MINIMUM COVERINGS OF CROWNS WITH CYCLES AND STARS

JENQ-JONG LIN

Department of Finance
Ling Tung University, Taichung 40852, Taiwan

e-mail: jjlin@teamail.ltu.edu.tw

AND

MIN-JEN JOU

Department of Information Technology
Ling Tung University, Taichung 40852, Taiwan

e-mail: mjjou@teamail.ltu.edu.tw

Abstract

Let \( F, G \) and \( H \) be graphs. A \((G,H)\)-decomposition of \( F \) is a partition of the edge set of \( F \) into copies of \( G \) and copies of \( H \) with at least one copy of \( G \) and at least one copy of \( H \). If \( F \) has a \((G,H)\)-decomposition, we say that \( F \) is \((G,H)\)-decomposable. A \((G,H)\)-decomposition of \( F \) with the smallest cardinality is a minimum \((G,H)\)-covering. This paper gives the solution of finding the minimum \((C_k,S_k)\)-covering of the crown \( C_{n,n-1} \).

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1. Introduction

Let \( F, G \) and \( H \) be graphs. A \( G \)-decomposition of \( F \) is a partition of the edge set of \( F \) into copies of \( G \). If \( F \) has a \( G \)-decomposition, we say that \( F \) is \( G \)-decomposable. A \((G,H)\)-decomposition of \( F \) is a partition of the edge set of \( F \) into copies of \( G \) and copies of \( H \) with at least one copy of \( G \) and at least one copy of \( H \). If \( F \) has a \((G,H)\)-decomposition, we say that \( F \) is \((G,H)\)-decomposable.
A \((G, H)\)-decomposition of \(F\) may not exist, a natural question of interest is to see: What is the minimum number of edges needed to be added to the edge set of \(F\) so that the resulting graph is \((G, H)\)-decomposable, and what does the collection of added edges look like? For \(R \subseteq F\), a \((G, H)\)-covering of \(F\) with *padding* \(R\) is a \((G, H)\)-decomposition of \(F + E(R)\). A \((G, H)\)-covering of \(F\) with the smallest cardinality is a minimum \((G, H)\)-covering. Moreover, the cardinality of the minimum \((G, H)\)-covering of \(F\) is called the \((G, H)\)-covering number of \(F\), denoted by \(c(F; G, H)\).

As usual \(K_n\) denotes the complete graph with \(n\) vertices and \(K_{m,n}\) denotes the complete bipartite graph with parts of sizes \(m\) and \(n\). A \(k\)-star, denoted by \(S_k\), is the complete bipartite graph \(K_{1,k}\). The vertex of degree \(k\) in \(S_k\) is the *center* of \(S_k\) and any vertex of degree 1 is an *end-vertex* of \(S_k\). Let \((y_1, y_2, \ldots, y_k)_x\) denote the \(k\)-star with center \(x\) and end-vertices \(y_1, y_2, \ldots, y_k\). A \(k\)-cycle (respectively, \(k\)-path), denoted by \(C_k\) (respectively, \(P_k\)), is a cycle (respectively, path) with \(k\) edges. Let \((v_1, v_2, \ldots, v_k)\) and \(v_1v_2 \cdots v_k\) denote the \(k\)-cycle and \((k-1)\)-path through vertices \(v_1, \ldots, v_k\) in order, respectively. A *spanning subgraph* \(H\) of a graph \(G\) is a subgraph of \(G\) with \(V(H) = V(G)\). A *1-factor* of \(G\) is a spanning subgraph of \(G\) with each vertex incident with exactly one edge. For positive integers \(\ell\) and \(n\) with \(1 \leq \ell \leq n\), the *crown* \(C_{n,\ell}\) is a bipartite graph with bipartition \((A, B)\) where \(A = \{a_0, a_1, \ldots, a_{n-1}\}\) and \(B = \{b_0, b_1, \ldots, b_{n-1}\}\), and edge set \(\{a_ib_j : i = 0, 1, \ldots, n-1, j \equiv i + 1, i + 2, \ldots, i + \ell \text{ (mod } n)\}\). In the sequel of the paper, \((A, B)\) always means the bipartition of \(C_{n,\ell}\) defined here. Note that \(C_{n,n-1}\) is the graph obtained from the complete bipartite graph \(K_{n,n}\) with a 1-factor removed.

The existence problems for \((C_k, S_k)\)-decomposition of \(K_{m,n}\) and \(C_{n,n-1}\) have been completely settled by Lee [1] and Lee and Lin [4], respectively. Lee [2] obtained the maximum packing and minimum covering of the balanced complete bipartite multigraph \(\lambda K_{n,n}\) with \((C_k, S_k)\). Lee and Chen [3] gave the maximum packing and minimum covering of \(\lambda K_{n}\) with \((P_k, S_k)\). This paper gives the solution of finding the minimum \((C_k, S_k)\)-covering of the crown \(C_{n,n-1}\).

## 2. Preliminaries

Let \(G = (V, E)\) be a graph. For sets \(A \subseteq V(G)\) and \(B \subseteq E(G)\), we use \(G[A]\) to denote the subgraph of \(G\) induced by \(A\) and \(G - B\) (respectively, \(G + B\)) to denote the subgraph obtained from \(G\) by deleting (respectively, adding) the edges in \(B\). When \(G_1, \ldots, G_t\) are graphs, not necessarily disjoint, we write \(G_1 \cup \cdots \cup G_t\) or \(\bigcup_{i=1}^t G_i\) for the graph with vertex set \(\bigcup_{i=1}^t V(G_i)\) and edge set \(\bigcup_{i=1}^t E(G_i)\). When the edge sets are disjoint, \(G = \bigcup_{i=1}^t G_i\) expresses the decomposition of \(G\) into \(G_1, \ldots, G_t\). For a graph \(G\) and a positive integer \(\lambda \geq 2\), we use \(\lambda G\) to denote
the multigraph obtained from $G$ by replacing each edge $e$ by $\lambda$ edges, each of which has the same ends as $e$.

The following results are essential to our proof.

**Lemma 1** [7]. For integers $m$ and $n$ with $m \geq n \geq 1$, the graph $K_{m,n}$ is $S_k$-decomposable if and only if $m \geq k$ and

$$
\begin{align*}
  m &\equiv 0 \pmod{k} \quad \text{if } n < k, \\
  mn &\equiv 0 \pmod{k} \quad \text{if } n \geq k.
\end{align*}
$$

**Lemma 2** [5]. $\lambda C_{n,\ell}$ is $S_k$-decomposable if and only if $k \leq \ell$ and $\lambda n \ell \equiv 0 \pmod{k}$.

**Lemma 3** [5]. Let $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,\ell}$. Suppose that $p$ and $q$ are positive integers such that $q < p \leq \ell$. If $\lambda n q \equiv 0 \pmod{p}$, then there exists a spanning subgraph $G$ of $\lambda C_{n,\ell}$ such that $\deg_G b_j = \lambda q$ for $0 \leq j \leq n-1$ and $G$ has an $S_p$-decomposition.

**Lemma 4** [6]. For positive integers $k$ and $n$, $C_{n,n-1}$ is $C_k$-decomposable if and only if $n$ is odd, $k$ is even, $4 \leq k \leq 2n$, and $n(n-1) \equiv 0 \pmod{k}$.

## 3. Covering Numbers

In this section the covering number of $C_{n,n-1}$ with $k$-cycles and $k$-stars is determined.

**Lemma 5** [4]. If $k$ is an even integer with $k \geq 4$, then $C_{k+1,k}$ is not $(C_k, S_k)$-decomposable.

**Lemma 6**. If $k$ is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is $(C_k, S_k)$-decomposable.

**Proof.** By Lemma 4, we have that $C_{k+1,k}$ is $C_k$-decomposable. Define a $k$-star $R = \langle b_1, b_2, \ldots, b_k \rangle_{a_0}$. Clearly, $C_{k+1,k} + E(R)$ is a $(C_k, S_k)$-covering with padding $R$. 

We obtain the following result by Lemmas 5 and 6.

**Corollary 7.** $c(C_{k+1,k}; C_k, S_k) = k + 2$.

**Lemma 8** [4]. If $k$ is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is $(C_k, S_k)$-decomposable.

**Lemma 9.** For integers $r$ and $k$ with $r \geq 3$ and $k > r(r+1)$, $C_{k+r+1,k+r}$ can be decomposed into one copy of $r(r+1)$-cycle and $k + 2r + 1$ copies of $k$-stars.
Proof. Let \( s = r(r + 1)/2 \). Trivially, \( k + r + 1 > s \). Let \( A_0 = \{a_0, a_1, \ldots, a_{s-1}\}, \ B_0 = \{b_0, b_1, \ldots, b_{s-1}\}, \ H_0 = C_{n,n-1}[A_0 \cup B_0], \ H_1 = C_{n,n-1}[\{A \setminus A_0\} \cup B_0], \) and \( H_2 = C_{n,n-1}[A \cup (B \setminus B_0)] \). Clearly, \( C_{k+r+1,k+r} = H_0 \cup H_1 \cup H_2 \). Note that \( H_0 \) is isomorphic to \( C_{s,s-1} \), \( H_1 \) is isomorphic to \( K_{k+r+1-s,s} \), and \( H_2 \) is isomorphic to \( C_{k+r+1-s,k+r-s} \cup K_{s,k+r+1-s} \). Let

\[
C = (b_1, a_0, b_2, a_1, b_3, a_2, \ldots, b_{s-1}, a_{s-2}, b_0, a_{s-1})
\]

and \( H = H_0 - E(C) \). Trivially, \( C \) is an \( r(r+1) \)-cycle in \( H_0 \) and \( H = C_{s,s-3} \). Note that \( r - 2 < s - r - 1 \) for \( r \geq 3 \) and \( s(r-2) = rs-r(r+1) = r(s-r-1) \). By Lemma 3, there exists a spanning subgraph \( X \) of \( H \) such that \( \text{deg}_X b_j = r - 2 \) for \( 0 \leq j \leq s - 1 \) and \( X \) has an \( S_{s-r-1} \)-decomposition \( \mathcal{H} \) with \( |\mathcal{H}| = r \). Furthermore, each \( S_{s-r-1} \) has its center in \( A_0 \) since \( \text{deg}_X b_j = r - 2 < s - r - 1 \). Suppose that the centers of the \((s-r-1)\)-stars in \( \mathcal{H} \) are \( a_1, \ldots, a_n \). Let \( S(u) \) be the \((s-r-1)\)-star with center \( a_u \) in \( \mathcal{H} \), and let \( Y = H - E(X) \cup H_1 \). Note that \( \text{deg}_Y b_j = (s - 3 - (r - 2)) + (k + r + 1 - s) = k \) for \( 0 \leq j \leq s - 1 \). Hence \( Y \) has an \( S_k \)-decomposition \( \mathcal{H}(1) \) with \( |\mathcal{H}(1)| = s \). For \( u \in \{1, \ldots, r\}, \) define \( S'(u) = H_2[\{a_u\} \cup (B \setminus B_0)] \) and \( Z = H_2 - E(\bigcup_{u=1}^r S'(u)) \). Clearly, \( S'(u) \) is a \((k + r + 1 - s)\)-star with center \( a_u \) in \( H_2 \), and \( S(u) \cup S'(u) \) is a \( k \)-star. There are \( r \) copies of such \( k \)-stars. Moreover, \( \text{deg}_Z b_j = k + r - r = k \) for \( s - j \leq k + r \), and it follows that \( Z \) has an \( S_k \)-decomposition \( \mathcal{H}(2) \) with \( |\mathcal{H}(2)| = k + r - s + 1 \). Thus there are \( s + r + k + r - s + 1 = k + 2r + 1 \) copies of \( k \)-stars. This completes the proof.

Lemma 10. Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k < n - 1 < 2k - 1 \). If \( (n-k)(n-k-1) < k \), then \( C_{n,n-1} \) has a \((C_k, S_k)\)-covering with padding \( P_{k-(n-k)(n-k-1)} \).

Proof. Let \( n - 1 = k + r \). From the assumption \( k < n - 1 < 2k - 1 \), we have \( 0 < r < k - 1 \). The proof is divided into two parts according to the value of \( r \).

Case 1. \( r \leq 2 \). Let \( A_0 = \{a_0, a_1, \ldots, a_{k+r}\}, \ A_1 = \{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}, \ B_0 = \{b_0, b_1, \ldots, b_{k-1}\}, \ B_1 = \{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\} \). Let \( D_0 = C_{n,n-1}[\{A_0 \cup \{a_k\} \cup (B_0 \cup \{b_k\})], \ D_1 = C_{n,n-1}[A_0 \cup B_1], \ D_2 = C_{n,n-1}[A_1 \cup B_0], \) and \( D_3 = C_{n,n-1}[A_1 \cup \{a_k\}] \cup (B_1 \cup \{b_k\}) \). Clearly, \( C_{n,n-1} = D_0 \cup D_1 \cup D_2 \cup D_3 \). Note that \( D_0 \) is isomorphic to \( C_{k+1,k} \), \( D_1 \) is isomorphic to \( K_{k,k} \), \( D_2 \) is isomorphic to \( K_{k,k} \), and \( D_3 \) is isomorphic to \( C_{r+1,r} \). By Lemma 2, we have that \( D_0 \) has a \( k \)-star decomposition \( \langle b_{j_1}, b_{j_2}, \ldots, b_{j_k} \rangle a_j \) for \( 0 \leq j \leq k \), where the subscripts of \( b \)'s are taken modulo \( k + 1 \) in the set of numbers \( \{0, 1, \ldots, k\} \). By Lemma 1, we obtain that \( D_1 \) and \( D_2 \) have \( k \)-star decompositions \( \langle a_0, a_1, \ldots, a_{k-1} \rangle b_j \) and \( \langle b_0, b_1, \ldots, b_{k-1} \rangle a_i \), for \( k + 1 \leq i, j \leq k + r \), respectively.

Subcase 1.1. \( r = 1 \). Define a \((k-2)\)-path \( R_1 \) as follows.

\[
R_1 = a_{k+1}b_1a_0b_2a_1b_3a_2 \cdots a_{k-3}b_{k-1}a_k
\]
where the subscripts of \(a\)'s and \(b\)'s are taken modulo \(n\). Then
\[
\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup D_3 \cup R_1
\]
\[
= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup \{a_kb_{k+1}, a_{k+1}b_k\} \cup R_1
\]
\[
= \langle b_0, b_1, \ldots, b_{k-2}, b_{k+1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-2}, b_{k}\rangle_{a_{k+1}} \cup a_kb_{k-1}a_{k+1} \cup R_1.
\]

Note that \(a_kb_{k-1}a_{k+1} \cup R_1\) is a \(k\)-cycle. Hence \(C_{k+2,k+1} + E(R_1)\) can be decomposed into \(k + 3\) copies of \(k\)-stars and one copy of \(k\)-cycle, that is, \(C_{k+2,k+1}\) has a \((C_k, S_k)\)-covering \(\mathcal{C}_1\) with \(|\mathcal{C}_1| = k + 4\) and padding \(R_1\).

**Subcase 1.2.** \(r = 2\). Define a \((k - 6)\)-path \(R_2\) as follows.
\[
R_2 = b_1a_0b_2a_1 \cdots b_{\frac{k-3}{2}}a_{\frac{k-1}{2}}b_{k+1},
\]
where the subscripts of \(a\)'s and \(b\)'s are taken modulo \(n\). Then
\[
\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup D_3 \cup R_2
\]
\[
= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup \{a_kb_{k+1}, a_{k}b_{k+2}, a_{k+1}b_k, a_{k+1}b_{k+2}, a_{k+2}b_k, a_{k+2}b_{k+1}\} \cup R_2
\]
\[
= \langle b_0, b_2, b_3, \ldots, b_{k-1}, b_{k+1}\rangle_{a_{k+2}} \cup b_{k+1}a_kb_{k+2}a_{k+1}b_1 \cup R_2.
\]

Note that \(b_{k+1}a_kb_{k+2}a_{k+1}b_k \cup R_2\) is a \(k\)-cycle. Hence \(C_{k+3,k+2} + E(R_2)\) can be decomposed into \(k + 5\) copies of \(k\)-stars and one copy of \(k\)-cycle, that is, \(C_{k+3,k+2}\) has a \((C_k, S_k)\)-covering \(\mathcal{C}_2\) with \(|\mathcal{C}_2| = k + 6\) and padding \(R_2\).

**Case 2.** \(r \geq 3\). Let \(s = r(r + 1)/2\) and \(H_0, H_1\) and \(H_2\) be the graphs defined in the proof of Lemma 9. Define a \((k - 2s)\)-path \(R_3\) as follows.
\[
R_3 = a_{s-1}b_{s-1}a_{s}b_{s+2} \cdots a_{\frac{k-3}{2}}b_{\frac{k+1}{2}}a_{k+r},
\]
where the subscripts of \(a\)'s and \(b\)'s are taken modulo \(n\).

Let \(S\) be the \(k\)-star with center \(b_1\) and \(C\) be the \(2s\)-cycle mentioned in Lemma 9. Then
\[
S \cup C \cup R_3
\]
\[
= (S - a_{k+r}b_1 + a_{s-1}b_1) \cup a_{k+r}b_1a_0b_2a_1b_3a_2 \cdots b_{s-1}a_{s-2}b_0a_{s-1} \cup R_3.
\]

Note that \(a_{k+r}b_1a_0b_2a_1b_3a_2 \cdots b_{s-1}a_{s-2}b_0a_{s-1} \cup R_3\) is a \(k\)-cycle. Hence \(C_{k+r+1,k+r} + E(R_3)\) can be decomposed into \(k + 2r + 1\) copies of \(k\)-stars and one copy of \(k\)-cycle, that is, \(C_{k+r+1,k+r}\) has a \((C_k, S_k)\)-covering \(\mathcal{C}_3\) with \(|\mathcal{C}_3| = k + 2r + 2\) and padding \(R_3\). This settles Case 2. 

Before plunging into the proof of the case of \((n - k)(n - k - 1) \geq k\), a result due to Lee and Lin [4] is needed.
Lemma 11 [4]. If \( k \) is an even integer with \( k \geq 4 \), then there exist \( k/2 - 1 \) edge-disjoint \( k \)-cycles in \( C_{k/2,k/2-1} \cup K_{k/2,k/2} \).

Lemma 12. Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k < n - 1 < 2k - 1 \). If \( (n - k)(n - k - 1) \geq k \), then \( C_{n,n-1} \) has a \( (C_k,S_k) \)-covering \( C \) with \( |C| = \lfloor n(n-1)/k \rfloor \).

**Proof.** Let \( n - 1 = k + r \). From the assumption \( k < n - 1 < 2k - 1 \), we have \( 0 < r < k - 1 \). Since \( (n - k)(n - k - 1) \geq k \), we assume that \( r(r + 1) = \alpha k + \beta \), where \( \alpha \geq 1 \) and \( 0 \leq \beta \leq k - 1 \). Let \( A''_1 = \left\{ a_0, a_1, \ldots, a_{k-1} \right\} \), \( A'_2 = A \setminus (A''_1 \cup A''_2) \), \( B'_0 = \left\{ b_0, b_1, \ldots, b_{k-1} \right\} \), \( B_1 = B \setminus B'_0 \). Let \( G_i = C_{n,n-1}[A''_1 \cup B'_0] \) for \( i \in \{0,1,2\} \) and \( G_3 = C_{n,n-1}[A \cup B''_1] \). Clearly, \( C_{n,n-1} = G_0 \cup G_1 \cup G_2 \cup G_3 \). Note that \( G_0 \) and \( G_1 \) are isomorphic to \( K_{k/2,k/2-1} \cup K_{k/2,k/2} \), \( G_2 \) is isomorphic to \( K_{r+1,k} \), which is \( S_k \)-decomposable by Lemma 1, and \( G_3 \) is isomorphic to \( K_{k,r+1} \cup K_{r+1,k} \). Let \( p_0 = [\alpha/2] \) and \( p_1 = [\alpha/2] \). In the following, we will show that, for each \( i \in \{0,1\} \), \( G_i \) can be decomposed into \( p_i \) copies of \( C_k \) and \( k/2 \) copies of \( S_{k-2p_i - 1} \), and \( G_3 \) can be decomposed into \( k/2 \) copies of \( S_{2p_1+1} \) and \( r + 1 \) copies of \( S_{k'} \), \( k' \leq k \), such that the \((k-2p_i-1)\)-stars and \((2p_i+1)\)-stars have their centers in \( A''_1 \).

We first show the required decomposition of \( G_i \) for \( i \in \{0,1\} \). Since \( r < k - 1 \), we have \( r + 1 < k \), and in turn \( \alpha < r \). Thus, \( p_0 = \left\lceil \frac{\alpha}{2} \right\rceil \leq \frac{\alpha+1}{2} \leq \frac{r+1}{2} = \frac{k-2}{2} - 1 \), which implies \( p_i \leq k/2 - 1 \) for \( i \in \{0,1\} \). This assures us that there exist \( p_i \) edge-disjoint \( k \)-cycles in \( G_i \) by Lemma 11. Suppose that \( Q_{i,0}, \ldots, Q_{i,p_i-1} \) are edge-disjoint \( k \)-cycles in \( G_i \). Let \( F_i = G_i - E \left( \bigcup_{h=0}^{p_i-1} Q_{i,h} \right) \) and \( X_{i,j} = F_i \left( \{a_{ik/2+j}\} \cup B'_0 \right) \) where \( i \in \{0,1\}, j \in \{0,\ldots,k/2-1\} \). Since \( \deg_{G_i} a_{ik/2+j} = k - 1 \) and each \( Q_{i,h} \) uses two edges incident with \( a_{ik/2+j} \) for each \( i \) and \( j \), we have \( \deg_{F_i} a_{ik/2+j} = k - 2p_i - 1 \). Hence \( X_{i,j} \) is a \((k-2p_i-1)\)-star with center \( a_{ik/2+j} \).

Next we show the required star decomposition of \( G_3 \). For \( j \in \{0,\ldots,k/2-1\} \), let

\[
X'_{i,j} = \begin{cases} 
    \langle b_{k+(2p_i+1)j}, b_{k+(2p_i+1)j+1}, \ldots, b_{k+(2p_i+1)j+p_1} \rangle_{a_{ik/2+j}}, & \text{if } i = 0, \\
    \langle b_{(p_0+3/2)(k+(2p_i+1)j)}, b_{(p_0+3/2)(k+(2p_i+1)j)+1}, \ldots, b_{(p_0+3/2)(k+(2p_i+1)j)+2p_1} \rangle_{a_{ik/2+j}}, & \text{if } i = 1,
\end{cases}
\]

where the subscripts of \( b \)'s are taken modulo \( r + 1 \) in the set of numbers \( \{k, k + 1, \ldots, k + r\} \). Since \( 2p_1 + 1 \leq 2p_0 + 1 \leq \alpha + 2 \leq r + 1 \), this assures us that there are enough edges for the construction of \( X'_{0,j} \) and \( X'_{1,j} \). Note that \( X'_{i,j} \) is a \((2p_1+1)\)-star and \( X_{i,j} \cup X'_{i,j} \) is a \( k \)-star for \( i \in \{0,1\}, j \in \{0,\ldots,k/2-1\} \).

On the other hand, let \( k - \beta = \tau(r + 1) + \rho \) where \( \tau \geq 0 \) and \( 0 \leq \rho \leq r \). We have that
\[ |E(G_3)| - |E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) | \]
\[ = (k + r)(r + 1) - (2p_0 + 2p_1 + 2)(k/2) \]
\[ = (k + r)(r + 1) - (\alpha + 1)k \]
\[ = (k + r)(r + 1) - (r + 1) - (k - \beta) \]
\[ = k(r + 1) - \tau(r + 1) - \rho = (k - \tau)(r + 1) - \rho \]
\[ = (k - \tau - 1)\rho + (k - \tau)(r + 1 - \rho). \]

Hence there exists a decomposition \( \mathcal{G} \) of \( G_3 - E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) \) into \( \rho \) copies of \((k - \tau - 1)\)-star with center \( b_w \) for \( w = k, k + 1, \ldots, k + \rho - 1 \) and \( r + 1 - \rho \) copies of \((k - \tau)\)-star with center \( b_w \) for \( w = k + \rho, k + \rho + 1, \ldots, k + r \), that is,

\[ Y_w = \begin{cases} 
S_{k-r-1}, & \text{if } w \in \{k, k + 1, \ldots, k + \rho - 1\}, \\
S_{k-r}, & \text{if } w \in \{k + \rho, k + \rho + 1, \ldots, k + r\}.
\end{cases} \]

Define a star \( Y'_w \) as follows.

\[ Y'_w = \begin{cases} 
\langle a_{w_1}, a_{w_2}, \ldots, a_{w_r}, a_{w_{r+1}} \rangle_{b_w}, & \text{if } w \in \{k, k + 1, \ldots, k + \rho - 1\}, \\
\langle a_{w_1}, a_{w_2}, \ldots, a_{w_r} \rangle_{b_w}, & \text{if } w \in \{k + \rho, k + \rho + 1, \ldots, k + r\},
\end{cases} \]

where \( b_w a_{w_t} \in E(X'_{i,j}) \) for \( 1 \leq t \leq \tau + 1 \). Since \( |E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) | = (\alpha + 1)k \), \( |B''_w| = r + 1 \) and \( (\tau + 1)(r + 1) = \tau(r + 1) + (r + 1) = (k - \beta - \rho) + (r + 1) < 2k \leq (\alpha + 1)k \), it follows that \( \tau + 1 < (\alpha + 1)k/(r + 1) \). This assures us that there are enough edges for the construction of \( Y'_w \). Note that \( Y_w + E(Y'_w) \) is a \( k \)-star. Hence \( C_{n,n-1} \) has a \((C_k, S_k)\)-covering \( \mathcal{C} \) with padding \( \bigcup_{w \in \{k, k+1, \ldots, k+r\}} Y'_w \) and \( |\mathcal{C}| = (k + r + 1) + (r + 1) + \alpha = k + 2r + 2 + \alpha = [n(n - 1)/k] \). This completes the proof.

Now, we are ready for the main result of this section.

**Theorem 13.** Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k \leq n - 1 \). Then

\[ c(C_{n,n-1}; C_k, S_k) = \begin{cases} 
[n(n - 1)/k], & \text{if } k < n - 1, \\
k + 2, & \text{if } k = n - 1.
\end{cases} \]

**Proof.** Since \( |E(C_{n,n-1})| = n(n - 1) \), we have that \( c(C_{n,n-1}; C_k, S_k) \geq [n(n - 1)/k] \). Let \( n - 1 = qk + r \), where \( q \) and \( r \) are integers with \( q \geq 1 \), \( 0 \leq r \leq k - 1 \). We consider the following two cases.

**Case 1.** \( q = 1 \). For \( r = 0 \), the result follows from Corollary 7. If \( r \neq 0 \), by Lemmas 8, 10 and 12, \( C_{k+r+1,k+r} \) has a \((C_k, S_k)\)-covering \( \mathcal{C} \) with \( |\mathcal{C}| = [(k + r + 1)(k + r)/k] \).
Case 2. \( q \geq 2 \). Note that
\[
C_{n,n-1} = C_{qk+r+1,qk+r} = C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}.
\]

Trivially, \(|E(C_{(q-1)k+1,(q-1)k})|, |E(K_{(q-1)k,k+r})| \) and \(|E(K_{k+r,(q-1)k})|\) are multiples of \( k \), by Lemmas 1 and 2, we have that \( C_{(q-1)k+1,(q-1)k} \), \( K_{(q-1)k,k+r} \) and \( K_{k+r,(q-1)k} \) have \( S_k \)-decompositions \( \mathcal{A}^{(1)} \), \( \mathcal{A}^{(2)} \) and \( \mathcal{A}^{(3)} \) with \(|\mathcal{A}^{(1)}| = (q-1)((q-1)k+1), |\mathcal{A}^{(2)}| = |\mathcal{A}^{(3)}| = (k+r)(q-1)\). For the case of \( r = 0 \), by Lemma 4, \( C_{k+1,k} \) has a \( C_k \)-decomposition \( \mathcal{C} \) with \(|\mathcal{C}| = k+1\). Hence \( C_{n,n-1} \) is \((C_k, S_k)\)-decomposable, that is, \( C_{n,n-1} \) has a \((C_k, S_k)\)-covering \( \bigcup_{i=1}^3 \mathcal{A}^{(i)} \cup \mathcal{C} \) with cardinality \((q-1)((q-1)k+1)+k(q-1)+k(q-1)+k+1 = n(qk+1) = n(n-1)/k\). For the other case of \( r \neq 0 \), by Lemmas 10 and 12, \( C_{k+r+1,k+r} \) has a \((C_k, S_k)\)-covering \( \mathcal{C}' \) with \(|\mathcal{C}'| = [(k+r+1)(k+r)/k]\). Hence \( \bigcup_{i=1}^3 \mathcal{A}^{(i)} \cup \mathcal{C}' \) is a \((C_k, S_k)\)-covering of \( C_{n,n-1} \) with cardinality \((q-1)((q-1)k+1) + (k+r)(q-1) + (k+r)(q-1) + [(k+r+1)(k+r)/k] = [(qk+r+1)(qk+r)/k] = [n(n-1)/k]\). This completes the proof.

References


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