MINIMUM COVERINGS OF CROWNS WITH CYCLES AND STARS

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Abstract

Let $F$, $G$ and $H$ be graphs. A \((G, H)\)-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$ and copies of $H$ with at least one copy of $G$ and at least one copy of $H$. If $F$ has a \((G, H)\)-decomposition, we say that $F$ is \((G, H)\)-decomposable.

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1. Introduction

Let $F$, $G$ and $H$ be graphs. A $G$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$. If $F$ has a $G$-decomposition, we say that $F$ is $G$-decomposable. A $(G, H)$-decomposition of $F$ is a partition of the edge set of $F$ into copies of $G$ and copies of $H$ with at least one copy of $G$ and at least one copy of $H$. If $F$ has a $(G, H)$-decomposition, we say that $F$ is $(G, H)$-decomposable.
A \((G, H)\)-decomposition of \(F\) may not exist, a natural question of interest is to see: What is the minimum number of edges needed to be added to the edge set of \(F\) so that the resulting graph is \((G, H)\)-decomposable, and what does the collection of added edges look like? For \(R \subseteq F\), a \((G, H)\)-covering of \(F\) with padding \(R\) is a \((G, H)\)-decomposition of \(F + E(R)\). A \((G, H)\)-covering of \(F\) with the smallest cardinality is a minimum \((G, H)\)-covering. Moreover, the cardinality of the minimum \((G, H)\)-covering of \(F\) is called the \((G, H)\)-covering number of \(F\), denoted by \(c(F; G, H)\).

As usual \(K_n\) denotes the complete graph with \(n\) vertices and \(K_{m,n}\) denotes the complete bipartite graph with parts of sizes \(m\) and \(n\). A \(k\)-star, denoted by \(S_k\), is the complete bipartite graph \(K_{1,k}\). The vertex of degree \(k\) in \(S_k\) is the center of \(S_k\) and any vertex of degree 1 is an end-vertex of \(S_k\). Let \((y_1, y_2, \ldots, y_k)_x\) denote the \(k\)-star with center \(x\) and end-vertices \(y_1, y_2, \ldots, y_k\). A \(k\)-cycle (respectively, \(k\)-path), denoted by \(C_k\) (respectively, \(P_k\)), is a cycle (respectively, path) with \(k\) edges. Let \((v_1, v_2, \ldots, v_k)\) and \(v_1v_2 \cdots v_k\) denote the \(k\)-cycle and \((k-1)\)-path through vertices \(v_1, \ldots, v_k\) in order, respectively. A spanning subgraph \(H\) of a graph \(G\) is a subgraph of \(G\) with \(V(H) = V(G)\). A 1-factor of \(G\) is a spanning subgraph of \(G\) with each vertex incident with exactly one edge. For positive integers \(\ell\) and \(n\) with \(1 \leq \ell \leq n\), the crown \(C_{n,\ell}\) is a bipartite graph with bipartition \((A, B)\) where \(A = \{a_0, a_1, \ldots, a_{n-1}\}\) and \(B = \{b_0, b_1, \ldots, b_{n-1}\}\), and edge set \(\{a_ib_j : i = 0, 1, \ldots, n - 1, j \equiv i + 1, i + 2, \ldots, i + \ell \pmod{n}\}\). In the sequel of the paper, \((A, B)\) always means the bipartition of \(C_{n,\ell}\) defined here. Note that \(C_{n,n-1}\) is the graph obtained from the complete bipartite graph \(K_{n,n}\) with a 1-factor removed.

The existence problems for \((C_k, S_k)\)-decomposition of \(K_{m,n}\) and \(C_{n,n-1}\) have been completely settled by Lee [1] and Lee and Lin [4], respectively. Lee [2] obtained the maximum packing and minimum covering of the balanced complete bipartite multigraph \(\lambda K_{m,n}\) with \((C_k, S_k)\). Lee and Chen [3] gave the maximum packing and minimum covering of \(\lambda K_n\) with \((P_k, S_k)\). This paper gives the solution of finding the minimum \((C_k, S_k)\)-covering of the crown \(C_{n,n-1}\).

2. Preliminaries

Let \(G = (V, E)\) be a graph. For sets \(A \subseteq V(G)\) and \(B \subseteq E(G)\), we use \(G[A]\) to denote the subgraph of \(G\) induced by \(A\) and \(G - B\) (respectively, \(G + B\)) to denote the subgraph obtained from \(G\) by deleting (respectively, adding) the edges in \(B\). When \(G_1, \ldots, G_t\) are graphs, not necessarily disjoint, we write \(G_1 \cup \cdots \cup G_t\) for the graph with vertex set \(\bigcup_{i=1}^t V(G_i)\) and edge set \(\bigcup_{i=1}^t E(G_i)\). When the edge sets are disjoint, \(G = \bigcup_{i=1}^t G_i\) expresses the decomposition of \(G\) into \(G_1, \ldots, G_t\). For a graph \(G\) and a positive integer \(\lambda \geq 2\), we use \(\lambda G\) to denote
the multigraph obtained from $G$ by replacing each edge $e$ by $\lambda$ edges, each of which has the same ends as $e$.

The following results are essential to our proof.

**Lemma 1** [7]. For integers $m$ and $n$ with $m \geq n \geq 1$, the graph $K_{m,n}$ is $S_k$-decomposable if and only if $m \geq k$ and

\[
\begin{cases}
  m \equiv 0 \pmod{k} & \text{if } n < k, \\
  mn \equiv 0 \pmod{k} & \text{if } n \geq k.
\end{cases}
\]

**Lemma 2** [5]. $\lambda C_{n,\ell}$ is $S_k$-decomposable if and only if $k \leq \ell$ and $\lambda n\ell \equiv 0 \pmod{k}$.

**Lemma 3** [5]. Let $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,\ell}$. Suppose that $p$ and $q$ are positive integers such that $q < p \leq \ell$. If $\lambda nq \equiv 0 \pmod{p}$, then there exists a spanning subgraph $G$ of $\lambda C_{n,\ell}$ such that $\deg_G b_j = \lambda q$ for $0 \leq j \leq n - 1$ and $G$ has an $S_p$-decomposition.

**Lemma 4** [6]. For positive integers $k$ and $n$, $C_{n,n-1}$ is $C_k$-decomposable if and only if $n$ is odd, $k$ is even, $4 \leq k \leq 2n$, and $n(n-1) \equiv 0 \pmod{k}$.

## 3. Covering numbers

In this section the covering number of $C_{n,n-1}$ with $k$-cycles and $k$-stars is determined.

**Lemma 5** [4]. If $k$ is an even integer with $k \geq 4$, then $C_{k+1,k}$ is not $(C_k, S_k)$-decomposable.

**Lemma 6.** If $k$ is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is $(C_k, S_k)$-decomposable.

**Proof.** By Lemma 4, we have that $C_{k+1,k}$ is $C_k$-decomposable. Define a $k$-star

$R = \langle b_1, b_2, \ldots, b_k \rangle_{a_0}$. Clearly, $C_{k+1,k} + E(R)$ is a $(C_k, S_k)$-covering with padding $R$. ■

We obtain the following result by Lemmas 5 and 6.

**Corollary 7.** $c(C_{k+1,k}; C_k, S_k) = k + 2$.

**Lemma 8** [4]. If $k$ is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is $(C_k, S_k)$-decomposable.

**Lemma 9.** For integers $r$ and $k$ with $r \geq 3$ and $k > r(r+1)$, $C_{k+r+1,k+r}$ can be decomposed into one copy of $r(r+1)$-cycle and $k + 2r + 1$ copies of $k$-stars.
Proof. Let \( s = r(r+1)/2 \). Trivially, \( k + r + 1 > s \). Let \( A_0 = \{a_0, a_1, \ldots, a_{s-1}\}, B_0 = \{b_0, b_1, \ldots, b_{s-1}\} \), \( H_0 = C_{n,n-1}[A_0 \cup B_0] \), \( H_1 = C_{n,n-1}[A_0 \cup B_0] \), and \( H_2 = C_{n,n-1}[A \cup (B \setminus B_0)] \). Clearly, \( C_{k+r+1,k+r} = H_0 \cup H_1 \cup H_2 \). Note that \( H_0 \) is isomorphic to \( C_{s,s-1} \), \( H_1 \) is isomorphic to \( K_{k+r+1-s,s} \), and \( H_2 \) is isomorphic to \( C_{k+r+1-s,k+r-s} \cup K_{s,k+r+1-s} \). Let

\[
C = (b_1, a_0, b_2, a_1, b_3, a_2, \ldots, b_{s-1}, a_{s-2}, b_0, a_{s-1})
\]

and \( H = H_0 - E(C) \). Trivially, \( C \) is an \((r+1)\)-cycle in \( H_0 \) and \( H = C_{s,s-3} \). Note that \( r - 2 < s - r - 1 \) for \( r \geq 3 \) and \( s(r-2) = rs - r(r+1) = r(s-r-1) \). By Lemma 3, there exists a spanning subgraph \( X \) of \( H \) such that \( \operatorname{deg}_X b_j = r - 2 \), for \( 0 \leq j \leq s - 1 \) and \( X \) has an \( S_{r-r-1} \)-decomposition \( \mathcal{H} \) with \( |\mathcal{H}| = r \). Furthermore, each \( S_{r-r-1} \) has its center in \( A_0 \) since \( \operatorname{deg}_X b_j = r - 2 < s - r - 1 \). Suppose that the centers of the \((s-r-1)\)-stars in \( \mathcal{H} \) are \( a_i, a_{i+1}, \ldots, a_{i+r} \). Let \( S(u) \) be the \((s-r-1)\)-star with center \( a_u \) in \( \mathcal{H} \), and let \( Y = H - E(X) \cup H_1 \). Note that \( \operatorname{deg}_Y b_j = (s - 3 - (r - 2)) + (k + r + 1 - s) = k \) for \( 0 \leq j \leq s - 1 \). Hence \( Y \) has an \( S_k \)-decomposition \( \mathcal{H}(1) \) with \( |\mathcal{H}(1)| = s \). For \( u \in \{1, 2, \ldots, r\} \), define \( S'(u) = H_2[(u)] \cup (B \setminus B_0) \) and \( Z = H_2 - E(\bigcup_{u=1}^r S'(u)) \). Clearly, \( S'(u) \) is a \((k + r + 1 - s)\)-star with center \( a_u \) in \( H_2 \), and \( S(u) \cup S'(u) \) is a \( k \)-star. There are \( r \) copies of such \( k \)-stars. Moreover, \( \operatorname{deg}_Z b_j = k + r - r - k = k \) for \( s \leq j \leq s + r \), and it follows that \( Z \) has an \( S_k \)-decomposition \( \mathcal{H}(2) \) with \( |\mathcal{H}(2)| = k + r - s + 1 \). Thus there are \( s + r + k + r - s + 1 = k + 2r + 1 \) copies of \( k \)-stars. This completes the proof.

Lemma 10. Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k < n - 1 < 2k - 1 \). If \((n-k)(n-k-1) < k \), then \( C_{n,n-1} \) has a \((C_k, S_k)\)-covering with padding \( P_{k-(n-k)(n-k-1)} \).

Proof. Let \( n - 1 = k + r \). From the assumption \( k < n - 1 < 2k - 1 \), we have \( 0 < r < k - 1 \). The proof is divided into two parts according to the value of \( r \).

Case 1. \( r \leq 2 \). Let \( A_0' = \{a_0, a_1, \ldots, a_{k+1}\}, A_1' = \{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}, B_0' = \{b_0, b_1, \ldots, b_{k+1}\}, B_1' = \{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\} \), \( D_0 = C_{n,n-1}[A_0' \cup \{a_k\}] \cup (B_0' \cup \{b_k\}) \), \( D_1 = C_{n,n-1}[A_0' \cup B_0'] \), \( D_2 = C_{n,n-1}[A_1' \cup B_0'] \) and \( D_3 = C_{n,n-1}[A_1' \cup B_1'] \). Clearly, \( C_{n,n-1} = D_0 \cup D_1 \cup D_2 \cup D_3 \). Note that \( D_0 \) is isomorphic to \( C_{k+1,k} \), \( D_1 \) is isomorphic to \( K_{k+r,k} \), \( D_2 \) is isomorphic to \( K_{r,k} \), and \( D_3 \) is isomorphic to \( C_{r+1,r} \). By Lemma 2, we have that \( D_0 \) has a \( k \)-star decomposition \( \langle b_{j+1}, b_{j+2}, \ldots, b_{j+k}\rangle_{a_j} \) for \( 0 \leq j \leq k \), where the subscripts of \( b \)'s are taken modulo \( k + 1 \) in the set of numbers \( \{0, 1, \ldots, k\} \). By Lemma 1, we obtain that \( D_1 \) and \( D_2 \) have \( k \)-star decompositions \( \langle a_0, a_1, \ldots, a_{k-1}\rangle_{b_j} \) and \( \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_i} \), \( i < j \leq k + r \), respectively.

Subcase 1.1. \( r = 1 \). Define a \((k-2)\)-path \( R_1 \) as follows.

\[
R_1 = a_{k+1}b_1a_0b_2a_1b_3a_2 \cdots a_{k-3}b_{k-1}a_k.
\]
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where the subscripts of $a$'s and $b$'s are taken modulo $n$. Then

$$\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup D_3 \cup R_1$$

$$= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup \{a_k b_{k+1}, a_{k+1} b_k\} \cup R_1$$

$$= \langle b_0, b_1, \ldots, b_{k-2}, b_{k+1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-2}, b_k\rangle_{a_{k+1}} \cup a_k b_{k-1} a_{k+1} \cup R_1.$$  

Note that $a_k b_{k-1} a_{k+1} \cup R_1$ is a $k$-cycle. Hence $C_{k+2, k+1} + E(R_1)$ can be decomposed into $k + 3$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+2, k+1}$ has a $(C_k, S_k)$-covering $\mathcal{C}_1$ with $|\mathcal{C}_1| = k + 4$ and padding $R_1$.

Subcase 1.2. $r = 2$. Define a $(k - 6)$-path $R_2$ as follows.

$$R_2 = b_1 a_0 b_2 a_1 \cdots b_{k-3} a_{k-4} b_{k-1},$$

where the subscripts of $a$'s and $b$'s are taken modulo $n$. Then

$$\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup D_3 \cup R_2$$

$$= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup \{a_k b_{k+1}, a_k b_{k+2}, a_{k+1} b_k, a_{k+1} b_{k+2}, a_k b_{k+1} b_{k+2}, a_{k+2} b_k, a_{k+2} b_{k+1}\} \cup R_2$$

$$= \langle b_0, b_2, b_3, \ldots, b_{k-1}, b_{k+1}\rangle_{a_{k+2}} \cup b_{k+1} a_k b_{k+2} a_k b_{k+1} b_{k+2} b_1 \cup R_2.$$  

Note that $b_{k+1} a_k b_{k+2} a_k b_{k+1} b_{k+2} b_1 \cup R_2$ is a $k$-cycle. Hence $C_{k+3, k+2} + E(R_2)$ can decomposed into $k + 5$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+3, k+2}$ has a $(C_k, S_k)$-covering $\mathcal{C}_2$ with $|\mathcal{C}_2| = k + 6$ and padding $R_2$.

Case 2. $r \geq 3$. Let $s = r(r + 1)/2$ and $H_0$, $H_1$ and $H_2$ be the graphs defined in the proof of Lemma 9. Define a $(k - 2s)$-path $R_3$ as follows.

$$R_3 = a_{s-1} b_{s+1} a_s b_{s+2} \cdots a_{\frac{k-s}{2}} b_{\frac{s}{2}} a_{k+r},$$

where the subscripts of $a$'s and $b$'s are taken modulo $n$.

Let $S$ be the $k$-star with center $b_1$ and $C$ be the $2s$-cycle mentioned in Lemma 9. Then

$$S \cup C \cup R_3$$

$$= (S - a_{k+r} b_1 + a_{s-1} b_1) \cup a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3.$$  

Note that $a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3$ is a $k$-cycle. Hence $C_{k+r+1, k+r} + E(R_3)$ can be decomposed into $k + 2r + 1$ copies of $k$-stars and one copy of $k$-cycle, that is, $C_{k+r+1, k+r}$ has a $(C_k, S_k)$-covering $\mathcal{C}_3$ with $|\mathcal{C}_3| = k + 2r + 2$ and padding $R_3$. This settles Case 2.

Before plunging into the proof of the case of $(n - k)(n - k - 1) \geq k$, a result due to Lee and Lin [4] is needed.
Lemma 11 [4]. If $k$ is an even integer with $k \geq 4$, then there exist $k/2 - 1$ edge-disjoint $k$-cycles in $C_{k/2,k/2-1} \cup K_{k/2,k/2}$.

Lemma 12. Let $k$ be a positive even integer and let $n$ be a positive integer with $4 \leq k < n - 1 < 2k - 1$. If $(n - k)(n - k - 1) \geq k$, then $C_{n,n-1}$ has a $(C_k,S_k)$-covering $\mathcal{C}$ with $|\mathcal{C}| = \lceil n(n-1)/k \rceil$.

**Proof.** Let $n - 1 = k + r$. From the assumption $k < n - 1 < 2k - 1$, we have $0 < r < k - 1$. Since $(n - k)(n - k - 1) \geq k$, we assume that $r(r + 1) = \alpha k + \beta$, where $\alpha \geq 1$ and $0 \leq \beta \leq k - 1$. Let $A''_0 = \{a_0,a_1,\ldots,a_{k-1}\}$, $A''_1 = \{a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\}$, $A''_2 = A \setminus (A''_0 \cup A''_1)$, $B''_0 = \{b_0,b_1,\ldots,b_{k-1}\}$, $B''_1 = B \setminus B''_0$. Let $G_i = C_{n,n-1}[A''_0 \cup B''_0]$ for $i \in \{0,1,2\}$ and $G_3 = C_{n,n-1}[A \cup B''_0]$. Clearly, $C_{n,n-1} = G_0 \cup G_1 \cup G_2 \cup G_3$. Note that $G_0$ and $G_1$ are isomorphic to $C_{k/2,k/2-1} \cup K_{k/2,k/2}$, $G_2$ is isomorphic to $K_{k+1,k}$, which is $S_k$-decomposable by Lemma 1, and $G_3$ is isomorphic to $K_{k,r+1} \cup C_{r+1,r}$. Let $p_0 = \lfloor \alpha/2 \rfloor$ and $p_1 = \lfloor \alpha/2 \rfloor$. In the following, we will show that, for each $i \in \{0,1\}$, $G_i$ can be decomposed into $p_i$ copies of $C_k$ and $k/2$ copies of $S_{k-2p_i-1}$, and $G_3$ can be decomposed into $k/2$ copies of $S_{2p_i+1}$ and $r + 1$ copies of $S_{k'}$, $k' \leq k$, such that the $(k - 2p_i - 1)$-stars and $(2p_i + 1)$-stars have their centers in $A''_i$.

We first show the required decomposition of $G_i$ for $i \in \{0,1\}$. Since $r < k - 1$, we have $r + 1 < k$, and in turn $0 < \beta < r$. Thus, $p_0 = \lceil \alpha/2 \rceil \leq \frac{\alpha+1}{2} \leq \frac{(r-1)+1}{2} \leq \frac{k-2}{2} = k - 1$, which implies $p_i \leq k/2 - 1$ for $i \in \{0,1\}$. This assures us that there exist $p_i$ edge-disjoint $k$-cycles in $G_i$ by Lemma 11. Suppose that $Q_{i,0},\ldots,Q_{i,p_i-1}$ are edge-disjoint $k$-cycles in $G_i$. Let $F_i = G_i - E \left( \bigcup_{h=0}^{p_i-1} Q_{i,h} \right)$ and $X_{i,j} = F_i \left[ \{a_{ik/2+j}\} \cup B''_0 \right]$ where $i \in \{0,1\}$, $j \in \{0,\ldots,k/2 - 1\}$. Since $\deg_{G_i} a_{ik/2+j} = k - 1$ and each $Q_{i,h}$ uses two edges incident with $a_{ik/2+j}$ for each $i$ and $j$, we have $\deg_{F_i} a_{ik/2+j} = k - 2p_i - 1$. Hence $X_{i,j}$ is a $(k - 2p_i - 1)$-star with center $a_{ik/2+j}$.

Next we show the required star decomposition of $G_3$. For $j \in \{0,\ldots,k/2 - 1\}$, let

$$X'_{i,j} = \begin{cases} \langle b_{k+(2p_0+1)j}, b_{k+(2p_0+1)j+1}, \ldots, b_{k+(2p_0+1)j+2p_0} \rangle_{a_{ij}} & \text{if } i = 0, \\ \langle b_{(p_0+3/2)k+(2p_1+1)j}, b_{(p_0+3/2)k+(2p_1+1)j+1}, \ldots, b_{(p_0+3/2)k+(2p_1+1)j+2p_1} \rangle_{a_{kj/2+j}} & \text{if } i = 1, \end{cases}$$

where the subscripts of $b$'s are taken modulo $r + 1$ in the set of numbers $\{k,k+1,\ldots,k+r\}$. Since $2p_1 + 1 \leq 2p_0 + 1 \leq \alpha + 2 \leq r + 1$, this assures us that there are enough edges for the construction of $X'_{0,j}$ and $X'_{1,j}$. Note that $X'_{i,j}$ is a $(2p_1 + 1)$-star and $X_{i,j} \cup X'_{i,j}$ is a $k$-star for $i \in \{0,1\}$, $j \in \{0,\ldots,k/2 - 1\}$.

On the other hand, let $k - \beta = \tau(r + 1) + \rho$ where $\tau \geq 0$ and $0 \leq \rho \leq r$. We have that
\[|E(G_3)| - |E\left(\bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j}\right)|
= (k + r)(r + 1) - (2p_0 + 2p_1 + 2)(k/2)
= (k + r)(r + 1) - (\alpha + 1)k
= (k + r)(r + 1) - r(r + 1) - (k - \beta)
= k(r + 1) - \tau(r + 1) - \rho = (k - \tau)(r + 1) - \rho
= (k - \tau - 1)\rho + (k - \tau)(r + 1 - \rho).
\]

Hence there exists a decomposition \( \mathcal{G} \) of \( G_3 - E\left(\bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j}\right) \) into \( \rho \) copies of \((k - \tau - 1)\)-star with center \( b_w \) for \( w = k, k + 1, \ldots, k + \rho - 1 \) and \( r + 1 - \rho \) copies of \((k - \tau)\)-star with center \( b_w \) for \( w = k + \rho, k + \rho + 1, \ldots, k + r \), that is,
\[Y_w = \begin{cases} S_{k - r - 1}, & \text{if } w \in \{k, k + 1, \ldots, k + \rho - 1\}, \\ S_{k - r}, & \text{if } w \in \{k + \rho, k + \rho + 1, \ldots, k + r\}. \end{cases}\]

Define a star \( Y'_w \) as follows.
\[Y'_w = \begin{cases} \langle a_{w_1}, a_{w_2}, \ldots, a_{w_r}, a_{w_{r+1}} \rangle_{b_w}, & \text{if } w \in \{k, k + 1, \ldots, k + \rho - 1\}, \\ \langle a_{w_1}, a_{w_2}, \ldots, a_{w_r} \rangle_{b_w}, & \text{if } w \in \{k + \rho, k + \rho + 1, \ldots, k + r\}, \end{cases}\]
where \( b_w a_{w_t} \in E(X'_{i,j}) \) for \( 1 \leq t \leq r + 1 \). Since \[|E\left(\bigcup_{i \in \{0,1\}} \bigcup_{j \in \{0,\ldots,k/2-1\}} X'_{i,j}\right)| = (\alpha+1)k, |B'_w| = r+1 \text{ and } (\tau+1)(r+1) = \tau(r+1) + (r+1) = (k - \beta - \rho) + (r + 1) < 2k \leq (\alpha+1)k, \]
it follows that \( \tau + 1 < (\alpha+1)k/(r + 1) \). This assures us that there are enough edges for the construction of \( Y'_w \). Note that \( Y_w + E(Y'_w) \) is a \( k \)-star.

Hence \( C_{n,n-1} \) has a \( (C_k, S_k) \)-covering \( \mathcal{C}_4 \) with padding \( \bigcup_{w \in \{k,k+1,\ldots,k+r\}} Y'_w \) and \(|\mathcal{C}_4| = (k + r + 1) + (r + 1) + \alpha = k + 2r + 2 + \alpha = [n(n-1)/k] \). This completes the proof. \( \blacksquare \)

Now, we are ready for the main result of this section.

**Theorem 13.** Let \( k \) be a positive even integer and let \( n \) be a positive integer with \( 4 \leq k \leq n - 1 \). Then
\[c(C_{n,n-1}; C_k, S_k) = \begin{cases} [n(n-1)/k], & \text{if } k < n - 1, \\ k + 2, & \text{if } k = n - 1. \end{cases}\]

**Proof.** Since \(|E(C_{n,n-1})| = n(n-1)\), we have that \( c(C_{n,n-1}; C_k, S_k) \geq [n(n-1)/k] \). Let \( n - 1 = qk + r \), where \( q \) and \( r \) are integers with \( q \geq 1, 0 \leq r \leq k - 1 \). We consider the following two cases.

**Case 1.** \( q = 1 \). For \( r = 0 \), the result follows from Corollary 7. If \( r \neq 0 \), by Lemmas 8, 10 and 12, \( C_{k+r+1,k+r} \) has a \( (C_k, S_k) \)-covering \( \mathcal{C} \) with \(|\mathcal{C}| = [(k + r + 1)(k + r)/k] \).
Case 2. \( q \geq 2 \). Note that
\[
C_{n,n-1} = C_{qk+r+1,qk+r} = C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}.
\]
Trivially, \(|E(C_{(q-1)k+1,(q-1)k})|\), \(|E(K_{(q-1)k,k+r})|\) and \(|E(K_{k+r,(q-1)k})|\) are multiples of \( k \), by Lemmas 1 and 2, we have that \( C_{(q-1)k+1,(q-1)k}, K_{(q-1)k,k+r} \) and \( K_{k+r,(q-1)k} \) have \( S_k \)-decompositions \( \mathcal{A}^{(1)} \), \( \mathcal{A}^{(2)} \) and \( \mathcal{A}^{(3)} \) with \( |\mathcal{A}^{(1)}| = (q-1)((q-1)k+1), |\mathcal{A}^{(2)}| = |\mathcal{A}^{(3)}| = (k+r)(q-1) \). For the case of \( r = 0 \), by Lemma 4, \( C_{k+1,k} \) has a \( C_k \)-decomposition \( \mathcal{E} \) with \( |\mathcal{E}| = k+1 \). Hence \( C_{n,n-1} \) is \((C_k,S_k)\)-decomposable, that is, \( C_{n,n-1} \) has a \((C_k,S_k)\)-covering \( \bigcup_{i=1}^{3} \mathcal{A}^{(i)} \cup \mathcal{E} \) with cardinality \((q-1)((q-1)k+1)+k(q-1)+k(q-1)+k+1 = q(qk+1) = n(n-1)/k \).

For the other case of \( r \neq 0 \), by Lemmas 10 and 12, \( C_{k+r+1,k+r} \) has a \((C_k,S_k)\)-covering \( \mathcal{E}' \) with \( |\mathcal{E}'| = [(k+r+1)(k+r)/k] \). Hence \( \bigcup_{i=1}^{3} \mathcal{A}^{(i)} \cup \mathcal{E}' \) is a \((C_k,S_k)\)-covering of \( C_{n,n-1} \) with cardinality \((q-1)((q-1)k+1)+(k+r)(q-1)+(k+r)(q-1)+[(k+r+1)(k+r)/k] = [(qk+r+1)(qk+r)/k] = [n(n-1)/k] \).

This completes the proof. 

\[
\begin{align*}
\text{References} \\
\end{align*}
\]

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