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3-TUPLE TOTAL DOMINATION NUMBER OF ROOK'S GRAPHS

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Abstract

A k-tuple total dominating set (kTDS) of a graph G is a set S of vertices in which every vertex in G is adjacent to at least k vertices in S. The minimum size of a kTDS is called the k-tuple total dominating number and it is denoted by $\gamma_{\times k,t}(G)$. We give a constructive proof of a general formula for $\gamma_{\times 3,t}(K_n \Box K_m)$.

Keywords: k-tuple total domination, Cartesian product of graphs, rook's graph, Vizing's conjecture.

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1. Introduction

Domination is well-studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [12, 13]. Among the many variations of domination, the one relevant to this paper is k-tuple total domination, which was introduced by Henning and Kazemi [15] as a generalization of [11]. Throughout this paper, we use standard notation for graphs, see for example [1]. All graphs considered here are finite, undirected, and simple.

For a graph $G = (V_G, E_G)$ and $k \geq 1$, a set $S \subseteq V_G$ is called a *k-tuple total dominating set* (kTDS) if every vertex $v \in V$ has at least k neighbours in S, i.e., $|N_G(v) \cap S| \geq k$. The *k-tuple total domination number*, which we denote by $\gamma_{\times k,t}(G)$, is the minimum cardinality of a kTDS of G. We use min-kTDS to refer to kTDSs of minimum size.

An immediate necessary condition for a graph to have a k-tuple total dominating set is that every vertex must have at least k neighbours. For example, for $k \geq 1$, a k-regular graph $G = (V_G, E_G)$ has only one k-tuple total dominating set, namely V_G itself.

In the history of domination problems, a lot of work has been done to study the class of Cartesian product of graphs and in particular of rook's graphs. Given two graphs G and H, their *Cartesian product* $G\Box H$ is the graph with vertex set $V_G \times V_H$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E_H$ or $v_1 = v_2$ and $u_1u_2 \in E_G$. For more information on the Cartesian product of graphs see [20]. We will be particularly interested in the case $K_n\Box K_m$, where K_n is the complete graph on n vertices. Such graph is known as the $n \times m$ *rook's graph*, as edges represent possible moves by a rook on an $n \times m$ chess board. The 3×4 rook's graph is drawn in Figure 1, along with a min-3TDS.

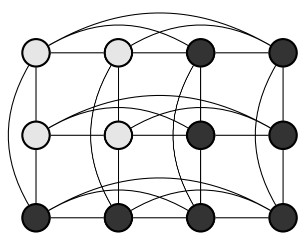

Figure 1. The 3×4 rook's graph, i.e., $K_3\Box K_4$. The dark vertices form a min-3TDS, so $\gamma_{\times 3,t}(K_3\Box K_4) = 8$.

In [23], Vizing studied the *domination number* of graphs, i.e., the minimal cardinality of a dominating set, and made an elegant conjecture that has subsequently become one the most famous open problems in domination theory.

Conjecture 1.1 (Vizing's Conjecture)**.** *For any graphs G and H,*

$$\gamma(G)\gamma(H) \leq \gamma(G\square H),$$

where $\gamma(G)$ and $\gamma(H)$ are the domination numbers of the graphs G and H, respectively.

Over more than forty years (see [2] and references therein), Vizing's Conjecture has been shown to hold for certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have gradually tightened. Additionally, researcher have explored inequalities (including Vizing-like inequalities) for different variations of domination [13]. A significant breakthrough occurred when in [9] Clark and Suen proved that

$$\gamma(G)\gamma(H) \leq 2\gamma(G\square H),$$

which led to the discovery of a Vizing-like inequality for total domination [16, 17], i.e.,

$$\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G\square H), \tag{1}$$

as well as for paired [4, 7, 18], and fractional domination [10], and the $\{k\}$-domination function (integer domination) [3, 8, 19], and total $\{k\}$-domination function [19].

Burchett, Lane, and Lachniet [6] and Burchett [5] found bounds and exact formulas for the k-tuple domination number and k-domination number of the rook's graph in square cases, i.e., $K_n\square K_n$ (where k-domination is similar to k-tuple total domination, but only vertices outside of the domination set need to be dominated). The k-tuple total domination number is known for $K_n \times K_m$ [14] and bounds are given for supergeneralized Petersen graphs [21]. In [22], the authors showed that the graph $K_n\square K_m$ is an extremal case in the study of kTDS of Cartesian product of graphs, motivating the study of the class of rook's graphs. Specifically, they showed that

$$\gamma_{\times k,t}(K_n\square K_m) \leq \gamma_{\times k,t}(G\square H),$$

when G and H are two graphs with n and m vertices, respectively. Moreover, they computed $\gamma_{\times 2,t}(K_n\square K_m)$ for all $m \geq n$.

This paper is organized as follows. In Section 2, we recall basic properties on kTDS. In Section 3, we describe a special class of 3TDS matrices. In Section 4, we describe several useful inequalities for $\gamma_{\times 3,t}(K_n\square K_m)$. In Section 5, we compute $\gamma_{\times 3,t}(K_n\square K_n)$, for any $n \geq 3$. In Section 6, we describe our main result: we determine the value of $\gamma_{\times 3,t}(K_n\square K_m)$ in Theorem 6.1 for all $m \geq n$.

2. PRELIMINARES

We recall some basic properties of kTDS and their relations with $(0,1)$-matrices. Assume the vertex set of the complete graph K_n is $[n] := \{1, \dots, n\}$. Given $D \subseteq V_{K_n} \times V_{K_m}$, we can associate to it an $n \times m$ $(0,1)$-matrix $S = (s_{ij})$ with $s_{ij} = 1$ if and only if $(i,j) \in D$. Let $S = (s_{ij})$ be an $n \times m$ $(0,1)$-matrix. Define

$$\begin{aligned} \kappa_S(i,j) &= \overbrace{\left(\sum\nolimits_{r\in[m]} s_{ir}\right)}^{i\text{-th row sum}} + \overbrace{\left(\sum\nolimits_{r\in[n]} s_{rj}\right)}^{j\text{-th column sum}} - 2s_{ij} \\ &= \mathfrak{r}_S(i) + \mathfrak{c}_S(j) - 2s_{ij}. \end{aligned}$$

If no confusion arises, we will simply write $\mathfrak{r}(i)$, $\mathfrak{c}(j)$ and $\kappa(i,j)$. Notice that $\mathfrak{r}(i)$ is the number of ones in the i-th row of S and, similarly, $\mathfrak{c}(j)$ is the number of ones in the j-th column of S. Moreover, we will denote by $|S|$ the number of ones in S.

An $n \times m$ $(0,1)$-matrix $S = (s_{ij})$ corresponds to a kTDS D of $K_n \square K_m$ if and only it satisfies

$$\kappa(i,j) \geq k$$

for all $i \in [n]$ and $j \in [m]$, which we call the *κ-bound.*

0	0	1	1
0	0	1	1
1	1	1	1

Figure 2. The 3TDS matrix corresponding to Figure 1.

We call an $n \times m$ $(0,1)$-matrix S a *kTDS matrix* if it satisfies the κ-bound for all $i \in [n]$ and $j \in [m]$. Furthermore, we call S a *min-kTDS matrix* if it has exactly $\gamma_{\times k,t}(K_n \square K_m)$ ones. Note that a kTDS matrix (respectively, min-kTDS matrix) remains a kTDS matrix (respectively, min-kTDS matrix) under permutations of its rows and/or columns.

Lemma 2.1. *For $n \geq 1$ and $m \geq 1$, an $n \times m$ kTDS matrix with an all-0 column or an all-0 row has at least kn or km ones, respectively.*

Proof. Let S be an $n \times m$ kTDS matrix. Assume there exists $1 \leq j_0 \leq m$ such that $\mathfrak{c}(j_0) = 0$. Then to achieve $\kappa(i, j_0) \geq k$ for any $i \in [n]$, we need $\mathfrak{r}(i) \geq k$. Since this is true for every row in S, we must have at least kn ones. A similar argument works if there exists $1 \leq i_0 \leq n$ such that $\mathfrak{r}(i_0) = 0$. ∎

There are instances when kn ones is the least number of ones in any $n \times m$ kTDS matrix. Some cases were established in Theorem 3.3 from [22]. We rewrite the part of the theorem relevant for this paper.

Proposition 2.2. *When $m \geq n \geq 2$ and $m \geq k$,*

$$\gamma_{\times k,t}(K_n \square K_m) \leq kn$$

with equality when $m \geq kn - 1$.

Proof. If $m \geq n \geq 2$ and $m \geq k$, the $n \times m$ $(0,1)$-matrix with ones in the last k columns and zeros elsewhere is a kTDS matrix with kn ones.

Assume $m \geq kn - 1$ and let S be an $n \times m$ kTDS matrix. If S has a column of zeros, then $|S| \geq kn$ by Lemma 2.1. If S has no column of zeros but $m \geq kn$, then $|S| \geq kn$. Thus, assume $m = kn - 1$ and $\mathfrak{c}(j) \geq 1$ for all $1 \leq j \leq m$. If $|S| < kn$, then $\mathfrak{c}(j) = 1$ for all $1 \leq j \leq m$. Therefore, if $s_{ij} = 1$, then $\mathfrak{r}(i) \geq k + 1$ to satisfy $\kappa(i,j) \geq k$. If this is true for every row, then $|S| \geq (k+1)n > kn$. Otherwise, there is a row of zeros, and Lemma 2.1 implies $|S| \geq km \geq kn$. ∎

Motivated by [6], given a $(0,1)$-matrix S, we can construct a graph $\Gamma(S)$ with vertices corresponding to the ones in S and edges between 2 ones belonging to the same row or column, if there are no other ones between them. The following gives one such example.

0	0	0	0	0	1	1
0	0	0	0	0	1	1
0	0	0	0	0	1	1
0	0	0	0	0	1	1
1	1	1	1	1	0	0

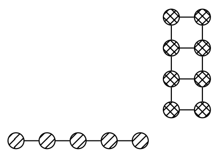

Figure 3. A $(0,1)$-matrix S and its graph $\Gamma(S)$.

In this way, every kTDS matrix S correspond to a graph, which has, in general, several (connected) components. If the set of vertices of a component of $\Gamma(S)$ is $\{(i_1, j_1), \ldots, (i_p, j_p)\}$, then we define the corresponding component of S as the submatrix of S formed by the intersection of rows $\{R_{i_1}, \ldots, R_{i_p}\}$ and columns $\{C_{j_1}, \ldots, C_{j_p}\}$, where R_d and C_d are the d-th row and column of S, respectively. We shade two components in the example above. In this example, the 5×7 matrix is the union of two components (a 4×2 component and a 1×5 component). A kTDS matrix S with a component H, up to permutations of the rows and columns of S, looks like one of the following

$$\begin{array}{|c|c|}\hline H & \emptyset \\ \hline \emptyset & ? \\ \hline \end{array}, \quad \begin{array}{|c|}\hline H \\ \hline \emptyset \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline H & \emptyset \\ \hline \end{array}, \text{ or } \begin{array}{|c|}\hline H \\ \hline \end{array},$$

where the question mark (?) denotes some $(0,1)$-submatrix, and $\emptyset$ denotes an all-0 submatrix. Components of kTDS matrices have the following properties:

- components have no all-0 rows and no all-0 columns,
- components are kTDS matrices in their own right.

Remark 2.3. If S is a 3TDS matrix with no all-0 rows and no all-0 columns, then in order to achieve the 3-bound, we have two possibilities: either it has at least 2 ones in each row or it has at least 2 ones in each column. Moreover, if S has at least 2 ones in each row (or column), the same is true for each of its components. Since we are interested in the study of rook's graphs with $m \geq n$, we will assume that each 3TDS matrix has at least 2 ones in each row.

In order to describe our main result, we will need the following 3TDS matrices. For $x \geq 1$ and $y \geq 1$ such that $x + y \geq 5$, we define $J(x, y)$ as the $x \times y$ all-1 matrix.

For $x \geq 5$, let $D(x, 3)$ be the $x \times 3$ 3TDS matrix whose first $x - 3$ rows coincide with $(0, 1, 1)$, the $(x - 2)$-th and $(x - 1)$-th rows coincide with $(1, 0, 1)$, and the last row coincides with $(1, 1, 0)$. Below we have the matrix $D(x, 3)$ for $x \in \{5, 6, 7\}$ depicted.

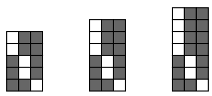

Figure 4. The matrix $D(x, 3)$ for $x \in \{5, 6, 7\}$.

3. On the Construction of Special 3TDS Matrices

We describe how to construct a special class of 3TDS matrices, looking with particular attention at the shape of their components. Moreover, we compute the number of ones in such matrices. Notice that these matrices are exactly the ones appearing in Table 1.

Proposition 3.1. *For any $m \geq n \geq 6$, except $(n, m) = (6, 6)$, there exists an $n \times m$ 3TDS matrix S with no all-0 rows and no all-0 columns with at least 2 ones in each row whose components, up to permutations of the rows and columns, are all $J(1, 4)$ or $J(3, 2)$, except possibly for one of the following cases:*

(i) *exactly one $J(4, 2)$ component;*

(ii) *exactly one $J(1, y)$ component with $5 \leq y \leq 6$;*

(iii) *exactly one $D(5, 3)$ component but no $J(4, 2)$ component;*

(iv) *exactly one $D(6, 3)$ component and no $D(5, 3)$ component or $J(4, 2)$ component or $J(1, y)$ component with $5 \leq y \leq 6$.*

Where only cases (i) *and* (ii), *and* (i) *and* (iii) *can appear simultaneously.*

Proof. Notice that by Table 1, it is enough to show that

- if S is an $n \times m$ 3TDS matrix with the properties we require, then we have a way to construct S' an $n \times (m+1)$ 3TDS matrix with the same properties;
- if S is an $n \times (n+1)$ 3TDS matrix with the properties we require, then we have a way to construct S' an $(n+1) \times (n+1)$ 3TDS matrix with the same properties.

Let now S be an $n \times m$ 3TDS matrix with the properties we require. We will apply the following rules to obtain S' an $n \times (m+1)$ 3TDS matrix with the same properties.

1. If S contains a $J(1,6)$ component and a $D(5,3)$ component, we obtain S' by transforming these two components in two $J(1,4)$ components and a $J(4,2)$ component, and leaving the other components unchanged.
2. If S contains a $J(1,4)$ component and a $D(6,3)$ component, we obtain S' by transforming these two components in one $J(1,4)$ component and two $J(3,2)$ components, and leaving the other components unchanged.
3. If S contains a $J(1,6)$ component, a $J(4,2)$ component and a $J(3,2)$ component, we obtain S' by transforming these three components in two $J(1,4)$ components and a $D(6,3)$ component, and leaving the other components unchanged.
4. If S contains a $J(1,6)$ component, two $J(3,2)$ components and no $J(4,2)$ or $D(5,3)$ components, we obtain S' by transforming these three components in two $J(1,4)$ components and a $D(5,3)$ component, and leaving the other components unchanged.
5. In all other cases not covered by the previous rules, since S always contains at least one $J(1,y)$ component with $4 \leq y \leq 5$, we obtain S' by transforming the $J(1,y)$ component in a $J(1,y+1)$ component, and leaving the other components unchanged.

Remark that if we can apply the rule, when $i \in \{1,\ldots,4\}$, then no other rule $j \in \{1,\ldots,4\}$ with $j \neq i$ can be applied. Moreover, we will use rule 5 in the case that none of the other rules can be used.

Notice that if S has only one $J(1,6)$ component and only one $J(4,2)$ component, then $n=5$. Similarly, if S has only one $J(1,6)$ component and only one $J(3,2)$ component, then $n=4$.

Let now S be an $n \times (n+1)$ 3TDS matrix with the properties we require. We will apply the following rules to obtain S' an $(n+1) \times (n+1)$ 3TDS matrix with the same properties.

1. If S contains a $J(1,y)$ component, with $y=5,6$ and a $J(4,2)$ components, we obtain S' by transforming these two components in a $J(1,y-1)$ component and a $D(5,3)$ component, and leaving the other components unchanged.

2. If S contains a $J(1, y)$ component, with $y = 5, 6$ and a $D(5, 3)$ component, we obtain S' by transforming these two components in a $J(1, y - 1)$ component and two $J(3, 2)$ components, and leaving the other components unchanged.
3. If S contains two $J(1, 4)$ components and a $D(6, 3)$ components, we obtain S' by transforming these three components in a $J(1, 6)$ component, a $J(3, 2)$ component and a $D(5, 3)$ component, and leaving the other components unchanged.
4. If S contains two $J(1, 4)$ components, a $J(4, 2)$ component and no $J(1, y)$ components, with $y = 5, 6$, we obtain S' by transforming these three components in a $J(1, 6)$ component and two $J(3, 2)$ components, and leaving the other components unchanged.
5. If S contains a $D(5, 3)$ components and no $J(1, y)$ components, with $y = 5, 6$, we obtain S' by transforming this component in a $D(6, 3)$ component, and leaving the other components unchanged.
6. In all other cases not covered by the previous rules, since S always contains at least one $J(3, 2)$ component, we obtain S' by transforming this component in a $J(4, 2)$ component, and leaving the other components unchanged.

Remark that, also for this second set of rules, if we can apply the rule, when $i \in \{1, \dots, 5\}$, then none other rule $j \in \{1, \dots, 5\}$ with $j \neq i$ can be applied. Moreover, we will use rule 6 in the case that none of the other rules can be used.

Notice that if S has only one $J(1, 4)$ component, only one $J(4, 2)$ component and no $J(1, y)$ components, with $y = 5, 6$, then $n = 5$, or $n = m = 8$ or $n > m$. Similarly, if S has only one $J(1, 4)$ component and one $D(6, 3)$ component, then $n = m = 7$ or $n > m$. ■

We can now compute the number of ones in a 3TDS matrix satisfying the requirements of the previous proposition.

Proposition 3.2. *For any integer $m \geq n \geq 6$, except $(n, m) = (6, 6)$, let $2n \equiv 3m + k \pmod{10}$, where $0 \leq k \leq 9$. Then the number of ones in a matrix of Proposition 3.1 is given by*

$$\begin{cases} \left\lceil \frac{8n+3m}{5} \right\rceil & \text{if } k = 0, 1, 2, 3, 4, 7, 8, 9; \\ \left\lceil \frac{8n+3m}{5} \right\rceil + 1 & \text{if } k = 5, 6. \end{cases}$$

Proof. Let S be an $n \times m$ 3TDS matrix with no all-0 rows and no all-0 columns with at least 2 ones in each row as described in Proposition 3.1. Let a be the number of $J(1, 4)$ components in S and let b be the number of $J(3, 2)$ components in S. To prove our statement we have to analyze six cases.

Case I. Assume S has only $J(1,4)$ and $J(3,2)$ components. Then

$$n = a + 3b,$$
$$m = 4a + 2b,$$

and the number of ones in S is $(4a + 6b) = (8n + 3m)/5$. In this case, we have $2n \equiv 3m \pmod{10}$.

Case II. Assume S has $J(1,4)$ components, $J(3,2)$ components and one $J(4,2)$ component. Then

$$n = a + 3b + 4,$$
$$m = 4a + 2b + 2,$$

and the number of ones in S is $(4a+6b+8) = (8n+3m+2)/5 = \lceil (8n + 3m)/5 \rceil$. In this case, we have $2n \equiv 3m + 2 \pmod{10}$.

Case III. Assume S has $J(1,4)$ components, $J(3,2)$ components and one $J(1,y)$ component with $5 \le y \le 6$. We have

$$n = a + 3b + 1,$$
$$m = 4a + 2b + y,$$

and the number of ones in S is

$$\begin{aligned}(4a + 6b) + y &= (8n + 3m + 2y - 8)/5 \\ &= \begin{cases} (8n + 3m + 2)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } y = 5; \\ (8n + 3m + 4)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } y = 6. \end{cases}\end{aligned}$$

In this case we have $2n \equiv 3m - 3y + 2 \pmod{10}$, i.e., $2n \equiv 3m + 7, 3m + 4 \pmod{10}$ when $y = 5, 6$, respectively.

Case IV. Assume S has $J(1,4)$ components, $J(3,2)$ components, a $J(1,y)$ component with $5 \le y \le 6$ and a $J(4,2)$ component. We have

$$n = a + 3b + 5,$$
$$m = 4a + 2b + y + 2,$$

and the number of ones in S is

$$\begin{aligned}(4a + 6b) + y + 8 &= (8n + 3m + 2y - 6)/5 \\ &= \begin{cases} (8n + 3m + 4)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } y = 5; \\ (8n + 3m + 6)/5 = \lceil (8n + 3m)/5 \rceil + 1 & \text{if } y = 6. \end{cases}\end{aligned}$$

In this case we have $2n \equiv 3m - 3y + 4 \pmod{10}$, i.e., $2n \equiv 3m + 9, 3m + 6 \pmod{10}$ when $y = 5, 6$, respectively.

Case V. Assume S has $J(1,4)$ components, $J(3,2)$ components and one $D(x,3)$ component with $5 \leq x \leq 6$. We have

$$\begin{aligned} n &= a + 3b + x, \\ m &= 4a + 2b + 3, \end{aligned}$$

and the number of ones in S is

$$\begin{aligned} (4a + 6b) + 2x &= (8n + 3m + 2x - 9)/5 \\ &= \begin{cases} (8n + 3m + 1)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } x = 5; \\ (8n + 3m + 3)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } x = 6. \end{cases} \end{aligned}$$

In this case we have $2n \equiv 3m + 2x + 1 \pmod{10}$, i.e., $2n \equiv 3m + 1, 3m + 3 \pmod{10}$ when $x = 5, 6$, respectively.

Case VI. Assume S has $J(1,4)$ components, $J(3,2)$ components, a $J(1,y)$ component with $5 \leq y \leq 6$ and a $D(5,3)$ component. We have

$$\begin{aligned} n &= a + 3b + 6, \\ m &= 4a + 2b + y + 3, \end{aligned}$$

and the number of ones in S is

$$\begin{aligned} (4a + 6b) + 10 + y &= (8n + 3m + 2y - 7)/5 \\ &= \begin{cases} (8n + 3m + 3)/5 = \lceil (8n + 3m)/5 \rceil & \text{if } y = 5; \\ (8n + 3m + 5)/5 = \lceil (8n + 3m)/5 \rceil + 1 & \text{if } y = 6. \end{cases} \end{aligned}$$

In this case we have $2n \equiv 3m - 3y + 3 \pmod{10}$, i.e., $2n \equiv 3m + 8, 3m + 5 \pmod{10}$ when $y = 5, 6$, respectively. ■

Remark 3.3. A direct computation shows that when $n \in \{4, 5\}$ and $(n, m) = (6, 6)$, we can compute the number of ones of the matrices in Table 1 with no all-0 rows and no all-0 columns with the formula of Proposition 3.2.

4. USEFUL INEQUALITIES FOR MIN-3TDS

We prove several inequalities for $\gamma_{\times 3,t}(K_n \Box K_m)$. Specifically, we show how $\gamma_{\times 3,t}$ changes when, in a 3TDS matrix, we increase the number of rows or columns in the general case, or both in the square case. The first lemma describes a lower bound for the number of ones in a 3TDS matrix.

Lemma 4.1. *Let $m \geq n \geq 3$. Then $\gamma_{\times 3,t}(K_n \Box K_m) \geq 2n+2$.*

Proof. Suppose that $\gamma_{\times 3,t}(K_n \Box K_m) \leq 2n+1$ and let S be an $n \times m$ 3TDS matrix with $|S| = 2n+1$. Since $2n+1 < 3n$, by Lemma 2.1, S has no all-0 rows or all-0 columns. Since by Remark 2.3 we can assume that $\mathfrak{r}(i) \geq 2$ for all $1 \leq i \leq n$, then S has one row with 3 ones and $n-1$ rows with 2 ones. Without loss of generality, we can assume that the first row of S has 3 ones in the first three entries. As a consequence, $\mathfrak{r}(i) = 2$ for all $2 \leq i \leq n$. For all $1 \leq j \leq 3$, $\kappa(1,j) = 3 + \mathfrak{c}(j) - 2 = \mathfrak{c}(j) + 1 \geq 3$, and hence $\mathfrak{c}(j) \geq 2$, i.e., each of the first three columns of S has at least 2 ones. Moreover, if $m \geq 4$, since S has no all-0 rows or all-0 columns, for all $4 \leq j \leq m$ there must exists $2 \leq i \leq n$ such that S has a one in position (i,j). Then $\kappa(i,j) = 2 + \mathfrak{c}(j) - 2 = \mathfrak{c}(j) \geq 3$, i.e., each of the last $m-3$ columns of S has at least 3 ones.

Assume $n = 3$. If $m \geq 4$, since $\mathfrak{c}(j) \geq 2$ for all $1 \leq j \leq 3$ and $\mathfrak{c}(j) \geq 3$ for all $4 \leq j \leq m$, then $|S| \geq 6 + 3(m-3) = 3m-3 > 2n-1$. We can then assume that $m = 3$ and that the zeros of S are in positions $(2,2)$ and $(3,1)$. However, $\kappa(2,1) = 2$ and so S is not a 3TDS matrix.

Assume now $n = 4$. Since $\mathfrak{c}(4) \geq 3$, the last column of S is $(0,1,1,1)^t$. Hence, we can assume that the remaining ones of S are in positions $(2,1)$, $(3,2)$ and $(4,3)$. However, $\kappa(2,1) = 2$ and so S is not a 3TDS matrix.

Assume now $n \geq 5$. Counting the ones of S by columns we obtain that $|S| \geq 6 + 3(m-3) = 3m-3$. However, since $m \geq n \geq 5$, $3m-3 > 2n+1$ and so S is not a 3TDS matrix. ∎

Remark 4.2. If $3 \leq n \leq 10$, then by Lemma 4.1, the $n \times n$ 3TDS matrices of Table 1 are min-3TDS and so $\gamma_{\times 3,t}(K_n \Box K_n) = 2n+2$.

We are now able to compute $\gamma_{\times 3,t}(K_3 \Box K_m)$ for all $m \geq 3$.

Lemma 4.3. *If $m \geq 3$, then*

$$\gamma_{\times 3,t}(K_3 \Box K_m) = \begin{cases} 8 & \text{if } m = 3,4; \\ 9 & \text{if } m \geq 5. \end{cases}$$

Proof. By Lemma 4.1, $\gamma_{\times 3,t}(K_3 \Box K_m) \geq 8$. Looking at the 3TDS matrices in Table 1, we obtain that $\gamma_{\times 3,t}(K_3 \Box K_3) = \gamma_{\times 3,t}(K_3 \Box K_4) = 8$.

Let now $m \geq 5$. Suppose that there exists S a $3 \times m$ 3TDS matrix with $|S| = 8$. By Remark 2.3, this implies that there exists $1 \leq i \leq 3$ such that $\mathfrak{r}(i) = 2$. Without loss of generality we can assume that $i = 3$ and that the last row of S coincides with $(0, \ldots, 0, 1, 1)$. If $m-1 \leq j \leq m$, then $\kappa(3,j) = 2 + \mathfrak{c}(j) - 2 \geq 3$. This implies that $\mathfrak{c}(m-1) = \mathfrak{c}(m) = 3$. Moreover, by Lemma 2.1, S has no all-0 rows or all-0 columns, and hence $\mathfrak{c}(j) \geq 1$ for all $1 \leq j \leq m-2$. This implies that $|S| = \sum_{j=1}^{m} \mathfrak{c}(j) \geq (m-2) + 6 > 8$, but this is a contradiction. ∎

The following result describes the relation between min-3TDS matrices that have the same number of rows but whose number of columns differs by one.

Lemma 4.4. *Let* $m \geq n \geq 3$. *Then*

$$\gamma_{\times 3,t}(K_n \Box K_m) \leq \gamma_{\times 3,t}(K_n \Box K_{m+1}) \leq \gamma_{\times 3,t}(K_n \Box K_m) + 1.$$

Proof. Firstly we will prove the first inequality $\gamma_{\times 3,t}(K_n \Box K_m) \leq \gamma_{\times 3,t}(K_n \Box K_{m+1})$.

If $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = 3n$, the first inequality holds by Proposition 2.2. Let S be an $n \times (m+1)$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n \Box K_m) - 1 < 3n$. By Lemma 2.1, S has no all-0 rows or all-0 columns. Furthermore, since $|S| < 3n$, then there exists $1 \leq j \leq m+1$ such that $\mathfrak{c}_S(j) \leq 2$. Hence, without loss of generality, we can assume that $j = m+1$. This fact is crucial for the rest of the proof.

If $n = 3$, the first inequality holds by Lemma 4.3. Assume now $m \geq n \geq 4$. If $\mathfrak{c}_S(m+1) = 1$, then we can assume that the last column of S is $(1, 0, \ldots, 0)^t$. Since $\kappa_S(1, m+1) = \mathfrak{r}_S(1) + 1 - 2 \geq 3$, then $\mathfrak{r}_S(1) \geq 4$. Consider S' the matrix obtained from S by deleting the last column. Notice that $\mathfrak{r}_{S'}(1) \geq 3$. S' is an $n \times m$ matrix with $|S| - 1 = \gamma_{\times 3,t}(K_n \Box K_m) - 2$ ones, and hence S' is not a 3TDS matrix, by definition of $\gamma_{\times 3,t}$. However, $\kappa_{S'}(i, j) = \kappa_S(i, j) \geq 3$, if $2 \leq i \leq n$ and $1 \leq j \leq m$. Since S' is not a 3TDS matrix, there exists $1 \leq j \leq m$ such that $\kappa_{S'}(1, j) \leq 2$. This implies that $\mathfrak{r}_{S'}(1) = 3$ and so that $\mathfrak{r}_S(1) = 4$. Since $m + 1 \geq 5$, the first row of S has at least one zero in the first m entries. We can construct S'' an $n \times m$ matrix obtained from S by deleting the last column and putting exactly 1 one in one of the zeros of the first row. By construction, S'' is an $n \times m$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n \Box K_m) - 1$ ones, but this is a contradiction.

If $\mathfrak{c}_S(m+1) = 2$, then we can assume that the last column of S is equal to $(1, 1, 0, \ldots, 0)^t$. Let S' be the matrix obtained from S by deleting the last column. S' is an $n \times m$ matrix with $|S| - 2 = \gamma_{\times 3,t}(K_n \Box K_m) - 3$ ones, and hence it is not a 3TDS matrix. However, $\kappa_{S'}(i, j) = \kappa_S(i, j) \geq 3$, if $3 \leq i \leq n$ and $1 \leq j \leq m$. This implies that at least one of the first two rows of S have exactly 3 ones. Assume it is the first one. Since $m + 1 \geq 5$, then the first row of S has at least 2 zeros in the first m entries. If any of the first m columns of S have 2 zeros in the first two rows, we can construct S'' an $n \times m$ matrix obtained from S by deleting the last column and putting 2 ones in the first two entries of such column. If such column does not exist, we can construct S'' an $n \times m$ matrix obtained from S by deleting the last column and putting exactly 1 one in one zero of the first row and, if the second row has a zero, 1 one there. By construction, S'' is an $n \times m$ 3TDS matrix with at most $|S| = \gamma_{\times 3,t}(K_n \Box K_m) - 1$ ones, but this is a contradiction. This proves the first inequality.

We are now ready to prove the second inequality, i.e., $\gamma_{\times 3,t}(K_n \Box K_{m+1}) \leq \gamma_{\times 3,t}(K_n \Box K_m) + 1$. Let S be a minimum $n \times m$ 3TDS matrix. By Lemma 4.1, we have that $\gamma_{\times 3,t}(K_n \square K_m) \geq 2n+2$ and hence there exists $1 \leq i \leq n$ such that $\mathfrak{r}_S(i) \geq 3$. Without loss of generality we can assume that $i = n$. Consider now S' an $n \times (m+1)$ matrix such that the first m columns coincide with S and the last column is $(0, \dots, 0, 1)^t$. By construction, S' is an $n \times (m+1)$ 3TDS matrix with $|S| + 1 = \gamma_{\times 3,t}(K_n \square K_m) + 1$ ones. ■

The next lemma describes the relation between min-3TDS matrices that have the same number of columns but whose number of rows differs by one.

Lemma 4.5. *Let $m > n \geq 3$, and assume $\gamma_{\times 3,t}(K_n \square K_m) < 3n$. Then*

$$\gamma_{\times 3,t}(K_n \square K_m) \leq \gamma_{\times 3,t}(K_{n+1} \square K_m) \leq \gamma_{\times 3,t}(K_n \square K_m) + 2.$$

Proof. Firstly we will prove the first inequality $\gamma_{\times 3,t}(K_n \square K_m) \leq \gamma_{\times 3,t}(K_{n+1} \square K_m)$.

Let S be an $(n+1) \times m$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n \square K_m) - 1$. Since $|S| < 3n$, by Remark 2.3 there exists $1 \leq i \leq n+1$ such that $\mathfrak{r}_S(i) = 2$. Without loss of generality, we can assume that $i = n+1$ and that the last row of S is $(0, \dots, 0, 1, 1)$. Consider S' the matrix obtained from S by deleting the last row. S' is an $n \times m$ matrix with $|S| - 2 = \gamma_{\times 3,t}(K_n \square K_m) - 3$ ones, and then it is not a 3TDS matrix. However, $\kappa_{S'}(i,j) = \kappa_S(i,j) \geq 3$, if $1 \leq i \leq n$ and $1 \leq j \leq m-2$. This implies that at least one of the last two columns of S have exactly 3 ones. Assume that this column is the last of S. Since $n+1 \geq 4$, the last column of S has at least one zero in the first n entries. If any of the first n rows of S have 2 zeros in the last two columns, we can construct S'' an $n \times m$ matrix obtained from S by deleting the last row and putting 2 ones in the last two entries of such row. If such row does not exist but the penultimate column of S has a zero, we can construct S'' an $n \times m$ matrix obtained from S by deleting the last row and putting exactly 1 one in one zero of the penultimate column and exactly 1 one in one zero of the last column. If the penultimate column has no zero, we can construct S'' an $n \times m$ matrix obtained from S by deleting the last row and putting exactly 1 one in one zero of the last column. By construction, S'' is an $n \times m$ 3TDS matrix with at most $|S| = \gamma_{\times 3,t}(K_n \square K_m) - 1$ ones, but this is a contradiction. This proves the first inequality.

We are now ready to prove the second inequality, i.e., $\gamma_{\times 3,t}(K_{n+1} \square K_m) \leq \gamma_{\times 3,t}(K_n \square K_m) + 2$. Let S be a minimum $n \times m$ 3TDS matrix. By assumption, we have that $\gamma_{\times 3,t}(K_n \square K_m) < 3n$ and hence there exists $1 \leq i \leq n$ such that $\mathfrak{r}_S(i) = 2$. Without loss of generality we can assume that $i = n$ and that the last row of S coincides with $(0, \dots, 0, 1, 1)$. Consider now S' an $(n+1) \times m$ matrix such that the first n rows coincide with S and the last row is $(0, \dots, 0, 1, 1)$. By construction, S' is an $(n+1) \times m$ 3TDS matrix with $|S| + 2 = \gamma_{\times 3,t}(K_n \square K_m) + 2$ ones. ■

We now describe the relation between square min-3TDS matrices whose number of rows and columns both differ by one.

Lemma 4.6. *Let $n \geq 3$. Then*

$$\gamma_{\times 3,t}(K_n \square K_n) + 2 \leq \gamma_{\times 3,t}(K_{n+1} \square K_{n+1}) \leq \gamma_{\times 3,t}(K_n \square K_n) + 3.$$

Proof. Firstly, we will prove the first inequality, i.e., $\gamma_{\times 3,t}(K_n \square K_n) + 2 \leq \gamma_{\times 3,t}(K_{n+1} \square K_{n+1})$.

If $n = 3, 4$, the first inequality follows from Remark 4.2. Assume $n \geq 5$. Suppose there exists S an $(n+1)\times(n+1)$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n \square K_n)+1$. By Proposition 2.2, $\gamma_{\times 3,t}(K_n \square K_n)+1 \leq 3n+1 < 3(n+1)$, and hence by Remark 2.3, there exists $1 \leq i \leq n+1$ such that $\mathfrak{r}_S(i) = 2$. Without loss of generality we can assume that $i = n+1$ and that the last row of S coincides with $(0, \dots, 0, 1, 1)$. Let now S' be the $n \times (n+1)$ matrix obtained from S by deleting the last row. S' has $|S| - 2 = \gamma_{\times 3,t}(K_n \square K_n) - 1$ ones, and hence it is not a 3TDS by Lemma 4.4. However, $\kappa_{S'}(i, j) = \kappa_S(i, j) \geq 3$ for all $1 \leq i \leq n$ and $1 \leq j \leq n-1$. Since S is a 3TDS matrix, if $n \leq j \leq n+1$, then $\kappa_S(n+1, j) = 2 + \mathfrak{c}_S(j) - 2 = \mathfrak{c}_S(j) \geq 3$, and hence $\mathfrak{c}_{S'}(j) \geq 2$. However, since S' is not a 3TDS matrix, there exist $1 \leq i \leq n$ and $n \leq j \leq n+1$ such that $\kappa_{S'}(i, j) = 2$, and hence $\mathfrak{c}_{S'}(n) = 2$ or $\mathfrak{c}_{S'}(n+1) = 2$. Without loss of generality, we can assume that $(i, j) = (1, n+1)$, and hence that the last column of S' is $(1, 1, 0, \dots, 0)^t$ and $\mathfrak{r}_{S'}(1) = 2$.

Assume that $\mathfrak{c}_{S'}(n) \geq 3$. If in S' there is a column with 2 zeros in the first two entries, we can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 2 ones in the first two entries of such column. If such column does not exist, then $\mathfrak{r}_{S'}(2) \geq 4$. Furthermore, since $|S| < 3(n+1)$, there must exists a column with 2 zeros, one in the first entry and the second one in the j-th position, for some $j \geq 3$. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in the first entry and 1 one in the j-th position of such column. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \square K_n) - 1$ ones, but this is a contradiction.

Assume now that $\mathfrak{c}_{S'}(n) = 2$. Denote by w the penultimate column of S'. There are four cases. If w has 2 zeros in the first two entries, then we can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 2 ones in the first two entries of w. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \square K_n) - 1$ ones, but this is a contradiction.

If $w = (1, 0, \dots)^t$, then the first row of S' is equal to $(0, \dots, 0, 1, 1)$. Since $|S| < 3(n+1)$, there must exists $1 \leq j \leq n-1$ such that $\mathfrak{c}_{S'}(j) = \mathfrak{c}_S(j) \geq 2$. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, j)$ and 1 one in position $(2, n)$. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \square K_n) - 1$ ones, but this is impossible.

Assume $w = (0, 1, \dots)^t$. If $\mathfrak{r}_{S'}(2) = 2$, then the second row of S' is equal to $(0, \dots, 0, 1, 1)$. Since $|S| < 3(n+1)$, there must exists $1 \le j \le n-1$ such that $\mathfrak{c}_{S'}(j) \ge 2$. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, n)$ and 1 one in position $(2, j)$. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, but this is a contradiction. If $\mathfrak{r}_{S'}(2) \ge 3$, but there is at least one zero in the second row of S', we can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, n)$ and exactly 1 one in one zero of the second row. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, but this is a contradiction. If $\mathfrak{r}_{S'}(2) = n+1$, we can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, n)$ and exactly 1 one in one zero of the first row. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, and this is a contradiction.

If $w = (1, 1, 0, \dots, 0)^t$, then the first row of S' is equal to $(0, \dots, 0, 1, 1)$. Since $|S| < 3(n+1)$, there must exists $1 \le j \le n-1$ such that $\mathfrak{c}_{S'}(j) \ge 2$, and we can assume that S' has ones in positions (p, j) and (q, j), for some $2 \le p < q \le n$. If $\mathfrak{r}_{S'}(2) = 2$, then the second row of S' is equal to $(0, \dots, 0, 1, 1)$. This implies that the first two entries of the j-th column of S' are zero. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, j)$, 1 one in position $(2, j)$, 1 one in position (p, n) and putting 1 zero in position (p, j). By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, however this is a contradiction. If $\mathfrak{r}_{S'}(2) = 3$, then, without loss of generality, we can assume that the second row of S' is equal to $(0, \dots, 0, 1, 1, 1)$. Since $\kappa_{S'}(2, n-1) = \kappa_S(2, n-1) \ge 3$, this implies that $\mathfrak{c}_{S'}(n-1) \ge 2$, and we can assume that S' has ones in positions $(2, n-1)$ and $(i, n-1)$, with $3 \le i \le n$. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, n-1)$ and 1 one in position (i, n). By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, but this is a contradiction. If $\mathfrak{r}_{S'}(2) \ge 4$, then, without loss of generality, we can assume that the second row of S' is equal to $(\dots, 1, 1, 1, 1)$. We can construct S'' an $n \times n$ matrix obtained from S' by deleting the last column and putting 1 one in position $(1, n-1)$ and 1 one in position $(1, n-2)$. By construction, S'' is an $n \times n$ 3TDS matrix with $|S'| = \gamma_{\times 3,t}(K_n \Box K_n) - 1$ ones, but this is a contradiction.

We are now ready to prove the second inequality, i.e., $\gamma_{\times 3,t}(K_{n+1} \Box K_{n+1}) \le \gamma_{\times 3,t}(K_n \Box K_n) + 3$. By Proposition 3.2, Remark 3.3 and Lemma 4.3, we have that $\gamma_{\times 3,t}(K_n \Box K_{n+1}) < 3n$. By Lemmas 4.4 and 4.5, we have that $\gamma_{\times 3,t}(K_{n+1} \Box K_{n+1}) \le \gamma_{\times 3,t}(K_n \Box K_{n+1}) + 2 \le \gamma_{\times 3,t}(K_n \Box K_n) + 3$. ■

Remark 4.7. Let $m \ge n \ge 3$ and S be an $n \times m$ 3TDS matrix with no all-0 columns or all-0 rows. Let $1 \le k \le n-1$ and assume that S has $2n + k$ ones.

In order to maximize the number of columns with at most 2 ones, we have to maximize the number of row with 3 or more ones. The way to do that is to maximize the number of row with exactly 4 ones. Since S has at least 2 ones in each row and $|S| = 2n + k$ we have at most $\left\lfloor \frac{k}{2} \right\rfloor$ row with 4 ones. This implies that S has at most $k + 2\left\lfloor \frac{k}{2} \right\rfloor$ columns that contain a one belonging to a row with at least 3 ones. Hence all the other columns contain at least 1 one belonging to a row with 2 ones, and so such columns all have at least 3 ones. This shows that $|S| \geq \left(k + 2\left\lfloor \frac{k}{2} \right\rfloor\right) + 3\left(m - k - 2\left\lfloor \frac{k}{2} \right\rfloor\right) = 3m - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor$.

5. The Square Case

We consider the case when $n = m$ and we give an explicit formula for $\gamma_{\times 3,t}(K_n \square K_n)$ that is independent from the component structure of square 3TDS matrices. Notice that our formula coincides with the number of ones of the square matrices appearing in Table 1.

Theorem 5.1. *For any integer $n \geq 3$, let $n \equiv r \pmod{10}$, where $0 \leq r \leq 9$. Then*

$$\gamma_{\times 3,t}(K_n \square K_n) = \begin{cases} 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2 & \text{if } r = 4,5,6,7,8,9; \\ 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil & \text{if } r = 0,1,2,3 \text{ and } n \neq 3; \\ 8 & \text{if } n = 3. \end{cases}$$

Proof. If $n = 3$, then $\gamma_{\times 3,t}(K_n \square K_n) = 8$ by Lemma 4.3. Assume $n \geq 4$. Since the description of Proposition 3.2 and Remark 3.3 coincides with our claim when $n = m$, we clearly have that

$$\gamma_{\times 3,t}(K_n \square K_n) \leq \begin{cases} 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2 & \text{if } r = 4,5,6,7,8,9; \\ 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil & \text{if } r = 0,1,2,3 \text{ and } n \neq 3. \end{cases}$$

Notice that when $n \equiv r \pmod{10}$ and $r = 1, 2$, then $\left(2n + 2\left\lfloor \frac{n}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil\right) + 2 = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil$. This implies that if $\gamma_{\times 3,t}(K_n \square K_n) = 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil$, then by Lemma 4.6, $\gamma_{\times 3,t}(K_{n+1} \square K_{n+1}) = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + \left\lceil \frac{r}{3} \right\rceil$. Similarly, when $r = 4, 5, 6, 7, 8$, then $\left(2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2\right) + 2 = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + 2$. This implies that if $\gamma_{\times 3,t}(K_n \square K_n) = 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2$, then by Lemma 4.6, $\gamma_{\times 3,t}(K_{n+1} \square K_{n+1}) = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + 2$. Moreover, when $r = 9$, $\left(2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2\right) + 2 = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor$. This implies that if $\gamma_{\times 3,t}(K_n \square K_n) = 2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 2$, then by Lemma 4.6, $\gamma_{\times 3,t}(K_{n+1} \square K_{n+1}) = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor$. However, when $r = 0$, then $\left(2n + 2\left\lfloor \frac{n}{10} \right\rfloor\right) + 3 = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + 1$, and hence in this situation we need to prove that $\gamma_{\times 3,t}(K_{n+1} \square K_{n+1}) = \gamma_{\times 3,t}(K_n \square K_n) + 3$. Similarly when $r = 3$, $\left(2n + 2\left\lfloor \frac{n}{10} \right\rfloor + 1\right) + 3 = 2(n+1) + 2\left\lfloor \frac{n+1}{10} \right\rfloor + 2$, and hence also in this situation we need to prove that $\gamma_{\times 3,t}(K_{n+1}\square K_{n+1}) = \gamma_{\times 3,t}(K_n\square K_n) + 3$. By Lemma 4.6, it is enough to show that if $r = 0, 3$, then $\gamma_{\times 3,t}(K_{n+1}\square K_{n+1}) > \gamma_{\times 3,t}(K_n\square K_n) + 2$.

Assume $r = 0$. Suppose there exists S an $(n+1)\times(n+1)$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n\square K_n) + 2 = 2(n+1) + k$, where $k = 2\left\lfloor \frac{n}{10} \right\rfloor$. By Remark 4.7, $|S| \geq 3(n+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor$. Notice that since $r = 0$, then $k = \frac{n}{5}$ and it is an even integer. This implies that $|S| \geq 3(n+1) - 4k$. However, since $k = \frac{n}{5}$, then $2(n+1) + k < 3(n+1) - 4k$ and hence S is not a 3TDS matrix.

Assume now $r = 3$. Suppose there exists S an $(n+1)\times(n+1)$ 3TDS matrix with $|S| = \gamma_{\times 3,t}(K_n\square K_n) + 2 = 2(n+1) + k$, where $k = 2\left\lfloor \frac{n}{10} \right\rfloor + 1$. By Remark 4.7, $|S| \geq 3(n+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor$. Notice that since $r = 3$, then $k = \frac{n-3}{5} + 1$ and it is an odd integer. This implies that $|S| \geq 3(n+1) - 4k + 2$. However, since $k = \frac{n-3}{5} + 1$, then $2(n+1) + k < 3(n+1) - 4k + 2$ and hence S is not a 3TDS matrix. ■

6. The General Case

We give a formula for $\gamma_{\times 3,t}(K_n\square K_m)$ that coincides with the number of ones of the 3TDS matrices in Table 1, but the argument is independent of the shape of the components in a 3TDS matrix.

Theorem 6.1. Let $m \geq n \geq 1$. Assume $(n,m) \notin \{(1,1),(1,2),(1,3),(2,2)\}$ and $2n \equiv 3m + k \pmod{10}$, where $0 \leq k \leq 9$. Then

$$\gamma_{\times 3,t}(K_n\square K_m) = \begin{cases} 4 & n = 1 \text{ and } m \geq 4; \\ 6 & n = 2 \text{ and } m \geq 3; \\ 8 & n = 3 \text{ and } m = 3, 4; \\ 3n & \text{if } m \geq \left\lfloor \frac{7n-1}{3} \right\rfloor - 1; \\ \left\lceil \frac{8n+3m}{5} \right\rceil & \text{if } k = 0,1,2,3,4,7,8,9; \\ \left\lceil \frac{8n+3m}{5} \right\rceil + 1 & \text{if } k = 5, 6. \end{cases}$$

Proof. If $(n,m) \in \{(1,1),(1,2),(1,3),(2,2)\}$, then there are no $n\times m$ 3TDS matrices. If $n = 1$ and $m \geq 4$, then any $1\times m$ $(0,1)$-matrix with exactly 4 ones is a min-3TDS matrix. If $n = 2$ and $m \geq 3$, then any $2\times m$ $(0,1)$-matrix with exactly 3 columns with 2 ones is a min-3TDS matrix. If $n = 3$ and $m = 3, 4$, by Lemma 4.3, $\gamma_{\times 3,t}(K_3\square K_3) = \gamma_{\times 3,t}(K_3\square K_4) = 8$.

Assume $n \geq 4$. By Propositions 2.2 and 3.2, and Remark 3.3, we have that

$$\gamma_{\times 3,t}(K_n\square K_m) \leq \begin{cases} \min\left\{\left\lceil \frac{8n+3m}{5} \right\rceil, 3n\right\} & \text{if } k = 0,1,2,3,4,7,8,9; \\ \min\left\{\left\lceil \frac{8n+3m}{5} \right\rceil + 1, 3n\right\} & \text{if } k = 5, 6. \end{cases}$$

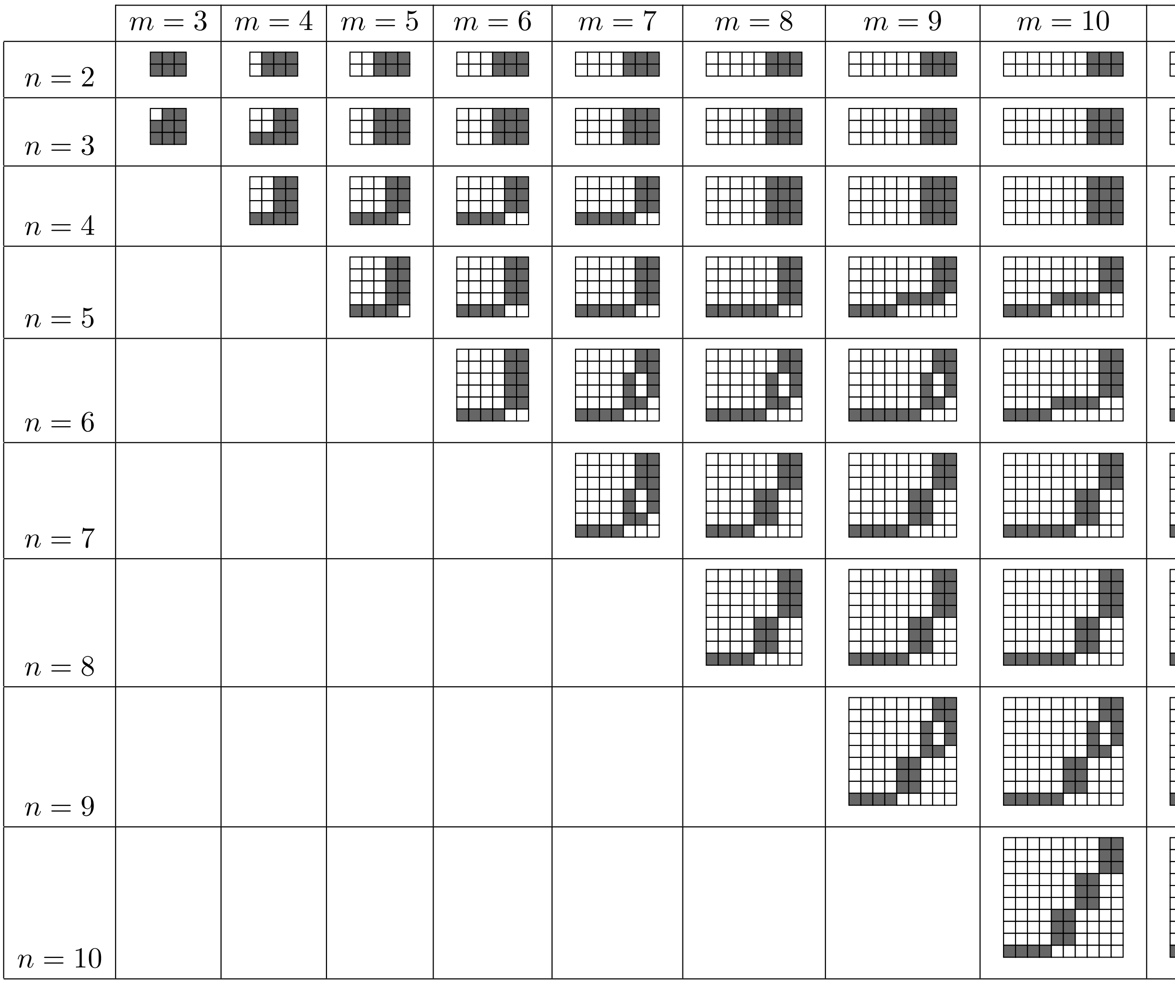

Table 1. Small min-3TDS matrices.

If $m > \frac{7n-1}{3} - 2$ (which occurs when $m \geq \lfloor \frac{7n-1}{3} \rfloor - 1$), then $\lceil \frac{8n+3m}{5} \rceil + 1 \geq 3n$, in which case the previous minimums coincide with $3n$. Assume now $m < \lfloor \frac{7n-1}{3} \rfloor - 1$. Since we assume $m \geq n$, we can write $m = n + d$, for some $d \geq 0$. When $n = m$, a direct computation shows that our formula coincides with the one of Theorem 5.1. Hence, using induction on m, it is enough to show that if $\gamma_{\times 3,t}(K_n \Box K_m)$ coincides with our formula, so does $\gamma_{\times 3,t}(K_n \Box K_{m+1})$. We will prove this with a case by case analysis.

Case I. Assume $2n \equiv 3m \pmod{10}$ and $m = n + d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil = 2n + \lceil \frac{n+3d}{5} \rceil = 2n + k$. Notice that $k = \frac{n+3d}{5}$ and it is an even integer. Since $2n \equiv 3m \pmod{10}$, then $2n \equiv 3(m+1) + 7 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k+1$. By hypothesis, $\gamma_{\times 3,t}(K_n \Box K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \Box K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n+k$. By Remark 4.7, $|S| \geq 3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor$. In this situation, $3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor = 3n + 3d + 3 - 4 \left(\frac{n+3d}{5} \right)$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case II. Assume $2n \equiv 3m + 7 \pmod{10}$ and $m = n + d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil = 2n + \lceil \frac{n+3d}{5} \rceil = 2n + k$. Notice that $k = \frac{n+3d+2}{5}$ and it is an odd integer. Since $2n \equiv 3m+7 \pmod{10}$, then $2n \equiv 3(m+1)+4 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k+1$. By hypothesis, $\gamma_{\times 3,t}(K_n \Box K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \Box K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n+k$. By Remark 4.7, $|S| \geq 3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor$. In this situation, $3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor = 3n + 3d + 3 - 4 \left(\frac{n+3d+2}{5} \right) + 2$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case III. Assume $2n \equiv 3m + 4 \pmod{10}$ and $m = n + d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil = 2n + \lceil \frac{n+3d}{5} \rceil = 2n + k$. Notice that $k = \frac{n+3d+4}{5}$ and it is an even integer. Since $2n \equiv 3m + 4 \pmod{10}$, then $2n \equiv 3(m+1) + 1 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + k$. By Lemma 4.4, $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case IV. Assume $2n \equiv 3m + 1 \pmod{10}$ and $m = n + d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil = 2n + \lceil \frac{n+3d}{5} \rceil = 2n + k$. Notice that $k = \frac{n+3d+1}{5}$ and it is an even integer. Since $2n \equiv 3m+1 \pmod{10}$, then $2n \equiv 3(m+1)+8 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k+1$. By hypothesis, $\gamma_{\times 3,t}(K_n \Box K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \Box K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n + k$. By Remark 4.7, $|S| \geq 3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor$. In this situation, $3(m+1) - 2k - 4 \lfloor \frac{k}{2} \rfloor =$

$3n+3d+3-4\left(\frac{n+3d+1}{5}\right)$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \square K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case V. Assume $2n \equiv 3m+8 \pmod{10}$ and $m = n+d$. Then we can write $\left\lceil \frac{8n+3m}{5} \right\rceil = 2n + \left\lceil \frac{n+3d}{5} \right\rceil = 2n+k$. Notice that $k = \frac{n+3d+3}{5}$ and it is an odd integer. Since $2n \equiv 3m+8 \pmod{10}$, then $2n \equiv 3(m+1)+5 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil + 1 = 2n + \left\lceil \frac{n+3d+3}{5} \right\rceil + 1$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + k + 1$. By hypothesis, $\gamma_{\times 3,t}(K_n \square K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \square K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n+k$. By Remark 4.7, $|S| \geq 3(m+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor$. In this situation, $3(m+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor = 3n+3d+3-4\left(\frac{n+3d+3}{5}\right)+2$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \square K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case VI. Assume $2n \equiv 3m+5 \pmod{10}$ and $m = n+d$. Then we can write $\left\lceil \frac{8n+3m}{5} \right\rceil + 1 = 2n + \left\lceil \frac{n+3d}{5} \right\rceil + 1 = 2n+k$. Notice that $k = \frac{n+3d}{5}$ and it is an odd integer. Since $2n \equiv 3m+5 \pmod{10}$, then $2n \equiv 3(m+1)+2 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \left\lceil \frac{n+3d+3}{5} \right\rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k$. By Lemma 4.4, $\gamma_{\times 3,t}(K_n \square K_{m+1}) = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case VII. Assume $2n \equiv 3m+2 \pmod{10}$ and $m = n+d$. Then we can write $\left\lceil \frac{8n+3m}{5} \right\rceil = 2n + \left\lceil \frac{n+3d}{5} \right\rceil = 2n+k$. Notice that $k = \frac{n+3d+2}{5}$ and it is an even integer. Since $2n \equiv 3m+2 \pmod{10}$, then $2n \equiv 3(m+1)+9 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \left\lceil \frac{n+3d+3}{5} \right\rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k+1$. By hypothesis, $\gamma_{\times 3,t}(K_n \square K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \square K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n+k$. By Remark 4.7, $|S| \geq 3(m+1)-2k-4\left\lfloor \frac{k}{2} \right\rfloor$. In this situation, $3(m+1)-2k-4\left\lfloor \frac{k}{2} \right\rfloor = 3n+3d+3-4\left(\frac{n+3d+2}{5}\right)$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \square K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case VIII. Assume $2n \equiv 3m+9 \pmod{10}$ and $m = n+d$. Then we can write $\left\lceil \frac{8n+3m}{5} \right\rceil = 2n + \left\lceil \frac{n+3d}{5} \right\rceil = 2n+k$. Notice that $k = \frac{n+3d+4}{5}$ and it is an odd integer. Since $2n \equiv 3m+9 \pmod{10}$, then $2n \equiv 3(m+1)+6 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil + 1 = 2n + \left\lceil \frac{n+3d+3}{5} \right\rceil + 1$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + k + 1$. By hypothesis, $\gamma_{\times 3,t}(K_n \square K_m) = 2n+k$. By Lemma 4.4, $2n+k \leq \gamma_{\times 3,t}(K_n \square K_{m+1}) \leq 2n+k+1$. Suppose there exists S an $n \times (m+1)$ 3TDS matrix with $|S| = 2n+k$. By Remark 4.7, $|S| \geq 3(m+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor$. In this situation, $3(m+1) - 2k - 4\left\lfloor \frac{k}{2} \right\rfloor = 3n+3d+3-4\left(\frac{n+3d+4}{5}\right)+2$ and it is strictly bigger than $2n+k$. This implies that S is not a 3TDS matrix and hence that $\gamma_{\times 3,t}(K_n \square K_{m+1}) = 2n+k+1 = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case IX. Assume $2n \equiv 3m+6 \pmod{10}$ and $m = n+d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil + 1 = 2n + \lceil \frac{n+3d}{5} \rceil + 1 = 2n+k$. Notice that $k = \frac{n+3d+1}{5}$ and it is an odd integer. Since $2n \equiv 3m+6 \pmod{10}$, then $2n \equiv 3(m+1)+3 \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k$. By Lemma 4.4, $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$.

Case X. Assume $2n \equiv 3m+3 \pmod{10}$ and $m = n+d$. Then we can write $\lceil \frac{8n+3m}{5} \rceil = 2n + \lceil \frac{n+3d}{5} \rceil = 2n+k$. Notice that $k = \frac{n+3d+3}{5}$ and it is an even integer. Since $2n \equiv 3m+3 \pmod{10}$, then $2n \equiv 3(m+1) \pmod{10}$. Moreover, $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n + \lceil \frac{n+3d+3}{5} \rceil$ and hence $\left\lceil \frac{8n+3(m+1)}{5} \right\rceil = 2n+k$. By Lemma 4.4, $\gamma_{\times 3,t}(K_n \Box K_{m+1}) = \left\lceil \frac{8n+3(m+1)}{5} \right\rceil$. ■

Directly from the formula of Theorem 6.1, we can generalize the statement of Lemma 4.6 and obtain the following.

Corollary 6.2. *Let $m \geq n \geq 3$. Then*

$$\gamma_{\times 3,t}(K_n \Box K_m) + 2 \leq \gamma_{\times 3,t}(K_{n+1} \Box K_{m+1}) \leq \gamma_{\times 3,t}(K_n \Box K_m) + 3.$$

Remark 6.3. Since, in general, $3n-1 > \lfloor \frac{7n-1}{3} \rfloor - 1$, we obtain a better bound than the one described in Proposition 2.2 for the case when $\gamma_{\times 3,t}(K_n \Box K_m) = 3n$.

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