

BOUNDS ON THE LOCATING-TOTAL DOMINATION NUMBER IN TREES

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Abstract

Given a graph $G = (V, E)$ with no isolated vertex, a subset S of V is called a total dominating set of G if every vertex in V has a neighbor in S . A total dominating set S is called a locating-total dominating set if for each pair of distinct vertices u and v in $V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of a locating-total dominating set of G is the locating-total domination number, denoted by $\gamma_t^L(G)$. We show that, for a tree T of order $n \geq 3$ and diameter d , $\frac{d+1}{2} \leq \gamma_t^L(T) \leq n - \frac{d-1}{2}$, and if T has l leaves, s support vertices and s_1 strong support vertices, then $\gamma_t^L(T) \geq \max \left\{ \frac{n+l-s+1}{2} - \frac{s+s_1}{4}, \frac{2(n+1)+3(l-s)-s_1}{5} \right\}$. We also characterize the extremal trees achieving these bounds.

Keywords: tree, total dominating set, locating-total dominating set, locating-total domination number.

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1. INTRODUCTION

In [5, 8], the authors introduced the concept of a locating-total dominating set in a graph. Locating-total dominating set has been studied, for example, in [1, 2, 3, 4, 9] and elsewhere. The problem of placing monitoring devices in a system such that every site (including the monitors themselves) in the system is adjacent to a monitor can be modelled by total domination in graphs. Applications where it is also important that if there is a problem in a device, its location can be uniquely identified by the set of monitors, can be modelled by a combination of total dominating sets and locating sets in graphs. In this paper, we consider locating-total domination in trees.

For notation and graph theory terminology in general we follow [6, 7]. Let $G = (V, E)$ be a graph with n vertices. For a vertex v in G , the set $N(v) = \{u \in V : uv \in E\}$ is called the *open neighborhood* of v and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v . The *degree* of v in G , denoted by $d(v)$, is equal to $|N(v)|$. A vertex of degree one is a *leaf* and the edge incident with a leaf is a *pendent edge*. A vertex adjacent to a leaf is a *support vertex* and a support vertex adjacent to at least two leaves is a *strong support vertex*. We will use $L(G)$, $S(G)$ and $S_1(G)$ to denote the set of leaves, support vertices and strong support vertices of G , respectively. The *distance* between two vertices u and v , denoted by $d(u, v)$, is the number of edges in a shortest path joining u and v . The *diameter* of G , denoted by $\text{diam}(G)$, is the maximum distance over all pairs of vertices of G . For two disjoint subsets A and B of V , let $[A, B] = \{uv \in E(G) : u \in A, v \in B\}$. Suppose G and H are two disjoint graphs, then the *disjoint union* of G and H , denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we simply write kG_1 for $G_1 + \cdots + G_k$.

For a subset $S \subseteq V$, let $G[S]$ be the subgraph induced by S . The *open neighborhood* of S is $N(S) = \bigcup_{v \in S} N(v)$ and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. S is called a *total dominating set* (TDS) of G if $N(S) = V$. A TDS S is a *locating-total dominating set* (LTDS) if for each pair of distinct vertices u and v in $V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of an LTDS of G is the *locating-total domination number* of G , denoted by $\gamma_t^L(G)$. An LTDS of cardinality $\gamma_t^L(G)$ is called a $\gamma_t^L(G)$ -*set*.

Let P_n and S_n be a path of order n and a star of order n , respectively. A *double star* $S_{p,q}$ is a tree obtained from S_{p+2} and S_{q+1} by identifying a leaf of S_{p+2} with the center of S_{q+1} , where $p, q \geq 1$.

Locating-total domination in trees has been studied in [2, 4, 8]. In this paper, we continue the study of it. We show that, for a tree T of order $n \geq 3$ and diameter d , $\frac{d+1}{2} \leq \gamma_t^L(T) \leq n - \frac{d-1}{2}$, and if T has l leaves, s support vertices and s_1 strong support vertices, then $\gamma_t^L(T) \geq \max \left\{ \frac{n+l-s+1}{2} - \frac{s+s_1}{4}, \frac{2(n+1)+3(l-s)-s_1}{5} \right\}$. We also characterize the extremal trees achieving these bounds.

2. LOWER BOUNDS ON THE LOCATING-TOTAL DOMINATION NUMBER IN TREES

The locating-total domination number of P_n was given in [8].

Theorem 1 [8]. For $n \geq 2$, $\gamma_t^L(P_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

In [9], a lower bound of $\gamma_t^L(G)$ involving diameter was given.

Theorem 2 [9]. If G is a connected graph of order at least 2, then $\gamma_t^L(G) \geq \frac{\text{diam}(G)+1}{2}$.

If G is a tree, we characterize all trees which achieve the lower bound.

Corollary 3. Suppose T is a tree of order at least 2, then $\gamma_t^L(T) \geq \frac{\text{diam}(T)+1}{2}$ and the equality holds if and only if $T = P_n$, where $n \equiv 0 \pmod{4}$.

Proof. Let $d = \text{diam}(T)$. From Theorem 2, $\gamma_t^L(T) \geq \frac{d+1}{2}$. If $T = P_n$, where $n \equiv 0 \pmod{4}$, then by Theorem 1, we have $\gamma_t^L(P_n) = \frac{n}{2} = \frac{d+1}{2}$.

Now assume T is a tree of order $n \geq 2$ and $\gamma_t^L(T) = \frac{d+1}{2}$. From the proof of Theorem 2, we have $d+1 \equiv 0 \pmod{4}$.

If $d = 3$, then $T = S_{a,b}$ for some $a, b \geq 1$. Since $\gamma_t^L(S_{a,b}) = n - 2$ and $\gamma_t^L(T) = \frac{d+1}{2} = 2$, we have $n = 4$ and $T = P_4$. Thus, we may assume $d \geq 7$.

Let D be a $\gamma_t^L(T)$ -set of T that contains a minimum number of leaves. Then for every support vertex v , exactly one leaf adjacent to v is not in D . Suppose $x, y \in V(T)$ with $d(x, y) = d$ and $P = v_0 v_1 \cdots v_d$ is the unique path joining x and y , where $v_0 = x$ and $v_d = y$. Then $d(x) = d(y) = 1$. For $i = 1, 2, \dots, \frac{d+1}{4}$, let T_i be the component of $T \setminus \bigcup_{i=1}^{(d-3)/4} \{v_{4i-1}, v_{4i}\}$ containing the vertex v_{4i-1} and let $V(T_i) = D_i$. Then $|D \cap D_i| \geq 2$ because $\{v_{4i-3}, v_{4i-2}\} \subseteq N(D)$. Thus, $|D| \geq \frac{2(d+1)}{4} = \frac{d+1}{2}$. Since $|D| = \gamma_t^L(T) = \frac{d+1}{2}$, we obtain $|D_i \cap D| = 2$ for $i = 1, 2, \dots, \frac{d+1}{4}$. Obviously, we have $v_1, v_{d-1} \in D$.

Fact 1. $d(v_1) = 2$.

Proof of Fact 1. Suppose $d(v_1) \geq 3$, then v_1 is a strong support vertex which is adjacent to exactly two leaves because $|D_1 \cap D| = 2$. Let z be the other leaf adjacent to v_1 . Thus we may assume $D \cap D_1 = \{z, v_1\}$. Now, for $v_0, v_2 \notin D$, we have $N(v_0) \cap D = N(v_2) \cap D = \{v_1\}$, a contradiction. \square

Fact 2. $D = \bigcup_{i=1}^{(d+1)/4} \{v_{4i-3}, v_{4i-2}\}$.

Proof of Fact 2. By Fact 1, we have $D \cap D_1 = \{v_1, v_2\}$ in order to totally dominate v_1 .

Suppose $v_4 \in D$. Then $D \cap D_2 = \{v_4, v_5\}$ in order to totally dominate v_4 and v_6 . Consequently, we have $D = \{v_1, v_2\} \cup (\bigcup_{i=2}^{(d+1)/4} \{v_{4i-4}, v_{4i-3}\})$, which induces $v_d \notin N(D)$, a contradiction. Thus, $v_4 \notin D$.

Suppose $v_5 \notin D$. In order to totally dominate v_4 , there must be two vertices $z_1, z_2 \in (V(T) \setminus V(P)) \cap D$ with $z_1 \in N(v_4)$ and $z_2 \in N(z_1)$. Since $|D_2 \cap D| = 2$, we have $v_6 \notin N(D)$, a contradiction. Thus, we have $v_5 \in D$.

Suppose $v_6 \notin D$. In order to totally dominate v_5 , there must be a vertex $z \in N(v_5) \cap D \setminus V(P)$. Then $D \cap D_2 = \{v_5, z\}$ and $N(v_4) \cap D = N(v_6) \cap D = \{v_5\}$, a contradiction. Thus, $v_6 \in D$ and $D_2 \cap D = \{v_5, v_6\}$.

By induction on i , we have $D \cap D_i = \{v_{4i-3}, v_{4i-2}\}$ for $i = 2, 3, \dots, \frac{d+1}{4}$. Thus, $D = \bigcup_{i=1}^{(d+1)/4} \{v_{4i-3}, v_{4i-2}\}$. \square

Fact 3. $V(T) = V(P)$.

Proof of Fact 3. Suppose $V(T) \setminus V(P) \neq \emptyset$. Since $D_i \cap D = \{v_{4i-3}, v_{4i-2}\}$ for $i = 1, 2, \dots, \frac{d+1}{4}$, there are no vertices in $V(T) \setminus V(P)$ adjacent to v_{4i-1} or v_{4i} for $i = 1, 2, \dots, \frac{d+1}{4}$.

Suppose there is $z \in V(T) \setminus V(P)$ with $zv \in E(T)$, where $v \in D_i \cap D$. Without loss of generality, we may assume that $z \in N(v_{4i-3})$ for some $i \in \{1, 2, \dots, \frac{d+1}{4}\}$. Then $N(v_{4i-4}) \cap D = N(z) \cap D = \{v_{4i-3}\}$, a contradiction. \square

Thus, $T = P = P_n$, where $n = d + 1 \equiv 0 \pmod{4}$. \blacksquare

Let \mathcal{F} be the family of trees obtained from t disjoint copies of P_4 and P_3 by first adding $t - 1$ edges in such a way that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once. Let ξ be the family of trees T that can be obtained from any tree T' by first attaching at least two leaves to each vertex of T' , and then subdividing each edge of T' exactly once if T' is nontrivial.

Theorem 4 [2]. *If T is a tree of order $n \geq 3$, $|L(T)| = l$ and $|S(T)| = s$, then*

$$\gamma_t^L(T) \geq \frac{2(n + l - s + 1)}{5},$$

with equality if and only if $T \in \mathcal{F}$.

Theorem 5 [4]. *If T is a tree of order $n \geq 3$ with l leaves and s support vertices, then $\gamma_t^L(T) \geq \frac{n+l+1}{2} - s$ and the equality holds if and only if $T \in \xi$.*

In the following, we give two new lower bounds on the locating-total domination number in trees. We also characterize the trees achieving those lower bounds. First, we need the following lemma. Let $T = (V, E)$ be a tree of order $n \geq 3$. Let $L(T) = L$, $S(T) = S$, $S_1(T) = S_1$, $S \setminus S_1 = S_2$ and A be a $\gamma_t^L(T)$ -set of

T that contains a minimum number of leaves. Then $S \subseteq A$ and for every $v \in S$, exactly one leaf adjacent to v is not in A . Let $B = \{v \notin A : |N(v) \cap A| = 1\}$ and $C = \{v \notin A : |N(v) \cap A| \geq 2\}$. Let $L_1 = L \cap A$, $Q_1 = A \setminus (L_1 \cup S)$, $L_2 = L \setminus L_1$ and $Q_2 = B \setminus L_2$. Then $A = L_1 \cup S \cup Q_1$, $B = L_2 \cup Q_2$, $V = A \cup B \cup C$. We have the following lemma.

Lemma 6. *Let $|L| = l$, $|S| = s$ and $|S_1| = s_1$. Then*

- (1) $|[A, B \cup C]| \geq |B| + 2|C| = 2n - 2|A| - |B|$;
- (2) $|[A, B \cup C]| = n - 1 - |E(T[A])| - |E(T[Q_2 \cup C])|$;
- (3) $|L_1| = l - s$, $|L_2| = s$, $|Q_1| = |A| - l$, $|Q_2| = |B| - s$;
- (4) $|Q_2| \leq |Q_1|$, $|B| \leq |A| - l + s$;
- (5) $|E(T[Q_2 \cup C])| \geq \frac{|Q_2|}{2}$ and the equality holds if and only if $T[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ and C is an independent set in $T[Q_2 \cup C]$;
- (6) $|E(T[S \cup Q_1])| \geq \frac{1}{2}(s - s_1 + |A| - l)$ and the equality holds if and only if $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S \cup Q_1]$;
- (7) $|E(T[A])| \geq \frac{|A|}{2}$ and the equality holds if and only if $T[A] \cong \frac{|A|}{2}K_2$.

Proof. (1)–(5) and (7) can be obtained by applying an argument similar to that of Lemma 3 we gave in [11] and can also be seen in [10].

(6) For every $v \in S_2 \cup Q_1$, $N(v) \cap (S \cup Q_1) \neq \emptyset$ by the definition of an LTDS. Thus,

$$|E(T[S \cup Q_1])| \geq \frac{1}{2} \sum_{v \in S_2 \cup Q_1} d_{T[S \cup Q_1]}(v) \geq \frac{1}{2}|S_2 \cup Q_1| = \frac{1}{2}(s - s_1 + |A| - l),$$

and the equality holds if and only if $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S \cup Q_1]$. \blacksquare

Let \mathcal{T}_1 denote the set $\{P_4\} \cup \{S_a : a \geq 3\}$. Let \mathcal{F}_1 be the family of trees obtained from r disjoint copies of trees in \mathcal{T}_1 by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 7. *Suppose T is a tree of order $n \geq 3$, $|L(T)| = l$, $|S(T)| = s$ and $|S_1(T)| = s_1$. Then*

$$\gamma_t^L(T) \geq \frac{2(n+1) + 3(l-s) - s_1}{5},$$

with equality if and only if $T \in \mathcal{F}_1$.

Proof. From Lemma 6(1) and (4), we obtain $|[A, B \cup C]| \geq 2n - 3|A| + l - s$. By Lemma 6(2), (3) and (6), $|[A, B \cup C]| \leq n - 1 - |E(T[A])| = n - 1 - |L_1| - |E(T[S \cup Q_1])| \leq n - 1 - (l - s) - \frac{1}{2}(s - s_1 + |A| - l)$. Thus $\gamma_t^L(T) = |A| \geq \frac{2(n+1)+3(l-s)-s_1}{5}$.

The equality $\gamma_t^L(T) = \frac{2(n+1)+3(l-s)-s_1}{5}$ holds if and only if $|E(T[Q_2 \cup C])| = 0$, $|N(v) \cap A| = 2$ for every vertex $v \in C$, $|Q_1| = |Q_2|$, $T[S \cup Q_1] \cong s_1 K_1 + \frac{|S_2 \cup Q_1|}{2} K_2$ and S_1 is an independent set in $T[S \cup Q_1]$. The equality $|E(T[Q_2 \cup C])| = 0$ implies $|Q_1| = |Q_2| = 0$ by Lemma 6(5). Thus, $A = L_1 \cup S$ and $T[S] \cong s_1 K_1 + \frac{s-s_1}{2} K_2$. Consequently, every connected component of $T[A \cup B]$ is either a P_4 , or a S_a , where $a \geq 3$. Thus, we have $T \in \mathcal{F}_1$. \blacksquare

Remark 8. The lower bound in Theorem 7 is no less than the lower bound in Theorem 4 because $\frac{2(n+1)+3(l-s)-s_1}{5} - \frac{2(n+l-s+1)}{5} = \frac{l-s-s_1}{5} \geq 0$. Note that we have the fact $\mathcal{F} \subset \mathcal{F}_1$.

Now let \mathcal{T}_2 denote the set $\{S_a : a \geq 3\} \cup \{P_b : b \geq 4 \text{ and } b \equiv 0 \pmod{4}\}$. For every $T \in \mathcal{T}_2$, if $T = P_b = v_1 v_2 \cdots v_b$ for some $b \geq 4$ and $b \equiv 0 \pmod{4}$, then we define $D_T = \bigcup_{i=1}^{b/4} \{v_{4i-2}, v_{4i-1}\}$; if $T = S_a$ for some $a \geq 3$, then we define $D_T = S(S_a)$. Let \mathcal{F}_2 be the family of trees obtained from r disjoint copies of trees in \mathcal{T}_2 by first adding $r - 1$ edges so that they are only incident with vertices in $\bigcup_{T \in \mathcal{T}_2} D_T$ and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 9. Suppose T is a tree of order $n \geq 3$, $|L(T)| = l$, $|S(T)| = s$ and $|S_1(T)| = s_1$. Then

$$\gamma_t^L(T) \geq \frac{n + l - s + 1}{2} - \frac{s + s_1}{4},$$

with equality if and only if $T \in \mathcal{F}_2$.

Proof. From Lemma 6(1), we obtain $|[A, B \cup C]| \geq 2n - 2|A| - |B|$. On the other hand,

$$\begin{aligned} |[A, B \cup C]| &= n - 1 - |E(T[A])| - |E(T[Q_2 \cup C])| \quad \text{by Lemma 6(2)} \\ &\leq n - 1 - |E(T[A])| - \frac{|Q_2|}{2} \quad \text{by Lemma 6(5)} \\ &= n - 1 - \frac{|B| - s}{2} - (|L_1| + |E(T[S \cup Q_1])|) \quad \text{by Lemma 6(3)} \\ &\leq n - 1 - \frac{|B| - s}{2} - ((l - s) + \frac{1}{2}(s - s_1 + |A| - l)) \quad \text{by Lemma 6(6)}. \end{aligned}$$

Combining this with $|[A, B \cup C]| \geq 2n - 2|A| - |B|$, we have

$$\frac{3}{2}|A| \geq n + 1 - \frac{|B|}{2} - s + \frac{l}{2} - \frac{s_1}{2}.$$

By Lemma 6(4), we have $2|A| \geq n + 1 + l - s - \frac{s+s_1}{2}$, which implies $\gamma_t^L(T) = |A| \geq \frac{n+l-s+1}{2} - \frac{s+s_1}{4}$.

The equality holds if and only if $|Q_1| = |Q_2|$, $T[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ and C is an independent set in $T[Q_2 \cup C]$, $T[S \cup Q_1] \cong s_1K_1 + \frac{|S_2 \cup Q_1|}{2}K_2$ and S_1 is an independent set in $T[S \cup Q_1]$, and $|N(u_2) \cap A| = 2$ for every $u_2 \in C$. For every $u_1 \in Q_2 \subseteq B$, $N(u_1) \cap A \subseteq Q_1$ by the definition of an LTDS and $|N(u_1) \cap Q_1| = 1$.

If $|Q_2| = 0$, then $T \in \mathcal{F}_1$ (by the same argument as that in the proof of Theorem 7) and therefore $T \in \mathcal{F}_2$ as $\mathcal{F}_1 \subset \mathcal{F}_2$.

Now we consider the case $|Q_1| = |Q_2| \neq 0$. Let $T_1, T_2, \dots, T_{\omega_1}$ be the components of $T[Q_1 \cup Q_2 \cup S_2]$. Note that T is a tree. Then for $i = 1, 2, \dots, \omega_1$, T_i is a path of order a_i with two leaves in S_2 and the other vertices in $Q_1 \cup Q_2$, where $a_i \equiv 2 \pmod{4}$. Thus, every component of $T[A \cup B]$ is in \mathcal{T}_2 . Suppose $X_1, X_2, \dots, X_{\omega_2}$ are the components of $T[A \cup B]$. For every X_j , if $X_j = P_{b_j} = v_1 v_2 \cdots v_{b_j}$ for some $b_j \geq 4$ and $b_j \equiv 0 \pmod{4}$, then we define $D_{X_j} = \bigcup_{i=1}^{b_j/4} \{v_{4i-2}, v_{4i-1}\}$, but if $X_j = S_{a_j}$ for some $a_j \geq 3$, then we define $D_{X_j} = S(X_j)$. Thus we have $S \cup Q_1 = \bigcup_{j=1}^{\omega_2} D_{X_j}$. Note that for every vertex $u \in C$, $|N(u) \cap A| = |N(u) \cap (S \cup Q_1)| = 2$ and T is a tree. Thus, $T \in \mathcal{F}_2$. ■

Remark 10. The lower bound in Theorem 9 is not less than the lower bound in Theorem 5 because $\frac{n+l-s+1}{2} - \frac{s+s_1}{4} - (\frac{n+l+1}{2} - s) = \frac{s-s_1}{4} \geq 0$. We also have $\xi \subset \mathcal{F}_2$, where ξ is defined in Theorem 5. On the other hand, if $n > \frac{3s+2l-s_1-2}{2}$, the lower bound in Theorem 9 is better than the lower bound in Theorem 7.

3. UPPER BOUNDS ON THE LOCATING-TOTAL DOMINATION NUMBER IN TREES

The next theorem gives an upper bound on $\gamma_t^L(T)$ of a tree of fixed order and diameter.

Theorem 11. *Suppose T is a tree of order $n \geq 3$ and diameter $d \geq 2$. Then $\gamma_t^L(T) \leq n - \frac{d-1}{2}$ and the equality holds if and only if $T = P_n$, where $n \equiv 2 \pmod{4}$.*

Proof. We first use an induction on the order n of T to show that $\gamma_t^L(T) \leq n - \frac{d-1}{2}$. If $n = 3$, then $\gamma_t^L(T) = 2 < n - \frac{d-1}{2}$. Next we assume that every tree T' of order $3 \leq n' < n$ and diameter $d' \geq 2$ satisfies $\gamma_t^L(T') \leq n' - \frac{d'-1}{2}$. Let T be a tree of order $n > 3$ and diameter $d \geq 2$.

Let $P = v_0 v_1 v_2 \cdots v_d$ be a path of length d in T . If $T = P$, then $d = n - 1$ and $\gamma_t^L(T) \leq n - \frac{d-1}{2}$ by Theorem 1. Now suppose $T \neq P$. Then there is a vertex v of P with $d(v) \geq 3$. Let u be a vertex of $T \setminus V(P)$ such that $d(u, v)$ is maximum. Then $u \in L(T)$. Let $N(u) = \{w\}$, $T' = T - u$ and D be a $\gamma_t^L(T')$ -set of T' . Then

$n' = n - 1$ and $d' = d$. By the inductive hypothesis, $\gamma_t^L(T') \leq n' - \frac{d'-1}{2}$. If $w \neq v$, then $w \in L(T')$ and $D \cup \{w\}$ is an LTDS of T ; if $w = v$ and $v \in D$, then $D \cup \{u\}$ is an LTDS of T ; if $w = v$ and $v \notin D$, then $D \cup \{v\}$ is an LTDS of T . In each case, we can find an LTDS of T with no more than $\gamma_t^L(T') + 1$ elements. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 1 \leq n' - \frac{d'-1}{2} + 1 = n - \frac{d-1}{2}$. This completes the proof of $\gamma_t^L(T) \leq n - \frac{d-1}{2}$.

By Theorem 1, if $T = P_n$, where $n \geq 4$ and $n \equiv 2 \pmod{4}$, then $\gamma_t^L(T) = \frac{n+2}{2} = n - \frac{d-1}{2}$. Conversely, suppose T is a tree with $\gamma_t^L(T) = n - \frac{d-1}{2}$, then $d \geq 4$ and d is odd. In order to prove $T = P_n$, where $n \geq 4$ and $n \equiv 2 \pmod{4}$, we proceed by induction on n . If $n \leq 6$, then $T = P_6$. Assume every tree T' of order $6 \leq n' < n$ and diameter $d' \geq 2$ with $\gamma_t^L(T') = n' - \frac{d'-1}{2}$ satisfies $T' = P_{n'}$, where $n' \geq 4$ and $n' \equiv 2 \pmod{4}$.

If T has a strong support vertex v , let $T' = T - y$, where y is a leaf adjacent to v . Then $n' = n - 1$, $d' = d$, $\gamma_t^L(T) \leq \gamma_t^L(T') + 1 \leq n' - \frac{d'-1}{2} + 1 = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $\gamma_t^L(T') = n' - \frac{d'-1}{2}$. By induction, $T' = P_{n'}$, where $n' \geq 4$ and $n' \equiv 2 \pmod{4}$. Suppose $T' = P_{n'} = v_1 v_2 \cdots v_{n'}$, where $v_2 = v$. Then $\{v_1, v_2\} \cup (\bigcup_{i=1}^{\lfloor \frac{n'}{4} \rfloor} \{v_{4i}, v_{4i+1}\})$ is an LTDS of T . Thus, $\gamma_t^L(T) \leq 2 + 2 \cdot \lfloor \frac{n'}{4} \rfloor = n' - \frac{d'-1}{2} < n - \frac{d-1}{2}$, a contradiction. Therefore, every support vertex in T is not strong.

Let $P = v_0 v_1 v_2 \cdots v_d$ be a path of length d in T . We root T at the vertex v_0 . Then we have the following two facts.

Fact 1. $d(v_2) = 2$.

Proof of Fact 1. Suppose $d(v_2) \geq 3$. If v_2 has a child $b \neq v_1$ which is a support vertex, let $T' = T \setminus \{v_0, v_1\}$. Then $n' = n - 2$ and $d' = d$. Let D' be a $\gamma_t^L(T')$ -set of T' that contains a minimum number of leaves. Then $v_2, b \in D'$ and $D' \cup \{v_1\}$ is an LTDS of T . Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 1 \leq n' - \frac{d'-1}{2} + 1 = n - 1 - \frac{d-1}{2} < n - \frac{d-1}{2}$, a contradiction. Therefore, every child of v_2 except v_1 is a leaf. Since T has no strong support vertices, $d(v_2) = 3$. Let c be a leaf adjacent to v_2 and $T' = T \setminus \{v_0, v_1, v_2, c\}$, then $n' = n - 4 \geq 3$ and $d - 3 \leq d' \leq d$. Let D' be a $\gamma_t^L(T')$ -set of T' , then $D' \cup \{v_1, v_2\}$ is an LTDS of T . Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq n' - \frac{d'-1}{2} + 2 \leq n - \frac{d-4}{2} + 2 = n - \frac{d}{2} < n - \frac{d-1}{2}$, a contradiction. \square

Fact 2. $d(v_3) = 2$.

Proof of Fact 2. Suppose $d(v_3) \geq 3$. Let $T' = T \setminus \{v_0, v_1, v_2\}$, then $n' = n - 3$ and $d - 2 \leq d' \leq d$. Let D' be a $\gamma_t^L(T')$ -set of T' , then $D' \cup \{v_1, v_2\}$ is an LTDS of T . Therefore, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq n' - \frac{d'-1}{2} + 2 \leq n - 1 - \frac{d-3}{2} = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $n' = n - 3$, $d' = d - 2$, $\gamma_t^L(T') = n - \frac{d'-1}{2}$ and v_3 is a support vertex in T . By induction, $T' = P_{n'}$, where $n' \geq 4$ and $n' \equiv 2 \pmod{4}$.

(mod 4). Now the set $\{v_1, v_2, v_3\} \cup_{i=1}^{(d-3)/4} \{v_{4i+1}, v_{4i+2}\}$ is a $\gamma_t^L(T)$ -set of T . Thus, $\gamma_t^L(T) = \gamma_t^L(T') + 1 = \frac{n+1}{2} < \frac{n+3}{2} = n - \frac{d-1}{2}$, a contradiction. \square

Now let $T' = T \setminus \{v_0, v_1, v_2, v_3\}$. Then $n' = n - 4 \geq 3$ and $d - 4 \leq d' \leq d$. Let D' be a $\gamma_t^L(T')$ -set of T' , then $D' \cup \{v_1, v_2\}$ is an LTDS of T . Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq n' - \frac{d'-1}{2} + 2 \leq n - 2 - \frac{d-5}{2} = n - \frac{d-1}{2}$. Since $\gamma_t^L(T) = n - \frac{d-1}{2}$, we have $n' = n - 4$, $d' = d - 4$, $\gamma_t^L(T') = n' - \frac{d'-1}{2}$ and $d_T(v_4) = 2$. By induction, $T' = P_{n'} = P_{n-4}$, where $n' \geq 4$ and $n' \equiv 2 \pmod{4}$. Therefore, $T = P_n$, where $n \equiv 2 \pmod{4}$. \blacksquare

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