcal{SL}$, $G = K_{2,3}$. Therefore, in the following, we only need to consider the cases $|V(G)| = 6$ and 7. Let $C = v_0 v_1 v_2 \cdots v_{c(G)-1} v_0$ be a longest cycle of $G$, where the subscripts are taken modulo $c(G)$ in the following.

*Case* 1. $|V(G)| = 6$. Then $4 \leq c(G) \leq 5$, otherwise, $G$ has an SCT, a contradiction.

*Subcase* 1.1. $c(G) = 4$. Since $|V(G)| = 6$ and $|V(C)| = 4$, $G - C = 2K_1$ or $K_2$. Let $V(G - C) = \{x, y\}$. Since $G$ is 2-connected and $G \notin \mathcal{SL}$, we can find a cycle containing vertices $x$ and $y$ with length more than 4, a contradiction.

*Subcase* 1.2. $c(G) = 5$. Then $C$ has no chords, otherwise, $G$ has a triangle, a contradiction. Since $|V(G)| = 6$ and $|V(C)| = 5$, $G - C = K_1$. Let $V(G - C) = \{x\}$. Since $G$ is 2-connected and triangle-free, $|N_G(x) \cap V(C)| = 2$ and $d_C(v_i, v_j) = 2$, where $v_i, v_j \in N_G(x) \cap V(C)$ $(v_i \neq v_j)$. Without loss of generality, let $N_G(x) = \{v_i, v_{i+2}\}$, then $G$ is isomorphic to $C_{5,2}$, which is depicted in Figure 3.

*Case* 2. $|V(G)| = 7$. Then $4 \leq c(G) \leq 6$, otherwise, $G$ has an SCT, a contradiction.

*Subcase* 2.1. $c(G) = 4$. Since $|V(G)| = 7$ and $|V(C)| = 4$, $G - C = 3K_1$ or $K_2 \cup K_1$ or $K_{1,2}$. Suppose that $G - C = K_2 \cup K_1$ or $K_{1,2}$. Since $G$ is 2-connected and triangle-free, there exists a path $x_1 \cdots x_k$ $(k = 2$ or $3)$ in $G - C$ with $v_i \in N_G(x_1) \cap V(C)$, $v_j \in N_G(x_k) \cap V(C)$ $(v_i \neq v_j)$. But now we can find a cycle containing vertices $x_1, \ldots, x_k$ with length more than 4, a contradiction. Then $G - C = 3K_1$. Let $V(G - C) = \{x, y, z\}$.

Since $G$ is 2-connected and triangle-free, $|N_G(x) \cap V(C)| = 2$ and $d_C(v_i, v_j) = 2$, where $v_i, v_j \in N_G(x) \cap V(C)$ $(v_i \neq v_j)$. Without loss of generality, let $N_G(x) = \{v_i, v_{i+2}\}$. By symmetry, we have $N_G(y) = \{v_i, v_{i+2}\}$ or $N_G(y) = \{v_{i+1}, v_{i+3}\}$. If $N_G(y) = \{v_{i+1}, v_{i+3}\}$, then $v_i x v_{i+2} v_{i+1} y v_{i+3} v_i$ is a cycle of length 6, a contradiction. So $N_G(y) = \{v_i, v_{i+2}\}$. Similarly, we have $N_G(z) = \{v_i, v_{i+2}\}$. Then $G$ is isomorphic to $K_{2,5}$, which is depicted in Figure 3.

*Subcase* 2.2. $c(G) = 5$. Then $C$ has no chords, otherwise, $G$ has a triangle, a contradiction. Since $|V(G)| = 7$ and $|V(C)| = 5$, $G - C = 2K_1$ or $K_2$. Suppose that $G - C = K_2$. Since $G$ is 2-connected and triangle-free, there exists an edge $xy$ in $G - C$ with $v_i \in N_G(x) \cap V(C)$, $v_j \in N_G(y) \cap V(C)$ $(v_i \neq v_j)$, and we can find a cycle containing vertices $x, y$ with length more than 5, a contradiction. Then $G - C = 2K_1$. Let $V(G - C) = \{x, y\}$.

Since $G$ is 2-connected and triangle-free, $|N_G(x) \cap V(C)| = |N_G(y) \cap V(C)| = 2$. Without loss of generality, let $N_G(x) = \{v_i, v_{i+2}\}$. Suppose that $v_i \in N_G(y)$ (by symmetry, it is similar for $v_{i+2} \in N_G(y)$). Since $G$ is triangle-free and $|N_G(y) \cap V(C)| = 2$, either $N_G(y) = \{v_i, v_{i+2}\}$ or $N_G(y) = \{v_i, v_{i+3}\}$. Then $v_i x v_{i+2} y v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4} v_i$ or $v_i v_{i+1} v_{i+2} x v_i v_{i+4} v_{i+3} y v_i$ is an SCT of $G$, a contradiction. Hence, $v_i, v_{i+2} \notin N_G(y)$. Then $N_G(y) \subset \{v_{i+1}, v_{i+3}, v_{i+4}\}$. Since $G$ is triangle-free and $|N_G(y) \cap V(C)| = 2$, $N_G(y) = \{v_{i+1}, v_{i+3}\}$ or $N_G(y) = \{v_{i+1}, v_{i+4}\}$. Then $v_i x v_{i+2} v_{i+1} y v_{i+3} v_{i+4} v_i$ or $v_i v_{i+1} y v_{i+4} v_{i+3} v_{i+2} x v_i$ is an SCT of $G$, a contradiction.

*Subcase* 2.3. $c(G) = 6$. By deleting all the chords of $C$ in $G$, the resulting 2-connected graph $G_0$ is a spanning subgraph of $G$. Obviously, $C$ is also a longest cycle of $G_0$. By Lemma 24, if $G_0$ has an SCT, then $G$ has an SCT. Therefore $G_0$ has no SCT. Note that if we add the deleted chords of $C$ to $G_0$ one by one, at each step we obtain at most one spanning subgraph of $G$ which has no SCT. Without loss of generality, we first assume that $C$ is an induced cycle of $G$, namely $G = G_0$.

Since $|V(G)| = 7$ and $|V(C)| = 6$, $G - C = K_1$. Let $V(G - C) = \{x\}$. Since $G$ is 2-connected triangle-free, $2 \leq |N_G(x) \cap V(C)| \leq 3$.

Suppose that $|N_G(x) \cap V(C)| = 2$. Let $v_i, v_j \in N_G(x)$ $(v_i \neq v_j)$. Since $G$ is triangle-free, $2 \leq d_C(v_i, v_j) \leq 3$. If $d_C(v_i, v_j) = 2$, then $G$ is isomorphic to $C_{6,2}$, which is depicted in Figure 3. If $d_C(v_i, v_j) = 3$, then $G$ is isomorphic to $C_{6,3}$, which is depicted in Figure 3.

Suppose that $|N_G(x) \cap V(C)| = 3$. Since $G$ is triangle-free, without loss of generality, let $N_G(x) = \{v_i, v_{i+2}, v_{i+4}\}$. Then $G$ is isomorphic to $W_3^*$, which is depicted in Figure 3.

Note that connecting any two nonadjacent vertices of $C_{6,2}$ will result in a triangle or a $W_3^*$ or an SCT of the new graph. Connecting any two nonadjacent vertices of $C_{6,3}$ or $W_3^*$ will result in a triangle or an SCT of the new graph. Hence, $G \in \{C_{6,2}, C_{6,3}, W_3^*\}$. The proof is completed. ■

***Proof of Theorem* 13.** If $G$ is a reduced graph, then by Theorem 23(c), $G$ is triangle-free. Then by Theorem 12, $G \in \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\}$. Note that $G$ is a simple graph. If $G$ has a collapsible subgraph $\Gamma$, then $|V(\Gamma)| \geq 3$. Let $G'$ be the reduction of $G$. Since $|V(G)| \leq 7$ and by $|V(\Gamma)| \geq 3$, $|V(G')| \leq 5$. Note that, by the same argument as in the first paragraph of the proof of Theorem 12, $G'$ must be 2-edge-connected. Hence Theorem 12 applies to $G'$. Then either $G'$ has an SCT or $G' = K_{2,3}$. For the first case, by Theorem 23(a), $G$ has an SCT, a contradiction. For the second case, $G'$ must be obtained from $G$ by contracting exactly a triangle. Then $G \in \mathcal{K}_{2,3}^{\triangle}$. The proof is completed. ■

## 3. Two Different Reduction Methods of a Graph and a Technical Lemma

### 3.1. The reduction of the core of a graph

Let $G$ be an essentially 2-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$. Then $D_1(G) \cup D_2(G)$ is an independent set. Let $E_1$ be the set of pendant edges in $G$. For each $x \in D_2(G)$, there are two edges $e_x^1$ and $e_x^2$ incident with $x$. Let $X_2(G) = \{e_x^1 \mid x \in D_2(G)\}$. Define

$$G_0 = G/(E_1 \cup X_2(G)).$$

In other words, $G_0$ is obtained from $G$ by deleting the vertices in $D_1(G)$ and replacing each path of length 2 whose internal vertex is a vertex in $D_2(G)$ by an edge. Note that $G_0$ may not be simple.

The vertex set $V(G_0)$ is regarded as a subset of $V(G)$. A vertex in $G_0$ is nontrivial if it is obtained by contracting some edges in $E_1 \cup X_2(G)$ or it is adjacent to a vertex in $D_2(G)$ in $G$. For instance, if $v \in D_2(G)$ and $N_G(v) = \{x, y\}$, and if $x_v$ is a vertex in $G_0$ obtained by contracting the edge $xv$, then both $x_v$ and $y$ are nontrivial in $G_0$ (although $x_v$ is a contracted vertex and $y$ is not a contracted vertex in $G_0$). Since $\overline{\sigma}_2(G) \geq 5$, all vertices in $D_2(G_0)$ are nontrivial.

Let $X = D_1(G) \cup D_2(G)$. In [16], $G_0$ is denoted by $I_X(G)$. In [15], Shao defined $G_0$ for essentially 3-edge-connected graphs $G$. Following [15], we call $G_0$ the *core* of $G$.

Let $G_0'$ be the reduction of $G_0$. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the maximum collapsible preimage of $v$ in $G_0$ and let $\Gamma(v)$ be the preimage of $v$ in $G$. Note that $\Gamma(v)$ is the graph induced by edges in $E(\Gamma_0(v))$ and some edges in $E_1 \cup X_2(G)$, for an example, see Figure 6. A vertex $v$ in $G_0'$ is a *nontrivial vertex* if $v$ is a contracted vertex (i.e., $|V(\Gamma(v))| > 1$) or $v$ is adjacent to a vertex in $D_2(G)$.

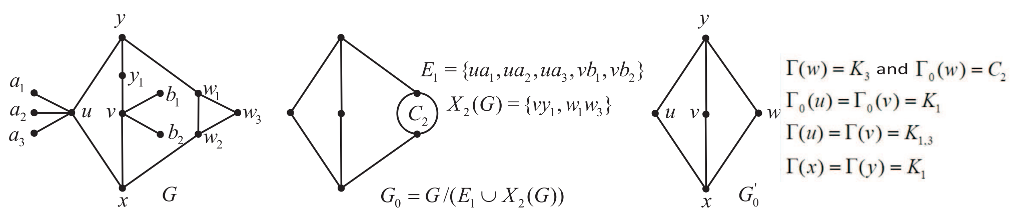


Figure 6. A process to obtain $G_0'$ from $G$ and the preimages of its vertices in $G$ and $G_0$.

Using Theorem 23, Veldman [16] and Shao [15] proved the following.

**Theorem 25.** *Let $G$ be a connected and essentially $k$-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$ where $k \in \{2, 3\}$ and $L(G)$ is not complete. Let $G_0$ be the core of $G$. Let $G_0'$ be the reduction of $G_0$. Then each of the following holds:*

(a) *$G_0$ is well-defined, nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq k$ and so $\kappa'(G_0') \geq \kappa'(G_0) \geq k$.*
(b) (*Lemma* 5 [16]) *$G$ has a DCT if and only if $G_0'$ has a DCT containing all the nontrivial vertices.*

In the following, let $H = L(G)$ and assume that $H$ is not complete. Then $|V(H)| = |E(G)|$ and $\overline{\sigma}_2(G) = \delta(H) + 2$. If $H = L(G)$ is a $k$-connected graph of order $n$ with $\delta(H) \geq 3$, then $G$ is essentially $k$-edge-connected with size $n$ and $\overline{\sigma}_2(G) \geq 5$. For each $v \in V(H)$, there is an edge $xy$ in $G$ corresponding to $v$ and $d_H(v) = d_G(x) + d_G(y) - 2$. For each edge $uv$ in $H$, there is a 2-path, $P_2 = xyz$ in $G$ such that $xy$ is corresponding to the vertex $u$ and edge $yz$ is corresponding to the vertex $v$ in $H$. Then $d_H(u) + d_H(v) = d_G(x) + 2d_G(y) + d_G(z) - 4$.

For any 2-path $P_2 = xyz$ in $G$, define $d_G(P_2) = d_G(x) + 2d_G(y) + d_G(z)$. Define

$$\delta_2(G) = \min\{d_G(P_2)|\ P_2 \text{ is a 2-path in } G\}.$$

Thus, for a graph $H = L(G)$,

$$\delta_2(G) = \overline{\sigma}_2(H) + 4. \tag{3.1}$$

For given integer $p > 0$ and $\epsilon$, if $\overline{\sigma}_2(H) \geq \frac{2n+\epsilon}{p}$, then the preimage $G$ of $H = L(G)$ has

$$\delta_2(G) \geq \frac{2n+\epsilon}{p} + 4. \tag{3.2}$$

### 3.2. More notation and a technical lemma

Let $G$, $G_0$ and $G_0'$ be the same as in previous definitions. For convenience, we use the following notation:

- ◇ $V^* = \{v \in V(G_0')|\ |V(\Gamma(v))| \geq 3\}$;
- ◇ $V_1 = \{v \in V(G_0')|\ |V(\Gamma(v))| = 1$ and $v$ is not adjacent to any vertices in $D_1(G) \cup D_2(G)\}$;
- ◇ $V_2 = \{v \in V(G_0') - V^*|\ |V(\Gamma(v))| = 2$ or $|V(\Gamma(v))| = 1$ and $v$ is adjacent to a vertex in $D_2(G)\}$;
  (Note that $V^* \cup V_2$ is the set of all nontrivial vertices in $G_0'$.)
- ◇ $\Phi = G_0'[V_1]$, the subgraph induced by $V_1$ in $G_0'$ if $V_1 \neq \emptyset$;
- ◇ $E_\Phi = E(\Phi)$, which is a matching under the conditions of Lemma 26 (see below);
- ◇ $V_\Phi = \{v \in V_1|\ v$ is incident with an edge in $E_\Phi\}$;
- ◇ $V_\Phi^0 = V_1 - V_\Phi$;
- ◇ $N_{\Phi,2} = \bigcup_{v \in V_\Phi \cup V_2} (N_{G_0'}(v) \cap V^*)$ if $V_\Phi \cup V_2 \neq \emptyset$ (otherwise, $N_{\Phi,2} = \emptyset$).

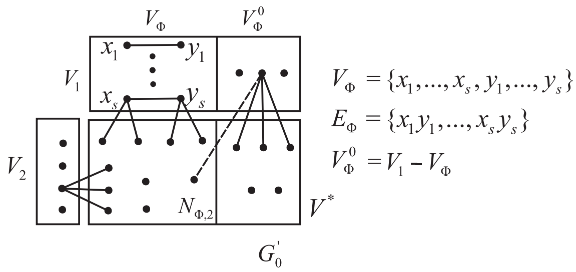


Figure 7. Decomposition of $V(G_0') = V^* \cup V_1 \cup V_2 = V^* \cup (V_\Phi \cup V_\Phi^0) \cup V_2$.

In the following, for given integer $p > 0$ and $\epsilon$, we use "$n \gg p$" for "$n$ is sufficiently large related to $p$ and $\epsilon$".

**Lemma 26** (Chen [4])**.** *Let $G$ be an essentially 2-edge-connected triangle-free graph and $G \neq K_{1,t}$ with size $n$ and $\overline{\sigma}_2(G) \geq 5$, and satisfying* (3.2) *and $n \gg p$. Assume that $G_0' \notin \mathcal{SL}$. For $V^*, N_{\Phi,2}, V_1, V_2, \Phi, E_\Phi, V_\Phi$, and $V_\Phi^0$ defined above, we have the following.*

(a) *For each $v \in V^*$, $|V(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v)$ and $|E(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - d_{G_0'}(v) - 1$.*

(b) *$D_2(G_0') \subseteq V^*$ and so $d_{G_0'}(v) \geq 3$ for $v \in V_1 \cup V_2$.*

(c) *If $E_\Phi \neq \emptyset$, for each $xy \in E_\Phi, (N_{G_0'}(x) - \{y\}) \cup (N_{G_0'}(y) - \{x\}) \subseteq N_{\Phi,2}$ and so $E_\Phi$ is a matching.*

(d) *For each vertex $v$ in $V_\Phi^0 \cup V_2, N_{G_0'}(v) \subseteq V^*$, and so $V_\Phi^0 \cup V_2$ is an independent set.*

(e) *If $|V_1 \cup V_2| \geq 3$, then $|V_\Phi^0 \cup V_2| + \frac{|V_\Phi|}{2} \leq 2|V^*| - 5$. If $|V_2| \geq 3$ or $V_\Phi \neq \emptyset$, then $|V_2| + \frac{|V_\Phi|}{2} \leq 2|N_{\Phi,2}| - 5$.*

(f) *$|V^*| \leq p$. Furthermore, if $|V^*| = p$ and $G_0' \neq K_{2,t}$ $(t \geq 2)$, then $|V(G_0')| \leq 2p - 5 - \frac{\epsilon}{2}$.*

(g) *For $v \in N_{\Phi,2}, |E(\Gamma(v))| \geq \delta_2(G) - 5p - 3$ and $|V^*| + |N_{\Phi,2}| \leq p$.*

(h) *If $V_2 \neq \emptyset$, then $|N_{\Phi,2}| \geq 3$. If $V_\Phi \neq \emptyset$, then $|N_{\Phi,2}| \geq 4$. Thus, $|N_{\Phi,2}| \geq 3$ if $|V_2 \cup V_\Phi| \neq 0$.*

The following lemma will be needed for the proofs of Theorems 15, 18 and Corollary 17.

**Lemma 27.** *Let $H$ be a 2-connected claw-free simple graph of order $n$ with $\delta(H) \geq 3$. If $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$ and $n$ is sufficiently large, then either $H$ is Hamiltonian, or $cl(H) = L(G)$, where $G$ is an essentially 2-edge-connected triangle-free graph and $G_0' \in \{K_{2,3}, C_{5,2}, W_3^*\}$ and $V(G_0') = V^* \cup V_\Phi^0$.*

***Proof.*** This is an improvement of a special case of Theorem 14 with $p = 5, \epsilon = -5$ and $k = 2$, since it narrows down $\mathcal{Q}_0(15, 2)$ to a subset of $\mathcal{Q}_0(7, 2)$. By Theorem

2, there is an essentially 2-edge-connected triangle-free graph $G$ such that the closure $cl(H) = L(G)$. Then $|E(G)| = |V(H)| = n$. Since $\delta(H) \geq 3$, $\overline{\sigma}_2(G) \geq 5$. Since $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$ and by (3.1), $\delta_2(G) \geq \frac{2n-5}{5} + 4$.

Suppose that $H$ is not Hamiltonian. Then $G \neq K_{1,t}$, otherwise, by Theorem 2, $H$ is Hamiltonian, a contradiction. Then $G_0'$ has no DCT containing all the nontrivial vertices of $G_0'$, otherwise, by Theorem 25(b) and Theorem 1, $H$ is Hamiltonian, a contradiction. Then $G_0' \notin \mathcal{SL}$. By Theorem 25(a), $G_0'$ is 2-edge-connected. By Theorem 23(c), $G_0'$ is triangle-free. Since $G_0' \notin \mathcal{SL}$ and $G_0'$ is triangle-free, $|V(G_0')| \geq 5$.

By Lemma 26(g), $2|N_{\Phi,2}| \leq |V^*| + |N_{\Phi,2}| \leq 5$ and so $|N_{\Phi,2}| \leq 2$. Then by Lemma 26(h), $|V_\Phi \cup V_2| = 0$. Therefore, $N_{\Phi,2} = \emptyset$, $V_\Phi^0 = V_1$ and $V(G_0') = V^* \cup V_\Phi^0$. By Lemma 26(f), $|V^*| \leq 5$.

*Case* 1. $|V^*| \leq 4$. Then $|V_\Phi^0| \leq 3$, otherwise, by Lemma 26(e), $|V^*| \geq 5$, a contradiction. Then $|V(G_0')| \leq |V_\Phi^0| + |V^*| \leq 7$. Since $G_0' \notin \mathcal{SL}$ and by Theorem 12, $G_0' \in \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\}$. If $G_0' \in \{C_{6,3}, C_{6,2}, K_{2,5}\}$, then $|D_2(G_0')| = 5$. Then by Lemma 26(b), $|V^*| \geq 5$, a contradiction. Hence, $G_0' \in \left\{K_{2,3}, C_{5,2}, W_3^*\right\}$.

*Case* 2. $|V^*| = 5$. Suppose that $G_0' = K_{2,t}$. Then $t$ is odd, otherwise, $G_0'$ has an SCT, a contradiction. By Lemma 26(b), $D_2(G_0') \subseteq V^*$. Then $t \leq 5$, otherwise, $|V^*| \geq 6$, a contradiction. Since $t$ ($t \leq 5$) is odd and by $|V(G_0')| \geq 5$, $G_0' = K_{2,3}$ or $K_{2,5}$. If $G_0' = K_{2,5}$, then by $D_2(K_{2,5}) \subseteq V^*$ and by $|V^*| = 5$, $D_2(K_{2,5}) = V^*$. Let $V^* = \{v_1, v_2, v_3, v_4, v_5\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$. Since $n = 10 + \sum_{i=1}^5 s_i \geq 10 + 5\left(\frac{\delta_2(G)}{2} - 3\right) = \frac{5}{2}\delta_2(G) - 5$, $\delta_2(G) \leq \frac{2n+10}{5}$, contrary to $\delta_2(G) \geq \frac{2n+15}{5}$. Hence, $G_0' = K_{2,3}$.

In the following, we assume that $G_0' \neq K_{2,t}$ for any integer $t$. Then by Lemma 26(f) with $p = 5$ and $\epsilon = -5$, $|V(G_0')| \leq 2p - 5 - \frac{\epsilon}{2} \leq 7.5$. Then $|V(G_0')| \leq 7$. Since $G_0' \notin \mathcal{SL}$ $\left(G_0' \neq K_{2,t}\right)$ and by Theorem 12, $G_0' \in \left\{C_{5,2}, W_3^*, C_{6,2}, C_{6,3}\right\}$.

Suppose that $G_0' = C_{6,2}$ or $C_{6,3}$. By Lemma 26(b), $D_2(G_0') \subseteq V^*$. Then by $|V^*| = 5$, $V^* = D_2(C_{6,2})$ or $D_2(C_{6,3})$. Let $V^* = \{v_1, v_2, v_3, v_4, v_5\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$. Since $n = 8 + \sum_{i=1}^5 s_i \geq 8 + 5\left(\frac{\delta_2(G)}{2} - 3\right) = \frac{5}{2}\delta_2(G) - 7$, $\delta_2(G) \leq \frac{2n+14}{5}$, contrary to $\delta_2(G) \geq \frac{2n+15}{5}$. Then $G_0' \neq C_{6,2}$ and $G_0' \neq C_{6,3}$.

Hence, $G_0' \in \{K_{2,3}, C_{5,2}, W_3^*\}$. The proof is completed. ∎

### 3.3. Veldman's reduction method

For an independent subset $X$ of $D(G)$, define $I_X(G)$ as the graph obtained from $G$ by deleting the vertices in $X$ of degree 1 and replacing each path of length 2 whose internal vertex is in $D_2(G) \cap X$ by an edge. Note that $I_X(G)$ may not be simple. We call $G$ $X$-*collapsible* if $I_X(G)$ is collapsible. A subgraph $H$ of $G$ is an $X$-*subgraph* of $G$ if $d_H(x) = d_G(x)$ for all $x \in X \cap V(H)$. An $X$-subgraph $H$ of $G$ is called $X$-collapsible if $H$ is $(X \cap V(H))$-collapsible. Let $R(X)$ be the set of vertices in $X$ that are not contained in an $X$-collapsible $X$-subgraph of $G$. Since $I_X(G)$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $L_1, \ldots, L_k$ such that $\bigcup_{i=1}^{k} V(L_i) = V(I_X(G))$, the graph $G$ has a unique collection of pairwise vertex-disjoint maximal $X$-collapsible $X$-subgraphs $H_1, \ldots, H_k$ such that $\left(\bigcup_{i=1}^{k} V(H_i)\right) \cup R(X) = V(G)$. The $X$-*reduction* of $G$ is the graph obtained from $G$ by contracting $H_1, \ldots, H_k$. Let $G''$ be the $X$-reduction of $G$ and $v \in V(G'')$. Then the *preimage* of $v$ is denoted by $\theta^{-1}(v)$. A vertex $v$ of $G''$ is called *nontrivial* if $\theta^{-1}(v)$ is not a vertex and *trivial* otherwise. The graph $G$ is $X$-*reduced* if there exists a graph $G^*$ and an independent subset $X^*$ of $D(G^*)$ such that $X = R(X^*)$ and $G$ is the $X^*$-reduction of $G^*$. An $X$-subgraph $H$ of $G$ is called $X$-*reduced* if $H$ is $(X \cap V(H))$-reduced.

**Remark 28.** If $X = \emptyset$, then the refinement method ($\emptyset$-reduction) is just the reduction method of Catlin. Let $G$ be an essentially 2-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$. Then $D(G)$ is an independent set. For the $D(G)$-reduction of $G$, if $R(D(G)) = \emptyset$, then the refinement method of the reduction of the core of the graph $G$ is just the $D(G)$-reduction method of Veldman.

In [16], Veldman obtained the following result.

**Theorem 29** (Veldman [16])**.** *Let $G$ be a connected simple graph of order $n$ and $p \geq 2$ an integer such that*

$$\overline{\sigma}_2(G) \geq 2\left( \lfloor n/p \rfloor - 1\right). \tag{3.3}$$

*If $n$ is sufficiently large relative to $p$, then*

$$\left|V\left(G''\right)\right| \leq \max\left\{p, \frac{3}{2}p - 4\right\}, \tag{3.4}$$

*where $G''$ is the $D(G)$-reduction of $G$. Moreover, for $p \leq 7$,* (3.4) *holds with equality only if* (3.3) *holds with equality.*

## 4. Proofs of Theorems 15, 18, 20 and Corollary 17

In this section, we shall present the proofs of Theorems 15, 18, 20 and Corollary 17.

***Proof of Theorem* 15.** Suppose that $H$ is not Hamiltonian. By Lemma 27, there is an essentially 2-edge-connected triangle-free graph $G$ such that the closure

$cl(H) = L(G)$ and $G_0' \in \{K_{2,3}, C_{5,2}, W_3^*\}$ and $V(G_0') = V^* \cup V_\Phi^0$. Since $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$ and by (3.1), $\delta_2(G) \geq \frac{2n-5}{5}+4$. By Lemma 26(f), $|V^*| \leq 5$. In the following, we label the vertices of $K_{2,3}$, $C_{5,2}$ and $W_3^*$ as the graphs in Figure 3.

*Case* 1. $|V^*| \leq 4$. Then $|V_\Phi^0| \leq 3$, otherwise, by Lemma 26(e), $|V^*| \geq 5$, a contradiction.

*Case* 1.1. $G_0' = K_{2,3}$. By Lemma 26(b), $D_2(G_0') \subseteq V^*$. Then $3 \leq |V^*| \leq 4$. Suppose that $|V^*| = 3$. Then $V^* = D_2(G_0')$. Let $V^* = \{v_1, v_2, v_3\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5}+4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2}-3 \geq \frac{2n-15}{10}$. Thus, $G \in \mathcal{K}_{2,3}(s_1, s_2, s_3, 0, 0)$. Furthermore, since $n = s_1 + s_2 + s_3 + 6 \geq 3\left(\frac{\delta_2(G)}{2} - 3\right)+6 = \frac{3}{2}\delta_2(G)-3$, $\delta_2(G) \leq \frac{2n+6}{3}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{2n-6}{3}$. Theorem 15(b) holds.

Suppose that $|V^*| = 4$. Since $D_2(G_0') \subseteq V^*$ and by $|D_2(G_0')| = 3$, without loss of generality, let $V^* = \{v_1, v_2, v_3, u_1\}$. Then $d_{G_0'}(v_i) = 2$, $d_{G_0'}(u_1) = 3$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$ and $s = |E(\Gamma(u_1))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-25}{10}$. Thus, $G \in \mathcal{K}_{2,3}(s_1, s_2, s_3, s, 0)$. Furthermore, since $n = s_1 + s_2 + s_3 + s + 6 \geq 3\left(\frac{\delta_2(G)}{2} - 3\right) + \left(\frac{\delta_2(G)}{2} - 4\right) + 6 = 2\delta_2(G) - 7$, $\delta_2(G) \leq \frac{n+7}{2}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{n-1}{2}$. Theorem 15(c) holds.

*Case* 1.2. $G_0' = C_{5,2}$. By Lemma 26(b), $D_2(G_0') \subseteq V^*$. Then $|V^*| = 4$. Let $V^* = \{v_1, v_2, v_3, v_4\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2}-3 \geq \frac{2n-15}{10}$. Thus, $G \in \mathcal{C}_{5,2}(s_1, s_2, s_3, s_4, 0)$. Furthermore, since $n = s_1 + s_2 + s_3 + s_4 + 7 \geq 4\left(\frac{\delta_2(G)}{2} - 3\right) + 7 = 2\delta_2(G) - 5$, $\delta_2(G) \leq \frac{n+5}{2}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{2n-6}{4}$. Theorem 15(d) holds.

*Case* 1.3. $G_0' = W_3^*$. By Lemma 26(b), $D_2(G_0') \subseteq V^*$. Then $3 \leq |V^*| \leq 4$. Suppose that $|V^*| = 3$. Then $7 = |V(G_0')| = |V^*| + |V_\Phi^0| \leq 3 + 3 = 6$, a contradiction. Then $|V^*| = 4$. Since $V_\Phi^0$ is an independent set and by $|V^*| = 4$, $V^* = \{v_1, v_2, v_3, v\}$. Then $d_{G_0'}(v_i) = 2$, $d_{G_0'}(v) = 3$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$ and $s = |E(\Gamma(v))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-25}{10}$. Thus, $G \in \mathcal{W}_3^*(s_1, s_2, s_3, s, 0)$. Furthermore, since $n = s_1 + s_2 + s_3 + s + 9 \geq 3\left(\frac{\delta_2(G)}{2} - 3\right) + \left(\frac{\delta_2(G)}{2} - 4\right) + 9 = 2\delta_2(G) - 4$, $\delta_2(G) \leq \frac{n+4}{2}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{n-4}{2}$. Theorem 15(e) holds.

*Case* 2. $|V^*| = 5$.

*Case* 2.1. $G_0' = K_{2,3}$. Since $|V^*| = 5$ and by $|V(G_0')| = 5$, $V(G_0') = V^*$. Let $V^* = \{v_1, v_2, v_3, u_1, u_2\}$. Then $d_{G_0'}(v_i) = 2$, $d_{G_0'}(u_i) = 3$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$, $s = |E(\Gamma(u_1))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-25}{10}$ and $r = |E(\Gamma(u_2))| \geq \frac{\delta_2(G)}{2}-4 \geq \frac{2n-25}{10}$. Thus, $G \in \mathcal{K}_{2,3}(s_1, s_2, s_3, s, r)$.

Furthermore, since $n = s_1+s_2+s_3+s+r+6 \geq 3\left(\frac{\delta_2(G)}{2} - 3\right)+2\left(\frac{\delta_2(G)}{2} - 4\right)+6 = \frac{5}{2}\delta_2(G) - 11$, $\delta_2(G) \leq \frac{2n+22}{5}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{2n+2}{5}$. Theorem 15(f) holds.

*Case* 2.2. $G'_0 = C_{5,2}$. By Lemma 26(b), $D_2(G'_0) \subseteq V^*$. Since $|V^*| = 5$ and by $|D_2(C_{5,2})| = 4$, there exists exactly one vertex $u_i \in D_3(C_{5,2})$ such that $u_i \in V^*$. Without loss of generality, let $V^* = D_2(G'_0) \cup \{u_1\} = \{v_1, v_2, v_3, v_4, u_1\}$. Then $d_{G'_0}(v_i) = 2$, $d_{G'_0}(u_1) = 3$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5} + 4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2} - 3 \geq \frac{2n-15}{10}$ and $s = |E(\Gamma(u_1))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-25}{10}$. Thus, $G \in \mathcal{C}_{5,2}(s_1, s_2, s_3, s_4, s)$. Furthermore, since $n = s_1 + s_2 + s_3 + s_4 + s + 7 \geq 4\left(\frac{\delta_2(G)}{2} - 3\right) + \left(\frac{\delta_2(G)}{2} - 4\right) + 7 = \frac{5}{2}\delta_2(G) - 9$, $\delta_2(G) \leq \frac{2n+18}{5}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{2n-2}{5}$. Theorem 15(g) holds.

*Case* 2.3. $G'_0 = W_3^*$. By Lemma 26(b), $D_2(G'_0) \subseteq V^*$. By Lemma 26(d), $V_\Phi^0$ is an independent set. Since $V(G'_0) = V^* \cup V_\Phi^0$ and by the fact $|V^*| = 5$ and $|D_2(W_3^*)| = 3$, there exist exactly two vertex $x, y \in D_3(W_3^*)$ such that $x, y \in V^*$. Since $V_\Phi^0$ is an independent set, without loss of generality, let $V^* = \{v_1, v_2, v_3, v, u_1\}$. Then $d_{G'_0}(v_i) = 2$, $d_{G'_0}(v) = d_{G'_0}(u_1) = 3$. By Lemma 26(a) and $\delta_2(G) \geq \frac{2n-5}{5}+4$, $s_i = |E(\Gamma(v_i))| \geq \frac{\delta_2(G)}{2}-3 \geq \frac{2n-15}{10}$, $s = |E(\Gamma(v))| \geq \frac{\delta_2(G)}{2}-4 \geq \frac{2n-25}{10}$ and $r = |E(\Gamma(u_1))| \geq \frac{\delta_2(G)}{2} - 4 \geq \frac{2n-25}{10}$. Thus, $G \in \mathcal{W}_3^*(s_1, s_2, s_3, s, r)$. Furthermore, since $n = s_1+s_2+s_3+s+r+9 \geq 3\left(\frac{\delta_2(G)}{2} - 3\right)+2\left(\frac{\delta_2(G)}{2} - 4\right)+9 = \frac{5}{2}\delta_2(G) - 8$, $\delta_2(G) \leq \frac{2n+16}{5}$. By (3.1), $\overline{\sigma}_2(H) \leq \frac{2n-4}{5}$. Theorem 15(h) holds. The proof is completed. ■

***Proof of Corollary* 17.** Since $\delta(H) \geq \frac{2n-5}{10}$, $\overline{\sigma}_2(H) \geq \frac{2n-5}{5}$. Since $n$ is sufficiently large, $\delta(H) \geq 3$. By Lemma 27, either $H$ is Hamiltonian, or $cl(H) = L(G)$, where $G$ is an essentially 2-edge-connected triangle-free graph and $G'_0 \in \{K_{2,3}, C_{5,2}, W_3^*\}$ and $V(G'_0) = V^* \cup V_\Phi^0$. Note that $D_2(G'_0) \subseteq V^*$ and $V_\Phi^0$ is an independent set. Obviously, $C_{5,2}$ can be contracted to a $K_{2,3}$ by contracting one edge of $C_{5,2}$. Suppose that $G$ can be contracted to a $K_{2,3}$. If $K_{2,3}$ has three nontrivial vertices, then $cl(H) \in \widetilde{\mathcal{G}}_3$. If $K_{2,3}$ has four nontrivial vertices, then $cl(H) \in \widetilde{\mathcal{G}}_4$. If all vertices of $K_{2,3}$ are nontrivial, then $cl(H) \in \widetilde{\mathcal{G}}'_5$. Suppose that $G'_0 = W_3^*$. Then $|V^*| \geq 4$. Then by Lemma 26(f), $4 \leq |V^*| \leq 5$. Then $cl(H) \in \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}'_5$. Hence, $cl(H) \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}'_5$. Since $H$ is a spanning subgraph of $cl(H)$, $H \in \widetilde{\mathcal{G}}_3 \cup \widetilde{\mathcal{G}}_4 \cup \widetilde{\mathcal{G}}'_5$. The proof is completed. ■

***Proof of Theorem* 18.** By Lemma 27, we have either $H$ is Hamiltonian or $cl(H) = L(G)$, where $G$ is an essentially 2-edge-connected triangle-free graph and $G'_0 \in \{K_{2,3}, C_{5,2}, W_3^*\}$. Obviously, $C_{5,2}$ can be contracted to a $K_{2,3}$ by contracting one edge of $C_{5,2}$. By the definition of $G'_0$, each vertex in $D_2(G'_0)$ is nontrivial. Then $G$ can be contracted such that each vertex of degree 2 in the resulting $K_{2,3}$ or $W_3^*$ is nontrivial. The proof is completed. ■

***Proof of Theorem*** **20.** Because $\overline{\sigma}_2(G) \geq 2\left(\lfloor \frac{n}{7} \rfloor - 1\right)$, if $n \geq 28$, then $\overline{\sigma}_2(G) \geq 6$ and $D(G)$ is an independent set. Let $G''$ be the $D(G)$-reduction of $G$. By (3.4), $|V(G'')| \leq 7$. Let $G_0$ be the core of $G$, $G_0'$ be the reduction of $G_0$. Since $G_0'$ is a refinement of the $D(G)$-reduction of $G$, $|V(G_0')| \leq |V(G'')| \leq 7$. If $G_0'$ has a DCT containing all the nontrivial vertices, then by Theorem 25(b), $G$ has a DCT. Then by Theorem 1, $L(G)$ is Hamiltonian. Therefore, in the following, we can assume that $G_0'$ has no DCT containing all the nontrivial vertices. Then $G_0'$ has no SCT. By Theorem 25(a), $G_0'$ is 2-edge-connected. Since $G_0'$ is the reduction of $G_0$, by Theorem 23(c), $G_0'$ is a triangle-free graph. Then by Theorem 12, $G_0' \in \{K_{2,3}, C_{5,2}, W_3^*, C_{6,2}, C_{6,3}, K_{2,5}\}$. Note that each graph in $\{C_{5,2}, C_{6,2}, C_{6,3}\}$ can be contracted to a $K_{2,3}$. Then $G_0'$ can be contracted to a graph in $\{K_{2,3}, K_{2,5}, W_3^*\}$. By the definition of $G_0'$, each vertex in $D_2(G_0')$ is nontrivial. Then $G$ can be contracted such that each vertex of degree 2 in the resulting $K_{2,3}$ or $K_{2,5}$ or $W_3^*$ is nontrivial. The proof is completed. ■

## 5. Concluding Remark

In this paper, we mainly focused on the Hamiltonicity of 2-connected claw-free graphs. In [4], Chen considered the Hamiltonicity of 2-connected claw-free graphs for a special case of Theorem 14 where $p = 4$. In this paper, our result (Theorem 18 for $p = 5$) is also a special case of Theorem 14. Obviously, Theorem 15 and Corollary 17 extend Theorems 7 and 10 while Theorem 10 does not imply Theorem 7.

Although the authors in [10], with the help of a computer, have considered the Hamiltonicity of 2-connected claw-free graphs of order $n \geq 153$ with $\delta(H) \geq \frac{n+39}{8}$ and the corresponding family of exceptions contains 318 infinite classes, their result also could not imply our results (Theorems 15 and 18).

In order to prove our main results, we characterized all 2-edge-connected simple graphs of order at most 7 which have no spanning closed trail. For $p \geq 6$, we may need to consider all 2-edge-connected simple graphs of order at most 10 (or more than 10) which have no spanning closed trail. Therefore, if one wants to deal with the case for $p \geq 6$ of Theorem 14, it would become very complicated.

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