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A NOTE ON CYCLES IN LOCALLY HAMILTONIAN AND LOCALLY HAMILTON-CONNECTED GRAPHS

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Abstract

Let $\mathcal{P}$ be a property of a graph. A graph G is said to be locally $\mathcal{P}$, if the subgraph induced by the open neighbourhood of every vertex in G has property $\mathcal{P}$. Ryjáček conjectures that every connected, locally connected graph is weakly pancyclic. Motivated by the above conjecture, van Aardt *et al.* [S.A.van Aardt, M. Frick, O.R. Oellermann and J.P.de Wet, *Global cycle properties in locally connected, locally traceable and locally Hamiltonian graphs*, Discrete Appl. Math. 205 (2016) 171–179] investigated the global cycle structures in connected, locally traceable/Hamiltonian graphs. Among other results, they proved that a connected, locally Hamiltonian graph G with maximum degree at least $|V(G)| - 5$ is weakly pancyclic. In this note, we improve this result by showing that such a graph with maximum degree at least $|V(G)|-6$ is weakly pancyclic. Furthermore, we show that a connected, locally Hamilton-connected graph with maximum degree at most 7 is fully cycle extendable.

Keywords: locally connected, locally Hamiltonian, locally Hamilton-connected, fully cycle extendability, weakly pancyclicity.

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1. Introduction

For definitions and notation we follow [2]. Let $G = (V, E)$ be a simple, undirected and connected graph. The order of G is denoted by $n(G)$, and let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively. If G is clear from the context we use n, δ and Δ, to denote these respective quantities. The *open neighbourhood* $N(v)$ of a vertex v of G is the set of all vertices adjacent to v. If $X \subseteq V(G)$, the subgraph induced by X is denoted by $\langle X \rangle$.

A *Hamilton cycle* (*path*) of a graph G is a spanning cycle (spanning path) of it, i.e., a cycle (path) that contains every vertex of G. A graph G is *Hamiltonian* (*traceable*) if it has a Hamilton cycle (path), and G is *Hamilton-connected* if there exists a Hamilton path between any two distinct vertices of G. A graph G is said to be *pancyclic* if G has a cycle of length m for every integer $3 \leq m \leq n$. The *girth* (respectively, *circumference*) of a graph G, denoted by $g(G)$ (respectively, $c(G)$), is defined as the length of a shortest (respectively, longest) cycle in G. A graph G that is not necessarily Hamiltonian but has cycles of every possible length from $g(G)$ to $c(G)$ is said to be *weakly pancyclic*. An even stronger notion than pancyclicity is that of "full cycle extendability", introduced by Hendry [8]. A cycle C in a graph G is *extendable* if there exists a cycle C' in G that contains all the vertices of C plus a single new vertex. A graph G is *cycle extendable* if every non-Hamiltonian cycle of G is extendable. If, in addition, every vertex of G belongs to a triangle, then G is *fully cycle extendable*.

For a given graph property $\mathcal{P}$, we call a graph G locally $\mathcal{P}$ if $\langle N(v) \rangle$ has property $\mathcal{P}$ for every $v \in V(G)$. The notion of locally Hamiltonian graphs was introduced by Skupień in 1965 [12], and in 1971 Chartrand and Pippert introduced locally connected graphs [5]. Note that many classical conditions that guarantee the existence of some specified structures in graphs can be expressed as local properties of graphs. For example, Dirac's minimum degree condition "$\delta(G) \geq \frac{n(G)}{2}$" may be written as "$|N(v)| \geq \frac{n(G)}{2}$" for every vertex v of G, and G is claw-free if and only if "$\alpha(\langle N(v) \rangle) \leq 2$" for every vertex v of G, where $\alpha(G)$ denotes the independence number of G. The properties of locally connected/traceable/Hamiltonian/isometric graphs have been extensively studied, for the details see, for example, [1, 3–6, 9–14]. The following conjecture is proposed by Ryjáček (see [15]).

Conjecture 1. *Every locally connected graph is weakly pancyclic.*

In [14], the authors studied the global cycle properties of connected, locally traceable and locally Hamiltonian graphs and proposed the following two weaker conjectures.

Conjecture 2. *Every locally traceable graph is weakly pancyclic.*

Conjecture 3. *Every locally Hamiltonian graph is weakly pancyclic.*

Ryjáček's conjecture seems very difficult to settle. However, some progress has been made for graphs with small maximum degree. The following result is obtained in [14].

Theorem 4 (Theorem 3.5 in [14])**.** *If G is a locally connected graph with $\Delta(G) \leq 5$, then G is weakly pancyclic.*

In [14], the authors confirmed Conjectures 2 and 3 for graphs with small maximum degree by giving stronger results, and confirmed Conjecture 3 for large maximum degree. For $k \geq 3$, the magwheel M_k is the graph obtained from the wheel W_k by adding, for each edge e on the rim of W_k, a vertex v_e and joining it to the two ends of the edge e (see [14]).

Theorem 5 (Theorem 4.1 in [14])**.** *Suppose G is a connected, locally traceable graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*

Theorem 6 (Theorem 5.1 in [14])**.** *Let G be a connected, locally Hamiltonian graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*

Theorem 7 (Theorem 6.3 in [14])**.** *If G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n - 5$, then G is weakly pancyclic.*

For our purpose here, we have just listed some results relevant to our results obtained in this paper. For more complete researches in this direction we refer the reader to the literature mentioned above. In this note, we improve Theorem 7 by proving that a connected locally Hamiltonian graph G with $\Delta(G) \geq n(G) - 6$ is weakly pancyclic. Furthermore, we show that a connected locally Hamilton-connected graph G with $\Delta(G) \leq 7$ is fully cycle extendable.

Theorem 8. *If G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n - 6$, then G is weakly pancyclic.*

Theorem 9. *Let G be a connected, locally Hamilton-connected graph of order $n \geq 3$ with $\Delta(G) \leq 7$. Then G is fully cycle extendable.*

2. Proof of Theorem 8

We first cite two useful lemmas from [14].

Lemma 10 (Lemma 6.1 in [14])**.** *If G is a locally Hamiltonian graph and $uv \in E(G)$, then $|N(u) \cap N(v)| \geq 2$.*

Lemma 11 (Lemma 6.2 in [14])**.** *If G is a connected, locally Hamiltonian graph of order n and maximum degree Δ, then G has cycles of length k for every k such that $3 \leq k \leq \min\{\Delta + 2, n\}$.*

***Proof of Theorem* 8.** By Theorem 7, we only need to consider the case $\Delta = n-6$. If $n \leq 12$, then $\Delta = n-6 \leq 6$, and the result holds from Theorem 6. Now assume $n \geq 13$. Note that by Lemma 11, G has cycles of length k for every k with $3 \leq k \leq n-4$. It suffices to show that if G has an $(n-i)$-cycle, then G also has an $(n-i-1)$-cycle for $i = 0, 1, 2$, respectively. Here we only prove that if G has an $(n-2)$-cycle, then G also has an $(n-3)$-cycle. The other two cases (for $i = 0, 1$) can be proved by a similar approach and we omit the proofs.

Now suppose that G has an $(n-2)$-cycle $C = v_0v_1v_2 \cdots v_{n-3}v_0$, but G has no $(n-3)$-cycle. Then C has no short chords (edges of the form v_iv_{i+2}) for $i \in \{0, 1, 2, \ldots, n-3\}$, where the subscripts are taken modulo $n-2$. For $0 \leq i < j \leq n-3$, we use $v_i\overrightarrow{C}v_j$ and $v_i\overleftarrow{C}v_j$ to denote respectively the paths $v_iv_{i+1}\cdots v_j$ and $v_iv_{i-1}\cdots v_j$, where again the subscripts are taken modulo $n-2$.

Let x and y be the vertices of G not on C. Since G has no $(n-3)$-cycle, we have that if $xv_l \in E(G)$, then $xv_{l+3} \notin E(G)$, where subscripts are expressed modulo $n-2$, and this in turn implies that if $d(x) = n-6$ or $d(y) = n-6$, then $n-2 = |C| \geq 2n-14$, but it is impossible since $n \geq 13$. So we may assume that C has a vertex of degree $n-6$, say, $d(v_0) = n-6$.

We state the following two observations before considering different cases. The proof of the Observation is straightforward.

Observation.

(1) *If* $v_i \in N(v_1) \cap N(v_2)$ *for* $v_i \in \{v_5, \ldots, v_{n-4}\}$*, then neither* v_{i-1} *nor* v_{i-2} *is a neighbour of* v_0.

(2) *If* $v_j \in N(v_{n-4}) \cap N(v_{n-3})$ *for* $v_j \in \{v_2, \ldots, v_{n-7}\}$*, then neither* v_{j+1} *nor* v_{j+2} *is a neighbour of* v_0.

Case 1. $\{x, y\} \cap N(v_1) \cap N(v_2) \neq \emptyset$ or $\{x, y\} \cap N(v_{n-4}) \cap N(v_{n-3}) \neq \emptyset$. Without loss of generality, we may assume that $x \in N(v_1) \cap N(v_2)$. Then, since G has no $(n-3)$-cycle, $v_{n-5} \notin N(v_0)$. Also $\{x, v_1, v_0, v_{n-7}, v_{n-6}, v_{n-5}\} \cap N(v_{n-4}) \cap N(v_{n-3}) = \emptyset$. Hence, by Lemma 10, $v_i \in N(v_{n-4}) \cap N(v_{n-3})$ for some $v_i \in \{v_2, v_3, \ldots, v_{n-8}\}$. Then, by Observation (2), $V(G) \setminus N(v_0) = \{v_0, v_2, v_{i+1}, v_{i+2}, v_{n-5}, v_{n-4}\}$. Thus, if $i \neq n-8$, then $v_{i+3} \in N(v_0)$, but then G contains an $(n-3)$-cycle $v_0v_{i+3}\overrightarrow{C}v_{n-3}v_i\overleftarrow{C}v_2xv_1v_0$. This proves that v_{n-8} is the only common neighbour of v_{n-4} and v_{n-3} on C, so by Lemma 10, $y \in N(v_{n-4}) \cap N(v_{n-3})$. But then again G has an $(n-3)$-cycle $v_{n-3}yv_{n-4}v_{n-8}\overleftarrow{C}v_2xv_1v_0v_{n-3}$.

Case 2. $\{x, y\} \cap N(v_1) \cap N(v_2) = \emptyset$ and $\{x, y\} \cap N(v_{n-4}) \cap N(v_{n-3}) = \emptyset$. It follows from Lemma 10 that $N(v_1) \cap N(v_2)$ has at least two vertices in $\{v_5, \ldots, v_{n-4}\}$, and $N(v_{n-4}) \cap N(v_{n-3})$ has at least two vertices in $\{v_2, \ldots, v_{n-7}\}$. It therefore follows from Observation (1) and (2) that v_0 has three consecutive non-neighbours v_{i-1}, v_i, v_{i+1} on C with $4 \leq i \leq n-6$, and $N(v_1) \cap N(v_2) = \{v_{i+1}, v_{i+2}\}$, while $N(v_{n-4}) \cap N(v_{n-3}) = \{v_{i-2}, v_{i-1}\}$. Thus, $V(G) \setminus N(v_0) = \{v_0, v_2, v_{i-1}, v_i, v_{i+1}, v_{n-4}\}$. Now, if $i \geq 5$, then $v_0v_{i-2} \in E(G)$, but then G has an $(n-3)$-cycle $v_0 v_{i-2} \overleftarrow{C} v_1 v_{i+1} \overrightarrow{C} v_{n-4} v_{i-1} v_{n-3} v_0$. If $i = 4$, then G also has an $(n-3)$-cycle $v_{n-4} v_3 v_2 v_{n-3} v_0 v_1 v_5 \overrightarrow{C} v_{n-4}$. This completes the proof of Theorem 8. ■

3. Proof of Theorem 9

***Proof of Theorem* 9.** Let $C = v_0 v_1 v_2 \cdots v_{t-1} v_0$ be a t-cycle in a graph G. As in the previous section, for $i < j$, we use $v_i \overrightarrow{C} v_j$ and $v_i \overleftarrow{C} v_j$ to denote the paths $v_i v_{i+1} \cdots v_j$ and $v_i v_{i-1} \cdots v_j$, respectively, where the subscripts are taken modulo t. A vertex on C is called an attachment vertex if it is adjacent to some vertex in $V(G) \setminus V(C)$. The following observation on non-extendable cycles in G is quite useful.

Lemma 12 [14]**.** *Let $C = v_0 v_1 \cdots v_{t-1} v_0$ be a non-extendable cycle in a graph G. Suppose v_i and $v_j (i < j)$ are two attachment vertices of C such that they are adjacent to a common off-cycle vertex. Then the following holds.*

(1) $j \neq i + 1$.

(2) *Neither $v_{i+1} v_{j+1}$ nor $v_{i-1} v_{j-1}$ is in $E(G)$.*

(3) *If $v_{i-1} v_{i+1} \in E(G)$, then neither $v_{j-1} v_i$ nor $v_{j+1} v_i$ is in $E(G)$.*

(4) *If $j = i + 2$, then v_{i+1} does not have two neighbours v_k, v_{k+1} on the path $v_{i+2} \cdots v_i$.*

The following lemma gives some basic properties of a Hamilton-connected graph, the proof is straightforward.

Lemma 13. *Let G be a Hamilton-connected graph of order $n \geq 4$. Then the following holds.*

(1) $\delta(G) \geq \kappa(G) \geq 3$.

(2) *$\omega(G - S) \leq |S| - 1$ for each $S \subseteq V$ with $|S| \geq 2$, where $\omega(G - S)$ is the number of components of $G - S$.*

(3) *$\alpha(G) < \frac{n}{2}$, where $\alpha(G)$ is the independence number of G.*

(4) *If n is odd and I is an independent set with $|I| = \frac{n-1}{2}$, then each vertex in $V \setminus I$ is adjacent to at least two vertices in I.*

If $\Delta(G) \leq 6$, then the result follows from Theorem 6. So suppose that $\Delta(G) = 7$.

Obviously, each vertex of G lies on a triangle, and so it suffices to show that every non-Hamiltonian cycle of G is extendable. Assume, to the contrary, that G contains a non-extendable cycle $C = v_0 v_1 \cdots v_{t-1} v_0$ where $t < n$. We may assume that v_0 has an off-cycle neighbour x. Then neither v_1 nor v_{t-1} is in $N(x)$. Next we show the following claim.

Claim. *Let $v_i \in V(C)$ with $1 < i < t-1$. If v_0 and v_i have a common off-cycle neighbour, then the following holds.*

(1) *If $i \neq 2$, then $v_iv_1 \notin E(G)$.*

(2) If $i \neq t-2$, then $v_iv_{t-1} \notin E(G)$.

Proof. Note that (1) and (2) are symmetric statements. Hence, we only prove (1). Suppose, to the contrary, that $i \neq 2$ and $v_iv_1 \in E(G)$. Since C is non-extendable, x is not adjacent to any of v_1, v_{t-1}, v_{i-1} and v_{i+1}. By Lemma 12(3), $v_{i+1}v_{i-1} \notin E(G)$. Thus $I_i = \{v_{i+1}, v_{i-1}, x\}$ is an independent set in $\langle N(v_i)\rangle$. By Lemma 13(3),(4), we know that $d(v_i)$ =7 and v_1 is adjacent to at least two vertices in I_i. Since $v_1x \notin E(G)$, we have $v_1v_{i+1} \in E(G)$ and $v_1v_{i-1} \in E(G)$. This is contrary to Lemma 12(2). The Claim is proved. □

Since $\langle N(v_0)\rangle$ is Hamilton-connected, it follows from Lemma 13(1) that x and v_1 each have at least 3 neighbours in $N(v_0)$. As $|N(v_0)| \leq 7$ and x and v_1 are not neighbours of each other, we infer that v_1 and x have at least one common neighbour in $N(v_0)$. But v_1 and x have no common off-cycle neighbour, so it follows from Claim (1) that $N(v_0) \cap N(v_1) \cap N(x) = \{v_2\}$. Thus $|N(v_0) \cap (N(v_1) \cup N(x))| \geq 5$ and by Lemma 12(2), $v_{t-1} \notin N(v_1)$. This implies that $|N(v_0) \cap (N(v_1) \cup N(x) \cup \{v_1, v_{t-1}, x\})| \geq 8$, contradicting that $|N(v_0)| \leq 7$. This completes the proof of Theorem 9. ■

4. Concluding Remark

In this paper, we prove that a connected, locally Hamiltonian graph G with $\Delta(G) \geq n(G)-6$ is weakly pancyclic, this is an improvement of the result obtained in [14]. Furthermore, we show that a connected, locally Hamilton-connected graph of order at least 3 and maximum degree at most 7 is fully cycle extendable. As we have mentioned, a lot of work has been done on the global cycle structures of connected, locally connected/traceable/Hamiltonian graphs, but we have not seen any research on the global cycle structures of connected, locally Hamilton-connected graphs. Since Hamilton-connectedness is a stronger graphical property, one could expect some stronger results on the global cycle structures of this kind of graphs.

Reference [10] by Pareek contains a theorem that states that if a connected, locally Hamiltonian graph has maximum degree 7, then the graph is Hamiltonian. Since locally Hamilton-connected graphs are locally Hamiltonian, that is a stronger result than Theorem 9 (although it does not establish that the graph is necessarily fully cycle extendable). However, in [7], it is shown that Pareek's proof is not valid. Therefore, Theorem 9 is a new result.

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