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# PACKING TREES IN COMPLETE BIPARTITE GRAPHS

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**Abstract**

An embedding of a graph $H$ in a graph $G$ is an injection (i.e., a one-to-one function) $\sigma$ from the vertices of $H$ to the vertices of $G$ such that $\sigma(x)\sigma(y)$ is an edge of $G$ for all edges $xy$ of $H$. The image of $H$ in $G$ under $\sigma$ is denoted by $\sigma(H)$. A $k$-packing of a graph $H$ in a graph $G$ is a sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of embeddings of $H$ in $G$ such that $\sigma_1(H), \sigma_2(H), \ldots, \sigma_k(H)$ are edge disjoint. We prove that for any tree $T$ of order $n$, there is a 4-packing of $T$ in a complete bipartite graph of order at most $n + 12$.

**Keywords:** packing, placement, edge-disjoint tree, bipartite graph.

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## 1. Introduction

Graphs in this article are simple and finite, which have no multiple edges nor loops. We use [4] for standard terminology and notation in graph theory. For any graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. If $G$ is a bipartite graph with a given bipartition, then denote it by $G = (X, Y; E)$. A tree is a connected graph without cycles. A star of order $n$ is a tree of order $n$ with a vertex of degree $n - 1$. We use $B_n$ to represent a complete bipartite graph $K_{a,b}$ with $n = a + b$. Thus $B_n$ is not uniquely defined for $n \geq 4$ because there is more than one positive integer partiton for $n = a + b$.

An *embedding* of a graph $H$ in a graph $G$ is an injection (i.e., a one-to-one function) $\sigma$ from the vertices of $H$ to the vertices of $G$ such that $\sigma(x)\sigma(y)$ is an edge of $G$ for all edges $xy$ of $H$. The image of $H$ in $G$ under $\sigma$ is denoted by $\sigma(H)$. A *k-packing* of a graph $H$ in a graph $G$ is a sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of $k$ embeddings of $H$ in $G$ such that $\sigma_1(H), \sigma_2(H), \ldots, \sigma_k(H)$ are edge disjoint.

There are many research results and theorems about packings of graphs. For surveys, see [9] and [10]. In the following, we list several results and conjectures about packing of trees in complete graphs and complete bipartite graphs, which are the motivation of our current work. The following Theorem 1 was proved by Burns and Schuster [1] in 1978.

**Theorem 1** [1]**.** *If $T$ is a tree of order $n$ that is not a star, then there is a 2-packing of $T$ in $K_n$.*

For a tree of order $n \geq 2$ to have a 3-packing in $K_n$, $K_n$ must have at least $3(n-1)$ edges as a tree of order $n$ has $n-1$ edges. Thus $n(n-1)/2 \geq 3(n-1)$ and so $n \geq 6$. In 1993, Wang and Sauer [6] characterized all the trees of order $n$ that have 3-packings in $K_n$.

**Theorem 2** [6]**.** *If $T$ is a tree of order $n \geq 6$, then with three exceptions, there is a 3-packing of $T$ in $K_n$.*

For a tree of order $n \geq 2$ to have a 4-packing in $K_n$, there must be $n \geq 8$ and its maximum degree must be at most $n-4$. In 2014, Haler and Wang [3] further characterized all the trees of order $n$ that have 4-packings in $K_n$.

**Theorem 3** [3]**.** *If $T$ is a tree of order $n \geq 8$ with maximum degree at most $n-4$, then with four exceptions, there is a 4-packing of $T$ in $K_n$.*

We refer readers to [6] and [3] for the exceptions in Theorem 2 and Theorem 3. The following conjecture proposed by Gyárfás and Lehel [2] is well-known as *Tree Packing Conjecture* in 1978. So far the results and theorems that are obtained by many mathematicians on this conjecture are still far from a complete solution to the conjecture.

**Tree Packing Conjecture.** *For any sequence $T_1, T_2, \ldots, T_n$ of trees such that $T_i$ is a tree of order $i$ for each $1 \leq i \leq n$, there is a packing of $T_1, T_2, \ldots, T_n$ in $K_n$.*

Since $T_1, T_2, \ldots, T_n$ together have $n(n-1)/2$ edges, the above conjecture implies that $K_n$ can be decomposed into edge disjoint trees $T_1, T_2, \ldots, T_n$. Wang [7] proved the following Theorem 4 in 1996.

**Theorem 4** [7]**.** *If $T$ is a tree of order $n$, then there is a 2-packing of $T$ in a complete bipartite graph $B_{n+1}$.*

In 2003, Orchel [5] further characterized all the trees of order $n$ that have 2-packings in a complete bipartite graph $B_n$. In 2009, Wang [8] proposed the following conjecture.

**Conjecture** 5 [8]**.** *If $T$ is a tree of order $n$, then for each positive integer $k$, there is a $k$-packing of $T$ in a complete bipartite graph $B_{n+k-1}$.*

In the above conjecture, if $k = n$, then $n$ copies of $T$ together contain exactly $n(n-1)$ edges, and this is the number of edges of $K_{n,n-1}$. Thus, Conjecture 5 implies that the complete bipartite graph $K_{n,n-1}$ can be decomposed into $n$ edge disjoint copies of $T$ for any given tree $T$ of order $n$. So far, the results about packing trees into complete bipartite graphs are also far from a complete solution to this conjecture.

To support this conjecture, Wang [8] proved the following theorem.

**Theorem 6** [8]**.** *If $T$ is a tree of order $n$, then there is a 3-packing of $T$ in a complete bipartite graph $B_{n+2}$.*

Thus, Conjecture 5 is true for $k = 2, 3$ by Theorem 4 and Theorem 5. Although this is a hard conjecture, our goal is to make some progress on it in the case $k = 4$. In this case, the conjecture says that there is a 4-packing of any given tree of order $n$ in a complete bipartite graph $B_{n+3}$. We have not found any counterexample to the conjecture in the case $k = 4$. Based on our experience working with $k = 4$, we find that we can prove that there is a 4-packing of any given tree of order $n$ in a complete bipartite graph $B_{n+12}$. As for how to reduce $B_{n+12}$ further to a $B_{n+l}$ for some $3 \le l \le 11$, it remains open for future. In this article, we will prove the following theorem.

**Main Theorem.** *If $T$ is a tree of order $n$, then there is a 4-packing of $T$ in a complete bipartite graph $B_{n+12}$.*

In the process of proving the Main Theorem, we often construct a 4-packing $(d, b, g, r)$ of a subgraph $H$ of $T$. We then modify this 4-packing $(d, b, g, r)$ and extend it to a 4-packing of $T$ in a $B_{n+12}$. While doing so, we need keep the order of the bipartition of $H$ to be agreeable with that of $T$ because $H$ may have more than one bipartitions. Therefore, for the sake of convenience, we regard a bipartite graph as a bipartite graph with a *given ordered bipartition*. Thus, if $(X, Y)$ is the given bipartition of $G$, then for any subgraph $H$ of $G$, it is already decided that $(X \cap V(H), Y \cap V(H))$ is the given bipartition of $H$.

Since we are dealing with embeddings of bipartite graphs in $B_n$, we need to keep the order of the two partites under embeddings, too. Therefore, we adopt this convention through out this article from now on. *For an embedding $\sigma$ of a bipartite graph $G$ (or a tree $T$) in $B_n$, we mean that $\sigma$ is an injection from $V(G)$ into $V(B_n)$ such that $\sigma(U) \subseteq X$ and $\sigma(W) \subseteq Y$ where $(U, W)$ and $(X, Y)$ are the given bipartitions of $G$ and $B_n$, respectively.* By doing so, we say that two vertices of $G$ have the *same parity* if they belong to the same partite of $G$ and otherwise we say that they have the *opposite* parity.

For a 4-packing of $G$ in $B_n$, we often use $(d, b, g, r)$ to represent the four embeddings and we say that $d(G)$ is a dark copy of $G$, $b(G)$ is a blue copy of $G$, $g(G)$ is a green copy of $G$, and $r(G)$ is a red copy of $G$. For a 4-packing $(d, b, g, r)$

of $G$ in $B_n$, we say that a vertex $x$ is 4-placed if $d(x)$, $b(x)$, $g(x)$ and $r(x)$ are distinct.

A *linear graph* is a graph such that each of its components is a path.

Given a bipartite graph $G$, we say that two or more vertices of $G$ are *strongly independent* if no two of them are adjacent and no two of them have any common neighbor. A *leaf* of $G$ is a vertex of $G$ that has degree 1. A *node* of $G$ is a vertex of $G$ that is adjacent to a leaf of $G$. A *supernode* of $G$ is a vertex $x$ of $G$ such that, with possibly one exception, every neighbor of $x$ is a leaf of $G$. If $G$ is a tree but not a star, we readily see that $G$ has at least two distinct supernodes by observing a longest path of $G$, that is, the second vertex and the second to the last vertex on the longest path are two supernodes.

For a tree $T$, the *trunk* of $T$, denoted by $T^*$, is the subtree of $T$ obtained from $T$ by removing all the leaves of $T$. By this definition, we see that a vertex of $T^*$ is a leaf of $T^*$ if and only if it is a supernode of $T$.

## 2. Proof of the Main Theorem

We use induction on the order of trees to prove the Main Theorem. The theorem is obviously true for a tree of order 1. Assume that the Main Theorem holds for trees of order less than $n$ with $n \geq 2$. Let $T = (V, E)$ be a tree of order $n$. The idea of the proof is as follows. If $T$ has four strongly independent leaves $u$, $x$, $y$, and $z$ of the same parity, we apply induction on $T - \{u, x, y, z\}$ to get a 4-packing of $T - \{u, x, y, z\}$ in a $B_{n-4+12} = B_{n+8}$. This 4-packing can be easily extended to a 4-packing of $T$ in $B_{n+12}$ by the following Lemma 1. If $T$ does not have four strongly independent leaves of the same parity, then $T$ has at most six supernodes and so $T^*$ has at most six leaves. This will allow us to classify the structure of $T$ into several cases. We then prove that the Main Theorem holds for each of these cases.

Clearly, $K_{4,n-1}$ has four edge disjoint copies of $K_{1,n-1}$. Thus, if $T$ is a star, there is a 4-packing of $T$ in a $B_{n+3}$, and the Main Theorem holds for $T$. Therefore, we may assume in the following that $T$ is not a star.

Let $n^*$ be the order of $T^*$. Let $\xi_i$ be the number of vertices of $T^*$ with degree $i$ in $T^*$. Then $\xi_1$ is the number of supernodes of $T$ with

$$\xi_1 \geq 2 \text{ and } n^* = \xi_1 + \xi_2 + \cdots + \xi_{n^*-1}. \tag{1}$$

Since the degree sum of a graph is equal to two times the number of its edges, we have

$$\xi_1 + 2\xi_2 + \cdots + (n^* - 1)\xi_{n^*-1} = 2(n^* - 1). \tag{2}$$

Combining (1) and (2), we obtain the number $\xi_1$ of supernodes of $T$ as follows

$$\xi_1 = 2 + \xi_3 + 2\xi_4 + 3\xi_5 + \cdots + (n^* - 3)\xi_{n^*-1}. \tag{3}$$

Let $\lambda(T)$ be the largest number of $k$ such that $T$ has $k$ nodes of the same parity. We now divide the proof of the Main Theorem into two parts.

**Part I.** $\lambda(T) \geq 4$

In this case, we need the following Lemma 1 which was proven in [3].

**Lemma 1** (Lemma 2.4 of [3])**.** *Let $G$ be a complete bipartite graph $K_{4,m}$ where $m \geq 4$. Let $A$ and $B$ be the two partite sets of $G$ with $|A| = 4$ and $|B| = m$. Let $B_1$, $B_2$, $B_3$, and $B_4$ be any given four subsets of $B$ with $|B_i| = 4$ for all $i \in \{1, 2, 3, 4\}$. Then $G$ has four disjoint matchings $M_1$, $M_2$, $M_3$, and $M_4$ such that $M_i$ matches $B_i$ into $A$ for all $i \in \{1, 2, 3, 4\}$.*

We apply Lemma 1 as follows. Let $y_1$, $y_2$, $y_3$, and $y_4$ be four distinct nodes of $T$ with the same parity. Let $x_1$, $x_2$, $x_3$, and $x_4$ be four strongly independent leaves of $T$ such that $x_i$ is adjacent with $y_i$ for each $i \in \{1, 2, 3, 4\}$. Let $H = T - \{x_1, x_2, x_3, x_4\}$. Then $H$ has order $n - 4$. By the induction hypothesis, there is a 4-packing $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ of $H$ in a $B_{n-4+12} = B_{n+8}$. We may choose $V(B_{n+8})$ such that $x_i$ is not in $B_{n+8}$ for $i \in \{1, 2, 3, 4\}$. Let $A = \{x_1, x_2, x_3, x_4\}$ and $B_i = \{\sigma_i(y_1), \sigma_i(y_2), \sigma_i(y_3), \sigma_i(y_4)\}$ for each $i \in \{1, 2, 3, 4\}$. Let $B = B_1 \cup B_2 \cup B_3 \cup B_4$ and say $|B| = m$. Let $G$ be the complete bipartite graph with partites $A$ and $B$. By Lemma 1, there exist four disjoint matchings $M_1$, $M_2$, $M_3$ and $M_4$ such that $M_i$ matches $B_i$ into $A$ for all $i \in \{1, 2, 3, 4\}$. For each $i \in \{1, 2, 3, 4\}$, let $f_i$ be the permutation of $A$ such that $M_i = \{f_i(x_1)\sigma_i(y_1), f_i(x_2)\sigma_i(y_2), f_i(x_3)\sigma_i(y_3), f_i(x_4)\sigma_i(y_4)\}$.

We now add $A$ to $B_{n+8}$ to form a $B_{n+12}$ such that $A$ has the opposite parity with $B_i (1 \leq i \leq 4)$. For each $i \in \{1, 2, 3, 4\}$, we extend $\sigma_i$ to an embedding of $T$ in $B_{n+12}$ such that $\sigma_i(x_1) = f_i(x_1)$, $\sigma_i(x_2) = f_i(x_2)$, $\sigma_i(x_3) = f_i(x_3)$ and $\sigma_i(x_4) = f_i(x_4)$. This means that we extend a copy $\sigma_i(H)$ of $H$ to a copy $\sigma_i(T)$ of $T$ by adding four independent edges to $\sigma_i(H)$ from $M_i$ which matches $B_i$ into $A$. As $M_1, M_2, M_3$ and $M_4$ are disjoint, we conclude that $\sigma_1(T), \sigma_2(T), \sigma_3(T)$ and $\sigma_4(T)$ are edge disjoint, i.e., $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is a 4-packing of $T$ in $B_{n+12}$. Thus, the Main Theorem holds for $T$.

**Part II.** $\lambda(T) \leq 3$

In this case, we will apply the following lemmas to prove the Main Theorem. We need introduce a notation $F'$ for every subgraph $F$ of $T$. $F'$ is obtained from $F$ by adding to $F$ all those edges of $T$ which are not in $F$ but join two vertices of $F$ in $T$. We call $F'$ the induced subgraph by $F$ in $T$.

**Lemma 2.** *Let $H$ be a subgraph of $T$ such that each vertex of $T - V(H)$ is a leaf of $T$ and is adjacent with some vertex of $H$. If there is a 4-packing of $H'$ in a $B_p$*

*such that each vertex $y$ of $H'$ with $xy \in E$ for some $x \in V(T)\backslash V(H)$ is 4-placed, then there is a 4-packing of $T$ in a $B_{p+q}$ where $q = |V(T)| - |V(H)|$ such that each 4-placed vertex of $H$ remains as 4-placed.*

***Proof.*** Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be a 4-packing of $H'$ in a $B_p$ such that if $xy \in E(T)$ with $x \in V(T)\backslash V(H)$, then $y$ is 4-placed. Note that $\sigma_1(y)$, $\sigma_2(y)$, $\sigma_3(y)$ and $\sigma_4(y)$ have the same parity in $B_p$ for all $y \in V(H)$. We obtain $B_{p+q}$ by adding each $x \in V(T)\backslash V(H)$ to $B_p$ such that if $xy \in E(T)$, then $x$ and $\sigma_1(y)$ have the opposite parity. Then for each $i \in \{1, 2, 3, 4\}$, we extend $\sigma_i$ to an embedding of $T$ in $B_{p+q}$ such that $\sigma_i(x) = x$ for each $x \in V(T)\backslash V(H)$. That is, if $xy \in E(T)$ with $x \in V(T)\backslash V(H)$, then $x\sigma_i(y)$ is an edge of $\sigma_i(T)$. Since $y$ is 4-placed for each edge $xy \in E(T)$ with $x \in V(T)\backslash V(H)$, i.e., $x\sigma_1(y), x\sigma_2(y), x\sigma_3(y)$ and $x\sigma_4(y)$ are distinct, we conclude that $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is a 4-packing of $T$ in $B_{p+q}$. ■

**Lemma 3.** *Let $P = x_1x_2 \cdots x_t$ be a path with $4 \le t \le 6$ or $9 \le t \le 10$. Then there is a 4-packing $(d, b, g, r)$ of $P$ in a $B_{t+4}$ such that every vertex of $P$ is 4-placed.*

***Proof.*** Let $V(B_{t+4}) = \{x_1, x_2, \ldots, x_{t+4}\}$ such that $\{x_i | i \text{ is odd}\}$ and $\{x_j | j \text{ is even}\}$ form a bipartition of $B_{t+4}$. We exhibit $(d, b, g, r)$ as follows.

If $t = 4$, we define

$$\begin{aligned} d(P) &= x_1x_2x_7x_4 \text{ with } d(x_1) = x_1, \\ b(P) &= x_3x_4x_1x_6 \text{ with } b(x_1) = x_3, \\ g(P) &= x_5x_6x_3x_8 \text{ with } g(x_1) = x_5, \\ r(P) &= x_7x_8x_5x_2 \text{ with } r(x_1) = x_7. \end{aligned}$$

If $t = 5$, we define

$$\begin{aligned} d(P) &= x_1x_2x_7x_4x_5 \text{ with } d(x_1) = x_1, \\ b(P) &= x_3x_4x_9x_6x_7 \text{ with } b(x_1) = x_3, \\ g(P) &= x_5x_6x_3x_8x_9 \text{ with } g(x_1) = x_5, \\ r(P) &= x_7x_8x_5x_2x_3 \text{ with } r(x_1) = x_7. \end{aligned}$$

If $t = 6$, we define

$$\begin{aligned} d(P) &= x_1x_2x_9x_4x_7x_6 \text{ with } d(x_1) = x_1, \\ b(P) &= x_3x_4x_1x_6x_9x_8 \text{ with } b(x_1) = x_3, \\ g(P) &= x_5x_6x_3x_8x_1x_{10} \text{ with } g(x_1) = x_5, \\ r(P) &= x_7x_8x_5x_{10}x_3x_2 \text{ with } r(x_1) = x_7. \end{aligned}$$

If $t = 9$, we define

$$\begin{aligned} d(P) &= x_1x_2x_{13}x_4x_{11}x_6x_9x_{10}x_5 \text{ with } d(x_1) = x_1, \\ b(P) &= x_7x_4x_1x_6x_{13}x_8x_{11}x_2x_9 \text{ with } b(x_1) = x_7, \\ g(P) &= x_5x_6x_3x_8x_1x_{10}x_7x_{12}x_{13} \text{ with } g(x_1) = x_5, \\ r(P) &= x_3x_{10}x_{11}x_{12}x_9x_4x_5x_8x_7 \text{ with } r(x_1) = x_3. \end{aligned}$$

If $t = 10$, we define

$$\begin{aligned} d(P) &= x_1x_2x_{13}x_{12}x_{11}x_6x_9x_8x_7x_{14} \text{ with } d(x_1) = x_1, \\ b(P) &= x_3x_4x_1x_6x_{13}x_8x_{11}x_{10}x_9x_{12} \text{ with } b(x_1) = x_3, \\ g(P) &= x_5x_6x_3x_8x_1x_{10}x_{13}x_{14}x_{11}x_4 \text{ with } g(x_1) = x_5, \\ r(P) &= x_7x_{10}x_5x_4x_9x_{14}x_1x_{12}x_3x_2 \text{ with } r(x_1) = x_7. \end{aligned}$$

■

**Remark.** Lemma 3 does not include the case $t \in \{7, 8\}$. Later in the proof, when Lemma 3 is needed and $t \in \{7, 8\}$, we have deleted an endvertex from $P$ or added some new vertices to $P$ to obtain a linear graph whose order is neither 7 nor 8 and then apply Lemma 3 to this linear graph.

**Lemma 4.** *Let $p$ be an integer with $p \geq 8$ and $P = x_1x_2 \cdots x_{2p}$ be a path of order $2p$. Then there is a 4-packing $(d, b, g, r)$ of $P$ in a $B_{2p}$ such that each vertex of $P$ is 4-placed.*

***Proof.*** Let $B_{2p}$ have the bipartition $\{x_1, x_3, \ldots, x_{2p-1}\}$ and $\{x_2, x_4, \ldots, x_{2p}\}$. Draw $B_{2p}$ in a plane as follows. Label the vertices of a regular $2p$-gon as $x_1, \ldots,$ $x_{2p}$ in order clockwise in an $xy$-plane and draw edges of $B_{2p}$ as stright line segements. Furthermore, the center of the regular $2p$-gon is at the origin $(0, 0)$. The distance from the center to the vertices is 1 and the vertex $x_{2p}$ is at $(0, 1)$. Thus, all the vertices are on the circle of radius 1 with its center at $(0, 0)$. Note that the vertex $x_p$ is at $(0, -1)$.

Take all the edges of $B_{2p}$ which are parallel to $x_{2p}x_1$ or $x_1x_2$. These edges form a Hamiltonian cycle $C$ of $B_{2p}$. If we remove the edge $x_1x_{2p}$ from $C$, we get a Hamiltonian path from $x_1$ to $x_{2p}$ in $B_{2p}$. Label this path as $L_1 = x_{i_1}x_{i_2} \cdots x_{i_{2p}}$ with $x_1 = x_{i_1}$, which can be viewed as a permutation of $\{x_1, x_2, \ldots, x_{2p}\}$ as well. Similarly, for each $x_i \in \{x_3, x_5, x_7\}$, using $x_{i-1}x_i$ and $x_ix_{i+1}$ in place of $x_{2p}x_1$ and $x_1x_2$ to repeat the above action, we get a Hamiltonian path $L_i$ from $x_i$ to $x_{i-1}$. Define $d(P) = L_1$ with $d(x_1) = x_1$, $b(P) = L_3$ with $b(x_1) = x_3$, $g(P) = L_5$ with $g(x_1) = x_5$ and $r(P) = L_7$ with $r(x_1) = x_7$.

In fact, we may rotate $L_1$ clockwise by $(360/p)^\circ$ three times to get $L_3, L_5$, and $L_7$ as follows

$$\begin{aligned} L_3 &= x_{i_1+2}x_{i_2+2} \cdots x_{i_{2p}+2}, \\ L_5 &= x_{i_1+4}x_{i_2+4} \cdots x_{i_{2p}+4}, \\ L_7 &= x_{i_1+6}x_{i_2+6} \cdots x_{i_{2p}+6}. \end{aligned}$$

Note that in the above writing, we define $x_{2p+i} = x_i$ for each $1 \leq i \leq 6$. As $2p \geq 16$, this implies that each vertex of $P$ is 4-placed. To see that $L_1$, $L_3$, $L_5$, and $L_7$ are edge disjoint, we observe that the edges of the regular $2p$-gon incident with vertices $x_1$, $x_3$, $x_5$, and $x_7$ are the chords of the half unit circle on the right side of $y$-axis because $p \geq 8$. Therefore, these edges have distinct slopes and so $L_1$, $L_3$, $L_5$, and $L_7$ are edge disjoint. ∎

Since a linear graph is always a subgraph of a path, the following lemma is an easy observation.

**Lemma 5.** *Let $r$, $s$, $p$ and $t$ be four positive integers such that $r \leq s \leq 2p \leq t$. Suppose that there is 4-packing of a path of order $s$ in a $B_{2p}$ such that each vertex of the path is 4-placed. Then there is a 4-packing of any linear graph of order $r$ in a $B_t$ such that every vertex of the linear graph is 4-placed.*

We are ready to prove the Main Theorem. As $\lambda(T) \leq 3$, it follows that $\xi_1 \leq 2\lambda \leq 6$. By (3), we see that $\xi_i = 0$ for $i \geq 7$. Thus,

$$2 \leq \xi_1 \leq 6 \text{ and } \xi_1 = 2 + \xi_3 + 2\xi_4 + 3\xi_5 + 4\xi_6. \tag{4}$$

First, suppose that $\xi_1 = 2$. Then $\xi_i = 0$ for all $i \geq 3$. Thus, $T^*$ is a path. By Lemmas 3 through 5, there is a 4-packing of $T^*$ in a $B_{n^*+6}$ such that each vertex of $T^*$ is 4-placed. By Lemma 2, there is a 4-packing of $T$ in $B_{n+6}$ where $n$ is the order of $T$. Therefore, we may assume that $\xi_1 \geq 3$, i.e., $\xi_i \geq 1$ for some $3 \leq i \leq 6$.

We define some terminology. For each supernode $u$ of $T$, there is a unique path $u_1u_2 \cdots u_kv$ in $T^*$ with $u = u_1$ such that $deg_{T^*}(v) \geq 3$ and $deg_{T^*}(u_i) = 2$ for all $2 \leq i \leq k$. In this case, we say that the supernode $u$ belongs to $v$. Moreover, if $k$ is even, we say that $u_1u_2 \cdots u_k$ is an *arm* of $v$. If $k$ is odd, we choose a fixed vertex $u'$ of $T$ that is a leaf of $T$ with $u'u_1 \in E$, and then we say that $u'u_1u_2 \cdots u_k$ is an arm of $v$. Clearly, by this definition, any arm is a path of *even order*, and any two arms do not intersect. We now divide the remaining proof into the following five cases.

*Case* 1. There is exactly one vertex $u$ of $T^*$ with $deg_{T^*}(u) \geq 3$. Thus, all the $\xi_1$ supernodes of $T$ belong to $u$. Let $H$ be the union of the $\xi_1$ arms of $u$. Let $(A, B)$ denote the bipartition of $H$. As each arm has an even order, $|A| = |B| = h$ for some $h \geq 3$. By Lemmas 3 through 5, there is a 4-packing $(d, b, g, r)$ of $H$ in a $B_{2h+6}$ such that each vertex of $H$ is 4-placed. We may assume that $H$ is a subgraph of $B_{2h+6}$ and $d$ is the identity embedding. Add $u$ and three new vertices $\omega_1$, $\omega_2$ and $\omega_3$ to $B_{2h+6}$ to form a $B_{2h+10}$ such that these four vertices have the same parity and $H + u$ is a subgraph of $B_{2h+10}$. Then extend $(d, b, g, r)$ to be a 4-packing of $H + u$ in $B_{2h+10}$ such that $d(u) = u$, $b(u) = \omega_1$, $g(u) = \omega_2$ and $r(u) = \omega_3$. Thus, $(d, b, g, r)$ is a 4-packing of $H + u$

is $B_{2h+10}$. By Lemma 2, there is a 4-packing of $T$ in a $B_{2h+10+q} = B_{n+9}$ where $q = |V(T)| - |V(H+u)| = n - 2h - 1$. This completes Case 1.

By Case 1, we may assume that $T^*$ has at least two vertices of degree at least 3 in $T^*$. As $\xi_1 \leq 6$, we see that $\xi_6 = 0$ by (4). Let $L$ be a longest path of $T^*$ from a vertex of degree at least 3 to another vertex of degree at least 3 in $T^*$. Say $L = uy_1y_2\cdots y_tv$. Say $deg_{T^*}(u) = i$ and $deg_{T^*}(v) = j$. Then there are $i-1$ supernodes of $T$ belonging to $u$ and $j-1$ supernodes of $T$ belonging to $v$.

Let $L_1 = x_1x_2\cdots x_a$ be a path of $T$ passing through $u$ such that $x_2$ and $x_{a-1}$ are two supernodes of $T$ belonging to $u$ and $x_1$ and $x_a$ are two leaves of $T$. Similarly, let $L_2 = z_1z_2\cdots z_b$ be a path of $T$ passing through $v$ such that $z_2$ and $z_{b-1}$ are two supernodes belonging to $v$. Since each arm has an even order and $L_1$ contains two arms and the vertex $u$, $L_1$ has order at least 5. Similarly, $L_2$ has order at least 5.

Let $A$ and $B$ denote the two partites of $L_1 \cup L_2$ with $|A| \geq |B|$. As $\lambda(T) \leq 3$, we see that if $L_1$ has an odd order, then $x_1$ and $x_a$ have the same parity and so one of $z_1$ and $z_b$ must have the opposite parity with $x_1$. This implies that $|A| - |B| \leq 1$. If $|A| = |B|$, Let $H_1$ be $L_1 \cup L_2$, and if $|A| - |B| = 1$, let $H_1 = L_1 \cup L_2 - w$ where $w$ is a leaf of $L_1 \cup L_2$ which belongs to $A$. Thus, the two partites of $H_1$ has the same size. Say $H_1$ has order $2h$. Then $h \geq 5$. Let $H_2$ be the union of all the arms of $u$ or $v$ that are not on any of $L_1$ and $L_2$. Let $(C, D)$ denote the bipartition of $H_2$. Then the two partites of $H_2$ have the same size. Say $H_2$ has order $2k$.

*Case* 2. There are exactly two distinct vertices of $T^*$ with degree at least 3 in $T^*$. If $t$ is even, let $H = H_1 \cup H_2 \cup (y_1y_2\cdots y_t)$. If $t$ is odd and $T$ has a leaf $y'$ adjacent to $y_t$, let $H = H_1 \cup H_2 \cup (y_1y_2\cdots y_ty')$, and if $t$ is odd and $T$ does not has a leaf $y'$ adjacent to $y_t$, let $H = H_1 \cup H_2 \cup (y_1y_2\cdots y_{t-1})$. Then, $H$ is a linear subgraph of $T$ and its two partites have the same size. Say $|V(H)| = m$. As $m \geq 10$ and by Lemmas 3 through 5, there is a 4-packing $(d, b, g, r)$ of $H$ in a $B_{m+4}$ such that all vertices of $H$ are 4-placed. We may assume that $H$ is a subgraph of $B_{m+4}$ and $d$ is the identity embedding. This 4-packing of $H$ does not consider the edges of $T^*$ that join a component of $H$ to another component of $H$. If these edges are added back to $H$ to form $H'$, this 4-packing of $H$ cannot be guaranteed to be a 4-packing of $H'$. These edges are incident with at least one $u$, $v$, and $y_t$. Therefore, we need to modify $(d, b, g, r)$ in the following. First, if $y_t$ is not in $H$, add $y_t$ to $H$ and define $d(y_t) = b(y_t) = g(y_t) = r(y_t) = y_t$.

Let $m'$ be the order of $H + y_t$. Note that if $y_t$ is already in $H$, then $m' = m$. Then add six new vertices $\omega_1, \ldots, \omega_6$ to $B_{m'+4}$ to form a $B_{m'+10}$ such that $\omega_1$, $\omega_2$ and $\omega_3$ have the same parity with $u$ and $\omega_4$, $\omega_5$ and $\omega_6$ has the same parity with $v$. Change the images of $u$ and $v$ under $b, g$ and $r$ as follows. Let

$$b(u) = \omega_1, g(u) = \omega_2, \text{ and } r(u) = \omega_3,$$

$$b(v) = \omega_4, g(v) = \omega_5, \text{ and } r(v) = \omega_6.$$

The images of all the other vertices of $(H + y_t)'$ are not changed under $(d, b, g, r)$. Clearly, if $y_t$ is not in $H$, then $y_t$ has only two neighbors $y_{t-1}$ and $v$ in $T$, and

$$\{d(y_{t-1}), b(y_{t-1}), g(y_{t-1}), r(y_{t-1})\} \cap \{d(v), b(v), g(v), r(v)\} = \emptyset.$$

Thus, $(d, b, g, r)$ becomes a 4-packing of $(H + y_t)'$ in $B_{m'+10}$ such that all the vertices of $H$ are 4-placed. By Lemma 2, $T$ has a 4-packing in a $B_{n+10}$. This completes Case 2.

By Case 2, we may assume that $T^*$ has at least three vertices of degree at least 3 in $T^*$. As $T^*$ is connected, any vertex of degree 3 in $T^*$ is connected by a path to a vertex on $y_1 \cdots y_t$ in $T^*$. Thus, $y_r$ has degree at least 3 in $T^*$ for some $1 \leq r \leq t$. As $\xi_1 \leq 6$ and by (4), we see that $\xi_5 = 0$ and if $deg_{T^*}(y_r) = 4$, then $deg_{T^*}(x) < 3$ for all $x \in V(T^*) \backslash \{u, v, y_r\}$. In the following cases, the arguments are very much similar to the argument in Case 2.

*Case* 3. Either $deg_{T^*}(y_r) = 4$ or $deg_{T^*}(y_s) = 3$ for some $1 \leq s \leq t$ and $s \neq r$. We define a new number $a$ such that $a = r$ if $deg_{T^*}(y_r) = 4$ and $a = s$ if $deg_{T^*}(y_r) = 3$ and $deg_{T^*}(y_s) = 3$ for some $1 \leq s \leq t$ and $s \neq r$. Thus, $T$ has two paths $Q_1 = u_1u_2 \cdots u_l y_r$ and $Q_2 = v_1v_2 \cdots v_c y_a$ such that $u_1u_2 \cdots u_l y_r$ and $v_1v_2 \cdots v_c$ are vertex disjoint and $u_2$ and $v_2$ are two supernodes of $T$ belonging $y_r$ and $y_a$, respectively. We may assume that $r \leq a$. Let

$$L_3 = u_1 \cdots u_l y_r y_{r+1} \cdots y_a v_c v_{c-1} \cdots v_1.$$

As $\lambda(T) \leq 3$, it is easy to see that if two of $L_1$, $L_2$ and $L_3$ have odd orders, then the leaves of one of them have the opposite parity with the leaves of the other and so the third path must have an even order. This implies that the two partites of $L_1 \cup L_2 \cup L_3$ have the same size or their sizes differ by 1. If the two partites of $L_1 \cup L_2 \cup L_3$ have the same size, we define $J_1 = L_1 \cup L_2 \cup L_3$. If the two partites of $L_1 \cup L_2 \cup L_3$ differ by 1 in size, we delete one leaf of $L_1 \cup L_2 \cup L_3$ from the larger partite and denote the resulting linear subgraph by $J_1$. Thus, $J_1$ has an even order $2q$ for some $q \geq 8$.

For each of $y_{r-1}y_{r-2} \cdots y_1$ and $y_{a+1}y_{a+2} \cdots y_t$, we delete its last vertex if it has an odd order and otherwise, we keep it. Thus, we obtain two paths of even orders and let $J_2$ be the union of these two paths. Let $H = J_1 \cup J_2$. By Lemma 4 and Lemma 5, there is a 4-packing $(d, b, g, r)$ of $H$ in a $B_m$ such that all vertices of $H$ are 4-placed, where $m$ is the order of $H$. We may assume that $H$ is a subgraph of $B_m$ and $d$ is the identity embedding.

Add $y_1$, $y_t$ and 12 new vertices $\omega_1, \ldots, \omega_{12}$ to $B_m$ to obtain a $B_{p+12}$ such that $\omega_1$, $\omega_2$ and $\omega_3$ have the same parity with $u$; $\omega_4$, $\omega_5$ and $\omega_6$ has the same parity with $y_{r-1}$; $\omega_7$, $\omega_8$ and $\omega_9$ have the same parity with $y_{a+1}$; and $\omega_{10}$, $\omega_{11}$ and $\omega_{12}$

have the same parity with $v$, where $p$ is the order of $H + y_1 + y_t$. Note that for $i \in \{1, t\}$, if $y_i$ is already in $H$, adding $y_i$ to $H$ does not increase the order of the resulting graph.

For each $y_i \in \{y_1, y_t\}$, if $y_i \notin V(H)$, we define $d(y_i) = b(y_i) = g(y_i) = r(y_i)$. Change the images of $u, y_{r-1}, y_{a+1}$ and $v$ under $b$, $g$, and $r$ as follows. Let

$$\begin{aligned}
&b(u) = \omega_1, g(u) = \omega_2, r(u) = \omega_3,\\
&b(y_{r-1}) = \omega_4, g(y_{r-1}) = \omega_5, r(y_{r-1}) = \omega_6,\\
&b(y_{a+1}) = \omega_7, g(y_{a+1}) = \omega_8, r(y_{a+1}) = \omega_9,\\
&b(v) = \omega_{10}, g(v) = \omega_{11}, r(v) = \omega_{12}.
\end{aligned}$$

The images of all the other vertices of $H+y_1+y_t$ are not changed. Thus, $(d, b, g, r)$ is a 4-packing of $(H + y_1 + y_t)'$ in a $B_{p+12}$ such that all the vertices of $H$ are 4-placed. As $\lambda(T) \leq 3$, every leaf of $T$ is adjacent to one of the six supernodes of $T$. By Lemma 2, there is a 4-packing of $T$ in a $B_{n+12}$. This completes Case 3.

*Case* 4. There are exactly three distinct vertices of $T^*$ with degree at least 3 in $T^*$. By Case 3, we may assume that $deg_{T^*}(y_r) = 3$. Let $Q_1 = u_1 \cdots u_l y_r$ be a path of $T$ such that $u_2$ is supernode of $T$ belonging to $y_r$. Consider three paths $y_{r-1}y_{r-2}\cdots y_1$, $y_{r+1}y_{r+2}\cdots y_t$ and $u_1 \cdots u_l y_r$. Let $I_1 = u_1u_2\cdots u_l y_r y_{r-1}\cdots y_1$. Then $I_1$ has order at least 3. Let $I_2 = y_{r+1}y_{r+2}\cdots y_t$ if this path has an even order. If $y_{r+1}y_{r+2}\cdots y_t$ has an odd order and there is a leaf of $T$ adjacent to $y_t$, let $I_2 = y_{r+1}\cdots y_t y'$. Otherwise, let $I_2 = y_{r+1}y_{r+2}\cdots y_{t-1}$. Let $H = H_1 \cup H_2 \cup I_1 \cup I_2$. Then $H$ has an order at least 13. By Lemma 4 and Lemma 5, there is a 4-packing $(d, b, g, r)$ of $H$ in a $B_{m+3}$ such that all vertices of $H$ are 4-placed, where $m$ is the order of $H$. We may assume that $H$ is a subgraph of $B_{m+3}$ and $d$ is the identity embedding. Add $y_t$ and nine new vertices $\omega_1, \ldots, w_9$ to $B_m$ and obtain a $B_{p+12}$, where $p$ is the order of $H + y_t$, such that $H + y_t$ is a subgraph of $B_{p+12}$, $\omega_1$, $\omega_2$ and $\omega_3$ have the same parity with $u$; $\omega_4$, $\omega_5$ and $\omega_6$ have the same parity with $v$; $\omega_7$, $\omega_8$ and $\omega_9$ have the same parity with $y_r$.

If $y_t$ is not in $H$, we define $d(y_t) = b(y_t) = g(y_t) = r(y_t) = y_t$. Change the images of $u$, $v$ and $y_r$ under $b$, $g$, and $r$ as follows.

$$\begin{aligned}
&b(u) = \omega_1, g(u) = \omega_2, r(u) = \omega_3,\\
&b(v) = \omega_4, g(v) = \omega_5, r(v) = \omega_6,\\
&b(y_r) = \omega_7, g(y_r) = \omega_8, r(y_r) = \omega_9.
\end{aligned}$$

The images of all the other vertices of $H + y_t$ are not changed. Thus, $(d, b, g, r)$ is a 4-packing of $(H + y_t)'$ in a $B_{p+12}$ such that all the vertices of $H$ are 4-placed. By Lemma 2, there is a 4-packing of $T$ in a $B_{n+12}$. This completes Case 4.

*Case* 5. There are at least four distinct vertices of $T^*$ with degree at least 3 in $T^*$. By (4), we see that $\xi_3 = 4$ and $\xi_4 = \xi_5 = 0$. Let $w$ be the fourth vertex of degree 3. By Case 3, we may assume that $w$ is not on $y_1y_2\cdots y_t$. As $T^*$ is connected, there is a path $Q = y_ru_1\cdots u_lw$ from $y_r$ to $w$ such that $y_i$ is not on this path for all $i \in \{1, 2, \ldots, t\}\backslash\{r\}$.

Let $L_3 = v_1v_2\cdots v_c$ be a path of $T$ passing through $w$ such that $v_2$ and $v_{c-1}$ are two supsernodes of $T$. Clearly, $c \geq 5$. Let $J_1 = L_1 \cup L_2 \cup L_3$ if the two partites have the same size. If their sizes differ by 1, we delete a leaf of $L_1 \cup L_2 \cup L_3$ to obtain a subgraph with its two partites having the same size and denote this resulting graph by $J_1$. As in Case 3, we see that $J_1$ has order $2q$ for some $q \geq 8$.

Consider three paths $y_{r-1}y_{r-2}\cdots y_1$, $y_{r+1}y_{r+2}\cdots y_t$ and $y_ru_1u_2\cdots u_l$. For each of these three paths, we delete its last vertex if it has an odd order and otherwise, we keep it. Thus, we obtain three paths of even orders and let $J_2$ be the union of these three paths of even order. Let $H = J_1 \cup J_2$. By Lemma 4 and Lemma 5, there is a 4-packing $(d, b, g, r)$ of $H$ in a $B_m$ such that all vertices of $H$ are 4-placed, where $m$ is the order of $H$. We may assume that $H$ is a subgraph of $B_m$ and $d$ is the identity embedding.

Let the order of $H + y_1 + y_t + u_l$ be $p$. Add $y_1, y_t$, $u_l$ and 12 new vertices $\omega_1, \ldots, \omega_{12}$ to $B_m$ to obtain a $B_{p+12}$ such that $\omega_1$, $\omega_2$ and $\omega_3$ have the same parity with $u$; $\omega_4$, $\omega_5$ and $\omega_6$ have the same parity with $v$; $\omega_7$, $\omega_8$ and $\omega_9$ have the same parity with $w$; and $\omega_{10}$, $\omega_{11}$ and $\omega_{12}$ have the same parity with $y_r$. Furthermore, $H + y_1 + y_t + u_l$ is a subgraph of $B_{p+12}$.

For each $x \in \{y_1, y_t, u_l\}$, if $x$ is not in $H$, we define $d(x) = b(x) = g(x) = r(x)$. Change or define the images of $u$, $v$, $w$, and $y_r$ under $d$, $b$, $g$, and $r$ as follows. Let $d(y_r) = y_r$ and

$$\begin{aligned}
&b(u) = \omega_1, g(u) = \omega_2, r(u) = \omega_3,\\
&b(v) = \omega_4, g(v) = \omega_5, r(v) = \omega_6,\\
&b(w) = \omega_7, g(w) = \omega_8, r(w) = \omega_9,\\
&b(y_r) = \omega_{10}, g(y_r) = \omega_{11}, r(y_r) = \omega_{12}.
\end{aligned}$$

The images of all the other vertices of $H$ are not changed. Thus, $(d, b, g, r)$ is a 4-packing of $(H + y_1 + y_t + u_l)'$ in $B_{p+12}$ such that all the vertices of $H + y_r$ are 4-placed. As $\lambda(T) \leq 3$ and $\xi_1 = 6$, each leaf of $T$ is adjacent to one of the six supernodes of $T$. By Lemma 2, there is a 4-packing of $T$ in a $B_{n+12}$.

This proves the Main Theorem.

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