MINIMUM COVERINGS OF CROWNS WITH CYCLES AND STARS

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Abstract

Let $F$, $G$ and $H$ be graphs. A $(G,H)$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$ and copies of $H$ with at least one copy of $G$ and at least one copy of $H$. If $F$ has a $(G,H)$-decomposition, we say that $F$ is $(G,H)$-decomposable. A $(G,H)$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$ and copies of $H$ with at least one copy of $G$ and at least one copy of $H$. If $F$ has a $(G,H)$-decomposition, we say that $F$ is $(G,H)$-decomposable. This paper gives the solution of finding the minimum $(C_k,S_k)$-covering of the crown $C_{n,n-1}$.

Keywords: cycle, star, covering, decomposition, crown.

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1. Introduction

Let $F$, $G$ and $H$ be graphs. A $G$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$. If $F$ has a $G$-decomposition, we say that $F$ is $G$-decomposable. A $(G,H)$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$ and copies of $H$ with at least one copy of $G$ and at least one copy of $H$. If $F$ has a $(G,H)$-decomposition, we say that $F$ is $(G,H)$-decomposable.

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A \((G, H)\)-decomposition of \(F\) may not exist, a natural question of interest is to see: What is the minimum number of edges needed to be added to the edge set of \(F\) so that the resulting graph is \((G, H)\)-decomposable, and what does the collection of added edges look like? For \(R \subseteq F\), a \((G, H)\)-covering of \(F\) with paddin \(R\) is a \((G, H)\)-decomposition of \(F + E(R)\). A \((G, H)\)-covering of \(F\) with the smallest cardinality is a minimum \((G, H)\)-covering. Moreover, the cardinality of the minimum \((G, H)\)-covering of \(F\) is called the \((G, H)\)-covering number of \(F\), denoted by \(c(F; G, H)\).

As usual \(K_n\) denotes the complete graph with \(n\) vertices and \(K_{m,n}\) denotes the complete bipartite graph with parts of sizes \(m\) and \(n\). A \(k\)-star, denoted by \(S_k\), is the complete bipartite graph \(K_{1,k}\). The vertex of degree \(k\) in \(S_k\) is the center of \(S_k\) and any vertex of degree 1 is an end-vertex of \(S_k\). Let \((y_1, y_2, \ldots, y_k)_x\) denote the \(k\)-star with center \(x\) and end-vertices \(y_1, y_2, \ldots, y_k\). A \(k\)-cycle (respectively, \(k\)-path), denoted by \(C_k\) (respectively, \(P_k\)), is a cycle (respectively, path) with \(k\) edges. Let \((v_1, v_2, \ldots, v_k)\) and \(v_1v_2 \cdots v_k\) denote the \(k\)-cycle and \((k - 1)\)-path through vertices \(v_1, \ldots, v_k\) in order, respectively. A spanning subgraph \(H\) of a graph \(G\) is a subgraph of \(G\) with \(V(H) = V(G)\). A 1-factor of \(G\) is a spanning subgraph of \(G\) with each vertex incident with exactly one edge. For positive integers \(\ell\) and \(n\) with \(1 \leq \ell \leq n\), the crown \(C_{n,\ell}\) is a bipartite graph with bipartition \((A, B)\) where \(A = \{a_0, a_1, \ldots, a_{n-1}\}\) and \(B = \{b_0, b_1, \ldots, b_{n-1}\}\), and edge set \(\{a_ib_j : i = 0, 1, \ldots, n - 1, j \equiv i + 1, i + 2, \ldots, i + \ell \mod n\}\). The paper, \((A, B)\) always means the bipartition of \(C_{n,\ell}\) defined here. Note that \(C_{n,n-1}\) is the graph obtained from the complete bipartite graph \(K_{n,n}\) with a 1-factor removed.

The existence problems for \((C_k, S_k)\)-decomposition of \(K_{m,n}\) and \(C_{n,n-1}\) have been completely settled by Lee [1] and Lee and Lin [4], respectively. Lee [2] obtained the maximum packing and minimum covering of the balanced complete bipartite multigraph \(\lambda K_{n,n}\) with \((C_k, S_k)\). Lee and Chen [3] gave the maximum packing and minimum covering of \(\lambda K_n\) with \((P_k, S_k)\). This paper gives the solution of finding the minimum \((C_k, S_k)\)-covering of the crown \(C_{n,n-1}\).

2. Preliminaries

Let \(G = (V, E)\) be a graph. For sets \(A \subseteq V(G)\) and \(B \subseteq E(G)\), we use \(G[A]\) to denote the subgraph of \(G\) induced by \(A\) and \(G - B\) (respectively, \(G + B\)) to denote the subgraph obtained from \(G\) by deleting (respectively, adding) the edges in \(B\). When \(G_1, \ldots, G_t\) are graphs, not necessarily disjoint, we write \(G_1 \cup \cdots \cup G_t\) or \(\bigcup_{i=1}^t G_i\) for the graph with vertex set \(\bigcup_{i=1}^t V(G_i)\) and edge set \(\bigcup_{i=1}^t E(G_i)\). When the edge sets are disjoint, \(G = \bigcup_{i=1}^t G_i\) expresses the decomposition of \(G\) into \(G_1, \ldots, G_t\). For a graph \(G\) and a positive integer \(\lambda \geq 2\), we use \(\lambda G\) to denote
the multigraph obtained from $G$ by replacing each edge $e$ by $\lambda$ edges, each of which has the same ends as $e$.

The following results are essential to our proof.

Lemma 1 [7]. For integers $m$ and $n$ with $m \geq n \geq 1$, the graph $K_{m,n}$ is $S_k$-decomposable if and only if $m \geq k$ and

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k, \\ mn \equiv 0 \pmod{k} & \text{if } n \geq k. \end{cases}$$

Lemma 2 [5]. $\lambda C_{n,\ell}$ is $S_k$-decomposable if and only if $k \leq \ell$ and $\lambda n \ell \equiv 0 \pmod{k}$.

Lemma 3 [5]. Let $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,\ell}$. Suppose that $p$ and $q$ are positive integers such that $q < p \leq \ell$. If $\lambda n q \equiv 0 \pmod{p}$, then there exists a spanning subgraph $G$ of $\lambda C_{n,\ell}$ such that $\deg_G b_j = \lambda q$ for $0 \leq j \leq n-1$ and $G$ has an $S_p$-decomposition.

Lemma 4 [6]. For positive integers $k$ and $n$, $C_{n,n-1}$ is $C_k$-decomposable if and only if $n$ is odd, $k$ is even, $4 \leq k \leq 2n$, and $n(n-1) \equiv 0 \pmod{k}$.

3. Covering numbers

In this section the covering number of $C_{n,n-1}$ with $k$-cycles and $k$-stars is determined.

Lemma 5 [4]. If $k$ is an even integer with $k \geq 4$, then $C_{k+1,k}$ is not $(C_k, S_k)$-decomposable.

Lemma 6. If $k$ is an even integer with $k \geq 4$, then $C_{k+1,k}$ has a $(C_k, S_k)$-covering with padding $S_k$.

Proof. By Lemma 4, we have that $C_{k+1,k}$ is $C_k$-decomposable. Define a $k$-star $R = \langle b_1, b_2, \ldots, b_k \rangle_{a_0}$. Clearly, $C_{k+1,k} + E(R)$ is a $(C_k, S_k)$-covering with padding $R$.

We obtain the following result by Lemmas 5 and 6.

Corollary 7. $c(C_{k+1,k}; C_k, S_k) = k + 2$.

Lemma 8 [4]. If $k$ is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is $(C_k, S_k)$-decomposable.

Lemma 9. For integers $r$ and $k$ with $r \geq 3$ and $k > r(r+1)$, $C_{k+r+1,k+r}$ can be decomposed into one copy of $r(r+1)$-cycle and $k + 2r + 1$ copies of $k$-stars.
Proof. Let $s = r(r + 1)/2$. Trivially, $k + r + 1 > s$. Let $A_0 = \{a_0, a_1, \ldots, a_{s-1}\}$, $B_0 = \{b_0, b_1, \ldots, b_{s-1}\}$, $H_0 = C_{n,n-1}[A_0 \cup B_0]$, $H_1 = C_{n,n-1}((A \setminus A_0) \cup B_0)$, and $H_2 = C_{n,n-1}[A \cup (B \setminus B_0)]$. Clearly, $C_{k+r+1,k+r} = H_0 \cup H_1 \cup H_2$. Note that $H_0$ is isomorphic to $C_{s,s-1}$, $H_1$ is isomorphic to $K_{k+r+1-s,s}$, and $H_2$ is isomorphic to $C_{k+r+1-s,k+r-s} \cup K_{s,k+r+1-s}$. Let

$$C = (b_1, a_0, b_2, a_1, b_3, a_2, \ldots, b_{s-1}, a_{s-2}, b_0, a_{s-1})$$

and $H = H_0 - E(C)$. Trivially, $C$ is an $r(r + 1)$-cycle in $H_0$ and $H = C_{s,s-3}$. Note that $r - 2 < s - r - 1$ for $r \geq 3$ and $s(r - 2) = rs - r(r + 1) = r(s - r - 1)$. By Lemma 3, there exists a spanning subgraph $X$ of $H$ such that $\deg_X b_j = r - 2$ for $0 \leq j \leq s - 1$ and $X$ has an $S_{s-r-1}$-decomposition $\mathcal{H}$ with $|\mathcal{H}| = r$. Furthermore, each $S_{s-r-1}$ has its center in $A_0$ since $\deg_X b_j = r - 2 < s - r - 1$. Suppose that the centers of the $(s-r-1)$-stars in $\mathcal{H}$ are $a_1, \ldots, a_s$. Let $S(u)$ be the $(s-r-1)$-star with center $a_u$ in $\mathcal{H}$, and let $Y = H - E(X) \cup H_1$. Note that $\deg_Y b_j = (s - 3 - (r - 2)) + (k + r + 1 - s) = k$ for $0 \leq j \leq s - 1$. Hence $Y$ has an $S_k$-decomposition $\mathcal{H}^{(1)}$ with $|\mathcal{H}^{(1)}| = s$. For $u \in \{1, \ldots, r\}$, define $S'(u) = H_2[[a_u] \cup (B \setminus B_0)]$ and $Z = H_2 - E(\bigcup_{u=1}^r S'(u))$. Clearly, $S'(u)$ is a $(k + r + 1 - s)$-star with center $a_u$ in $H_2$, and $S(u) \cup S'(u)$ is a $k$-star. There are $r$ copies of such $k$-stars. Moreover, $\deg_Z b_j = k + r - r = k$ for $s \leq j \leq k + r$, and it follows that $Z$ has an $S_k$-decomposition $\mathcal{H}^{(2)}$ with $|\mathcal{H}^{(2)}| = k + r - s + 1$. Thus there are $s + r + k + r - s + 1 = k + 2r + 1$ copies of $k$-stars. This completes the proof.

Lemma 10. Let $k$ be a positive even integer and let $n$ be a positive integer with $4 \leq k < n - 1 < 2k - 1$. If $(n - k)(n - k - 1) < k$, then $C_{n,n-1}$ has a $(C_k, S_k)$-covering with padding $P_{k - (n - k)(n - k - 1)}$.

Proof. Let $n - 1 = k + r$. From the assumption $k < n - 1 < 2k - 1$, we have $0 < r < k - 1$. The proof is divided into two parts according to the value of $r$.

Case 1. $r \leq 2$. Let $A'_0 = \{a_0, a_1, \ldots, a_{k+1}\}$, $A'_1 = \{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}$, $B'_0 = \{b_0, b_1, \ldots, b_{k+1}\}$, $B'_1 = \{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\}$. Let $D_0 = C_{n,n-1}[A'_0 \cup \{a_k\}] \cup (B'_0 \cup \{b_k\})$, $D_1 = C_{n,n-1}[A'_0 \cup B'_1]$, $D_2 = C_{n,n-1}[A'_1 \cup B'_0]$ and $D_3 = C_{n,n-1}[A'_1 \cup \{a_k\}] \cup (B'_1 \cup \{b_k\})$. Clearly, $C_{n,n-1} = D_0 \cup D_1 \cup D_2 \cup D_3$. Note that $D_0$ is isomorphic to $C_{k+1,k}$, $D_1$ is isomorphic to $K_{k,k}$, $D_2$ is isomorphic to $K_{r,k}$ and $D_3$ is isomorphic to $C_{r+1,r}$. By Lemma 2, we have that $D_0$ has a $k$-star decomposition $\langle b_{j_1}, b_{j_2}, \ldots, b_{j_k}a_j \rangle$ for $0 \leq j \leq k$, where the subscripts of $b$'s are taken modulo $k + 1$ in the set of numbers $\{0, 1, \ldots, k\}$. By Lemma 1, we obtain that $D_1$ and $D_2$ have $k$-star decompositions $\langle a_0, a_1, \ldots, a_{k-1}b_j \rangle$ and $\langle b_0, b_1, \ldots, b_{k-1}a_i \rangle$ for $0 \leq i, j \leq k + r$, respectively.

Subcase 1.1. $r = 1$. Define a $(k - 2)$-path $R_1$ as follows.

$$R_1 = a_{k+1}b_1a_0b_2a_1b_3a_2 \cdots a_{k-3}b_{k-1}a_k.$$
where the subscripts of $a$’s and $b$’s are taken modulo $n$. Then

$$\langle b_0, b_1, \ldots, b_k \rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_k \rangle_{a_{k+1}} \cup D_3 \cup R_1$$

$$= \langle b_0, b_1, \ldots, b_k \rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_k \rangle_{a_{k+1}} \cup \{a_k b_{k+1}, a_k b_k \} \cup R_1$$

$$= \langle b_0, b_1, \ldots, b_k \rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_k \rangle_{a_{k+1}} \cup a_k b_{k-1} a_{k+1} \cup R_1.$$ 

Note that $a_k b_k a_{k-1} \cup R_1$ is a $k$-cycle. Hence $C_{k+2, k+1} + E(R_1)$ can be decomposed into $k + 3$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+2, k+1}$ has a $(C_k, S_k)$-covering $\mathcal{C}_1$ with $|\mathcal{C}_1| = k + 4$ and padding $R_1$.

Subcase 1.2. $r = 2$. Define a $(k - 6)$-path $R_2$ as follows.

$$R_2 = b_1 a_0 b_2 a_1 \cdots b_{\frac{k}{2} - 3} a_{\frac{k}{2} - 4} b_{k-1},$$

where the subscripts of $a$’s and $b$’s are taken modulo $n$. Then

$$\langle b_0, b_1, \ldots, b_k \rangle_{a_{k+2}} \cup D_3 \cup R_2$$

$$= \langle b_0, b_1, \ldots, b_k \rangle_{a_{k+2}} \cup \{a_k b_{k+1}, a_k b_{k+2}, a_{k+1} b_k, a_{k+1} b_{k+2}, a_{k+2} b_k, a_{k+2} b_{k+1} \} \cup R_2$$

$$= \langle b_0, b_2, b_3, \ldots, b_k, b_{k+1} \rangle_{a_{k+2}} \cup b_{k+1} a_k b_{k+2} a_{k+1} b_k a_{k+2} b_k \cup R_2.$$ 

Note that $b_{k+1} a_k b_{k+2} a_{k+1} b_k a_{k+2} b_k \cup R_2$ is a $k$-cycle. Hence $C_{k+3, k+2} + E(R_2)$ can decomposed into $k + 5$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+3, k+2}$ has a $(C_k, S_k)$-covering $\mathcal{C}_2$ with $|\mathcal{C}_2| = k + 6$ and padding $R_2$.

Case 2. $r \geq 3$. Let $s = r(r + 1)/2$ and $H_0, H_1$ and $H_2$ be the graphs defined in the proof of Lemma 9. Define a $(k - 2s)$-path $R_3$ as follows.

$$R_3 = a_{s-1} b_s a_s b_{s+2} \cdots a_{\frac{k}{2} - 2} b_2 a_{k+r},$$

where the subscripts of $a$’s and $b$’s are taken modulo $n$.

Let $S$ be the $k$-star with center $b_1$ and $C$ be the $2s$-cycle mentioned in Lemma 9. Then

$$S \cup C \cup R_3$$

$$= (S - a_{k+r} b_1 + a_{s-1} b_1) \cup a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3.$$ 

Note that $a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3$ is a $k$-cycle. Hence $C_{k+r+1, k+r} + E(R_3)$ can be decomposed into $k + 2r + 1$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+r+1, k+r}$ has a $(C_k, S_k)$-covering $\mathcal{C}_3$ with $|\mathcal{C}_3| = k + 2r + 2$ and padding $R_3$. This settles Case 2.

Before plunging into the proof of the case of $(n - k)(n - k - 1) \geq k$, a result due to Lee and Lin [4] is needed.
Lemma 11 [4]. If \( k \) is an even integer with \( k \geq 4 \), then there exist \( k/2 - 1 \) edge-disjoint \( k \)-cycles in \( C_{k/2,k/2-1} \cup K_{k/2,k/2} \).

Lemma 12. Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k < n - 1 < 2k - 1 \). If \( (n-k)(n-k-1) \geq k \), then \( C_{n,n-1} \) has a \( (C_k,S_k) \)-covering \( \mathcal{C} \) with \( |\mathcal{C}| = \lfloor n(n-1)/k \rfloor \).

**Proof.** Let \( n-1 = k + r \). From the assumption \( k < n - 1 < 2k - 1 \), we have \( 0 < r < k - 1 \). Since \( (n-k)(n-k-1) \geq k \), we assume that \( r(r+1) = \alpha k + \beta \), where \( \alpha \geq 1 \) and \( 0 \leq \beta < k - 1 \). Let \( A''_0 = \{a_0,a_1,\ldots,a_{k-1}\} \), \( A''_1 = \left\{a_{k-1}+1,\ldots,a_{k-1}\right\} \), \( A''_2 = A \setminus (A''_0 \cup A''_1) \), \( B''_0 = \{b_0,b_1,\ldots,b_{k-1}\} \), \( B''_1 = B \setminus B''_0 \). Let \( G_i = C_{n,n-1}[A''_i \cup B''_j] \) for \( i \in \{0,1,2\} \) and \( G_3 = C_{n,n-1}[A \cup B''_0] \). Clearly, \( C_{n,n-1} = G_0 \cup G_1 \cup G_2 \cup G_3 \). Note that \( G_0 \) and \( G_1 \) are isomorphic to \( C_{k/2,k/2-1} \cup K_{k/2,k/2} \). \( G_2 \) is isomorphic to \( K_{r+1,k} \), which is \( S_{k'} \)-decomposable by Lemma 1, and \( G_3 \) is isomorphic to \( K_{k,r+1} \cup C_{r+1,r} \). Let \( p_0 = \lceil \alpha/2 \rceil \) and \( p_1 = \lfloor \alpha/2 \rfloor \). In the following, we will show that, for each \( i \in \{0,1\} \), \( G_i \) can be decomposed into \( p_1 \) copies of \( C_k \) and \( k/2 \) copies of \( S_{k-2p_1-1} \), and \( G_3 \) can be decomposed into \( k/2 \) copies of \( S_{2p_1+1} \) and \( r+1 \) copies of \( S_{k'} \), \( k' \leq k \), such that the \( (k-2p_1-1) \)-stars and \( (2p_1+1) \)-stars have their centers in \( A''_i \).

We first show the required decomposition of \( G_i \) for \( i \in \{0,1\} \). Since \( r < k - 1 \), we have \( r + 1 < k \), and in turn \( \alpha < r \). Thus, \( p_0 = \left\lfloor \frac{\alpha}{2} \right\rfloor \leq \frac{\alpha + 1}{2} \leq \frac{r+1}{2} = \frac{k-2}{2} = \beta - 1 \), which implies \( p_i \leq k/2 - 1 \) for \( i \in \{0,1\} \). This assures us that there exist \( p_i \) edge-disjoint \( k \)-cycles in \( G_i \) by Lemma 11. Suppose that \( Q_{i,0},\ldots,Q_{i,p_i-1} \) are edge-disjoint \( k \)-cycles in \( G_i \). Let \( F_i = G_i - E \left( \bigcup_{h=0}^{p_i-1} Q_{i,h} \right) \) and \( X_{i,j} = F_i \left[ (a_{ik/2+j}) \cup B''_0 \right] \) where \( i \in \{0,1\} \), \( j \in \{0,\ldots,k/2-1\} \). Since \( \deg_{G_i} a_{ik/2+j} = k - 1 \) and each \( Q_{i,h} \) uses two edges incident with \( a_{ik/2+j} \) for each \( i \) and \( j \), we have \( \deg_{F_i} a_{ik/2+j} = k - 2p_1 - 1 \). Hence \( X_{i,j} \) is a \((k-2p_1-1)\)-star with center \( a_{ik/2+j} \).

Next we show the required star decomposition of \( G_3 \). For \( j \in \{0,\ldots,k/2-1\} \), let

\[
X'_{i,j} = \begin{cases} 
\langle b_{k+(2p_0+1)}j, b_{k+(2p_0+1)+1}, \ldots, b_{k+(2p_0+1)+2p_0} \rangle_{a_j}, & \text{if } i = 0, \\
\langle b_{(p_0+3/2)k+(2p_1+1)}, b_{(p_0+3/2)k+(2p_1+1)+1}, \ldots, b_{(p_0+3/2)k+(2p_1+1)+2p_1} \rangle_{a_{k/2+j}}, & \text{if } i = 1,
\end{cases}
\]

where the subscripts of \( b \)'s are taken modulo \( r+1 \) in the set of numbers \( \{k,k+1,\ldots,k+r\} \). Since \( 2p_1 + 1 \leq 2p_0 + 1 \leq \alpha + 2 \leq r + 1 \), this assures us that there are enough edges for the construction of \( X'_{0,j} \) and \( X'_{1,j} \). Note that \( X'_{i,j} \) is a \((2p_1+1)\)-star and \( X_{i,j} \cup X'_{i,j} \) is a \( k \)-star for \( i \in \{0,1\} \), \( j \in \{0,\ldots,k/2-1\} \).

On the other hand, let \( k - \beta = \tau(r+1) + \rho \) where \( \tau \geq 0 \) and \( 0 \leq \rho \leq r \). We have that
\[ |E(G_{3})| - |E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) | \]
\[ = (k+r)(r+1) - (2p_0 + 2p_1 + 2)(k/2) \]
\[ = (k+r)(r+1) - (\alpha + 1)k \]
\[ = (k+r)(r+1) - r(r+1) - (k-\beta) \]
\[ = k(r+1) - \tau(r+1) - \rho = (k-\tau)(r+1) - \rho \]
\[ = (k-\tau-1)\rho + (k-\tau)(r+1 - \rho). \]

Hence there exists a decomposition \( \mathcal{G} \) of \( G_{3} - E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) \) into \( \rho \) copies of \((k-\tau-1)-\)star with center \( b_w \) for \( w = k, k+1, \ldots, k+\rho - 1 \) and \( r+1-\rho \) copies of \((k-\tau)-\)star with center \( b_w \) for \( w = k+\rho, k+\rho+1, \ldots, k+r \), that is,
\[
Y_w = \begin{cases} 
S_{k-r-1}, & \text{if } w \in \{k, k+1, \ldots, k+\rho - 1\}, \\
S_{k-r}, & \text{if } w \in \{k+\rho, k+\rho+1, \ldots, k+r\}. 
\end{cases}
\]

Define a star \( Y'_w \) as follows.
\[
Y'_w = \begin{cases} 
\langle a_{w_1}, a_{w_2}, \ldots, a_{w_{\tau}}, a_{w_{\tau+1}} \rangle_{b_{w}}, & \text{if } w \in \{k, k+1, \ldots, k+\rho - 1\}, \\
\langle a_{w_1}, a_{w_2}, \ldots, a_{w_r} \rangle_{b_{w}}, & \text{if } w \in \{k+\rho, k+\rho+1, \ldots, k+r\}, 
\end{cases}
\]
where \( b_w a_{w_i} \in E(X'_{i,j}) \) for \( 1 \leq t \leq \tau + 1 \). Since \[ |E \left( \bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j} \right) | \]
\[ = (\alpha+1)k, \] \( |B'_w| = r+1 \) and \((\tau+1)(r+1) = \tau(r+1) + (r+1) = (k-\beta-\rho) + (r+1) < 2k \leq (\alpha+1)k \), it follows that \( \tau + 1 < (\alpha + 1)k / (r+1) \). This assures us that there are enough edges for the construction of \( Y'_w \). Note that \( Y_w + E(Y'_w) \) is a \( k \)-star.

Hence \( C_{n,n-1} \) has a \( (C_k, S_k) \)-covering \( \mathcal{C}_4 \) with padding \( \bigcup_{w \in \{k,k+1,\ldots,k+r\}} Y'_w \) and \( |\mathcal{C}_4| = (k+r+1) + (r+1) + \alpha = k + 2r + 2 + \alpha = n(n-1)/k \). This completes the proof.

Now, we are ready for the main result of this section.

**Theorem 13.** Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k \leq n-1 \). Then
\[
c(C_{n,n-1}; C_k, S_k) = \begin{cases} 
[n(n-1)/k], & \text{if } k < n-1, \\
k + 2, & \text{if } k = n-1. 
\end{cases}
\]

**Proof.** Since \( |E(C_{n,n-1})| = n(n-1) \), we have that \( c(C_{n,n-1}; C_k, S_k) \geq [n(n-1)/k] \). Let \( n-1 = qk + r \), where \( q \) and \( r \) are integers with \( q \geq 1, 0 \leq r \leq k-1 \). We consider the following two cases.

**Case 1.** \( q = 1 \). For \( r = 0 \), the result follows from Corollary 7. If \( r \neq 0 \), by Lemmas 8, 10 and 12, \( C_{k+r+1,k+r} \) has a \( (C_k, S_k) \)-covering \( \mathcal{C} \) with \( |\mathcal{C}| = [(k+r+1)(k+r)/k] \). 

Case 2. \( q \geq 2 \). Note that
\[
C_{n,n-1} = C_{qk+r+1,qk+r} = C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}.
\]
Trivially, \(|E(C_{(q-1)k+1,(q-1)k})|, |E(K_{(q-1)k,k+r})|\) and \(|E(K_{k+r,(q-1)k})|\) are multiples of \( k \), by Lemmas 1 and 2, we have that \( C_{(q-1)k+1,(q-1)k}, K_{(q-1)k,k+r} \) and \( K_{k+r,(q-1)k} \) have \( S_k \)-decompositions \( \mathcal{A}^{(1)} \), \( \mathcal{A}^{(2)} \) and \( \mathcal{A}^{(3)} \) with \(|\mathcal{A}^{(1)}| = (q-1)((q-1)k+1), |\mathcal{A}^{(2)}| = |\mathcal{A}^{(3)}| = (k+r)(q-1)\). For the case of \( r = 0 \), by Lemma 4, \( C_{k+1,k} \) has a \( C_k \)-decomposition \( \mathcal{C} \) with \(|\mathcal{C}| = k+1\). Hence \( C_{n,n-1} \) is \((C_k,S_k)\)-decomposable, that is, \( C_{n,n-1} \) has a \((C_k,S_k)\)-covering \( \bigcup_{i=1}^{3} \mathcal{A}^{(i)} \cup \mathcal{C} \) with cardinality \((q-1)((q-1)k+1)+k(q-1)+k(q-1)+k+q(k+1) = n(n-1)/k\).
For the other case of \( r \neq 0 \), by Lemmas 10 and 12, \( C_{k+r+1,k+r} \) has a \((C_k,S_k)\)-covering \( \mathcal{C}' \) with \(|\mathcal{C}'| = [(k+r+1)(k+r)/k]\). Hence \( \bigcup_{i=1}^{3} \mathcal{A}^{(i)} \cup \mathcal{C}' \) is a \((C_k,S_k)\)-covering of \( C_{n,n-1} \) with cardinality \((q-1)((q-1)k+1)+(k+r)(q-1)+(k+r)(q-1)+[(k+r+1)(k+r)/k] = [(qk+r+1)(qk+r)/k] = [n(n-1)/k]\).
This completes the proof. \( \blacksquare \)

References


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