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ON $L(2,1)$-LABELINGS OF ORIENTED GRAPHS

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Abstract

We extend a result of Griggs and Yeh about the maximum possible value of the $L(2,1)$-labeling number of a graph in terms of its maximum degree to oriented graphs. We consider the problem both in the usual definition of the oriented $L(2,1)$-labeling number and in some variants we introduce.

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1. Introduction

An $L(2,1)$-*labeling*, or $L(2,1)$-*coloring*, of a graph G is a function $f : V(G) \to \{0,\ldots,k\}$ such that $|f(u) - f(v)| \geq 2$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq 1$, if there is a path of length two joining u and v. The minimum value of k among the $L(2,1)$-labelings of G is denoted by $\lambda_{2,1}(G)$, and it is called the $L(2,1)$-*labeling number* of G. This notion was introduced by Yeh [1], and it traces back to the frequency assignment problem of wireless networks introduced by Hale [5].

The definitions above can be extended to oriented graphs (a directed graph whose underlying graph is simple), namely: if G is an oriented graph, an $L(2,1)$-labeling of G is a function $f : V(G) \to \{0,\ldots,k\}$ such that $|f(u) - f(v)| \geq 2$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq 1$, if there is a *directed* path of length two joining u and v. The corresponding $L(2,1)$-labeling number is usually denoted by $\overrightarrow{\lambda}_{2,1}(G)$. These labelings were first considered by Chang and Liaw [2], and the

$L(2,1)$-labeling problem has been extensively studied since then in both undirected and directed versions. We refer the interested reader to the excellent surveys of Calamoneri (an earlier version published in [10] with an updated online version in [6]) and Yeh [7].

One of the most basic results about $L(2,1)$-labelings, which appeared in the seminal paper of Griggs and Yeh [3], is an asymptotically sharp upper bound on $\lambda_{2,1}(G)$ as a function of Δ, the maximum degree of the graph. On the one hand, they proved that there is a greedy $L(2,1)$-labeling of G with $k \leq \Delta^2 + 2\Delta$; on the other hand, every $L(2,1)$-labeling of the incidence graph of a projective plane requires $k \geq \Delta^2 - \Delta$. They conjectured that the stronger bound $\lambda_{2,1}(G) \leq \Delta^2$ holds for every G, which was proved by Havet *et al.* [8] for sufficiently large values of Δ.

In this note, we will address the problem of bounding the $L(2,1)$-labeling number asymptotically in directed graphs. Our results are divided into two sections: in Section 2, we will consider the asymptotic value of the $L(2,1)$-labeling number of oriented graphs as it is defined above. In Section 3, we introduce alternative definitions of this number and deal with the corresponding problems in these new settings.

2. Classical Directed Graph Version

Even though there is a bound on $\overrightarrow{\lambda}_{2,1}(G)$ in terms of $\lambda_{2,1}(H)$, where G is an oriented graph and H is its underlying graph, namely, $\overrightarrow{\lambda}_{2,1}(G) \leq \lambda_{2,1}(H)$, it is usually far from sharp. Indeed, these two quantities behave quite differently: while it is easy to see that $\lambda_{2,1}(H) \geq \Delta(H)+1$ (as every vertex in a neighborhood of a vertex in H must be labeled with a different number), there is no such phenomenon in the oriented case, in which the neighborhood of any vertex can be locally colored with two colors, one for the in-neighborhood and other for the out-neighborhood. In fact, there is no lower bound on $\overrightarrow{\lambda}_{2,1}(G)$ in terms of its maximum degree: for instance, every directed tree T satisfies $\overrightarrow{\lambda}_{2,1}(T) \leq 4$ [2]. On the other hand, for an undirected tree T, $\Delta(T) + 1 \leq \lambda_{2,1}(T) \leq \Delta(T) + 2$ [3]. Similar contrasting results hold for broader classes of oriented planar graphs (see, e.g., [4]).

Motivated by these differences, we show in the following theorem that, for oriented graphs, we can give a sharper bound on $\overrightarrow{\lambda}_{2,1}(G)$ as a function of the maximum degree inside a block (i.e., a maximal biconnected subgraph) of the underlying graph of G (in contrast to its global maximum degree). We also show a construction that yields a lower bound asymptotically equal to half of the upper bound.

Theorem 1. *Let G be an oriented graph with the following property: for every block B of its underlying graph, all the in- and out-degrees of the vertices of $G[B]$, the subgraph of G induced by $V(B)$, are bounded by k. Then $\overrightarrow{\lambda}_{2,1}(G) \leq 2k^2+6k$.*

Proof. We proceed by induction on the number of blocks of H, the underlying graph of G. If H has only one block (that is, it is 2-connected), it is clear that we can color G greedily using at most $2k^2+6k+1$ colors, since the first (respectively, second) directed neighborhood of any vertex v in G contains at most $2k$ (respectively, $2k^2$) vertices, and each of those vertices forbids at most three (respectively, one) colors for v.

On the other hand, if H contains at least two blocks, let v be a cut vertex with the property that at most one of the blocks containing v contains a cut vertex distinct from v. It is clear that such a vertex exists from the tree structure of the blocks of H. Let $B_1, \ldots, B_t$ be the blocks containing v such that v is the only cut vertex of B_i.

We apply induction on the graph $G' = G - \bigcup_{i=1}^{t}(V(B_i)\setminus\{v\})$ to get a coloring of it using at most $2k^2+6k+1$ colors. We are left with the vertices of the blocks B_i (except v) to color.

Let A and B be, respectively, the set of uncolored vertices that point to and from v in G. It is clear that the size of any connected component in A and B is at most k and that the only paths joining these components pass through v. In this way, as v has at most $2k$ colored neighbors in G at this point, we have at least $2k^2+6k+1-2k-3 \geq 2k$ distinct free colors for the vertices in A and B. Let some of the free colors be $c_1 < c_2 < \cdots < c_{2k}$. We use colors $c_1, c_3, \ldots, c_{2k-1}$ for A and $c_2, c_4, \ldots, c_{2k}$ for B, coloring each vertex in a connected component with a distinct color.

Now that $A \cup B$ is colored, we have to color the vertices of $\bigcup_{i=1}^{t} B_i$ at distance at least two from v. We can color these vertices greedily as before, since its neighbors and second neighbors lie inside a block of H, in which the maximum degree is k. ■

The construction, as we show in the next theorem, is more sophisticated than the corresponding one for the undirected case.

Theorem 2. *There is an oriented graph G such that its underlying graph is 2-connected, every in-degree and out-degree in G is bounded by $k+O(1)$ and $\overrightarrow{\lambda}_{2,1}(G) \geq k^2+O(k)$.*

Proof. Let $V(G) = \mathbb{Z}_k^2$, the set of pairs of integers modulo k, where $k \geq 4$ is a positive integer. To simplify the notation, we write ab for the pair $(a,b) \in \mathbb{Z}_k^2$. The arcs of G are defined as follows.

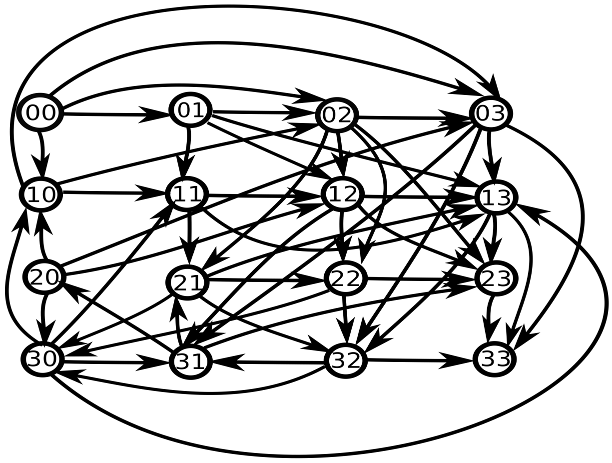
(i) $ab \to bc$, if $c > a$.

(ii) $ab \to (b+1)c$, if $b < k-1$, $c \leq a$ and $c \neq a-1$.

(iii) $ab \to a(b+1)$, if $b < k-1$ and $a \neq b+2$.

(iv) $ab \to (a+1)b$, if $a < k-1$ and $a \neq b+1$.

As an example, the oriented graph that corresponds to $k = 4$ is illustrated below.



It is easy to check that G does not contain opposite arcs and both the in-degree and out-degree of its vertices are bounded by $k+1$. Furthermore, it will be clear from the proof that its underlying graph is 2-connected.

Note that to prove the theorem it suffices to show that, for every pair of vertices ab, cd with $a, b, c, d \notin \{0, k-1\}$, there is a directed path of length at most 2 from ab to cd or vice-versa. Therefore, we assume this condition holds in what follows.

We can find paths of length at most 2 joining ab and cd as follows.

1. If $a < c$ and $b < d$: $ab \to bc \to cd$.
2. If $a > c$ and $b > d$: $cd \to da \to ab$.
3. If $a < c$ and $b > d$: $cd \to (d+1)a \to ab$, except if
   (i) $c = a+1$: $ab \to (b+1)a \to (a+1)d$;
   (ii) $b = d+1$: $cd \to (d+1)(a-1) \to a(d+1)$.
4. If $a > c$ and $b < d$: $ab \to (b+1)c \to cd$, except if
   (i) $a = c+1$: $(c+1)b \to (b+1)(c-1) \to cd$;
   (ii) $d = b+1$: $ab \to (b+1)(c-1) \to c(b+1)$.
5. If $a = c$ and, say, $b < d$ (without loss of generality): $ab \to (b+1)a \to ad$, except if
   (i) $d = b+1$ and $a \neq b+2$: $ab \to a(b+1)$;

(ii) $d = b + 1$ and $a = b + 2$: $a(b + 1) \to ab$.

6. If, say, $a < c$ (without loss of generality) and $b = d$: $ab \to b(c - 1) \to cb$, except if
   (i) $c = a + 1$ and $a \neq b + 1$: $ab \to (a + 1)b$;
   (ii) $c = a + 1$ and $a = b + 1$: $(a + 1)b \to ab$. ■

3. Other Directed Versions

Many different generalizations of the $L(2,1)$-labeling problem have been investigated. The $L(h,k)$-labeling is probably the most famous of them: it is defined as a coloring of the vertices of a graph (either undirected or directed) with integers $\{0,\ldots,n\}$ for which adjacent vertices get colors at least h apart, and vertices connected by a path of length 2 get colors at least k apart. When the interval is considered as a cycle (and hence, for instance, the colors 0 and n are just 1 apart), we get yet another new variant. Again, we refer to the survey of Calamoneri [6] as a comprehensive reference in the area.

In this section, we propose other versions of the problem. A path of lenght two admits three pairwise non-isomorphic orientations: $a \to b \to c$, $a \to b \leftarrow c$, and $a \leftarrow b \to c$; we call these paths P_1, P_2 and P_3, respectively. In this terminology, we can rephrase the definition of an $L(2,1)$-labeling of an oriented graph G as follows: an assignment $f : V(G) \to \{0,\ldots,k\}$ such that $|f(u) - f(v)| \geq 2$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq 1$, if there is a P_1 in G joining u and v.

We study the corresponding problems that arise when we replace P_1 in this definition by P_2 or P_3, or, even more generally, by a subset S of $\{P_1, P_2, P_3\}$. We denote the corresponding minimum value of k by $\lambda_S(G)$. Some of the choices of S lead us back to previous questions, namely, $\lambda_\emptyset(G) = 2\chi(G) - 1$; $\lambda_{\{P_1,P_2,P_3\}}(G) = \lambda_{2,1}(H)$, where H is the underlying graph of G; and $\lambda_{\{P_1\}}(G) = \overrightarrow{\lambda}_{2,1}(G)$. Also, by the symmetry of P_2 and P_3, we have just the following three cases left to consider: $S = \{P_2\}$, $S = \{P_2, P_3\}$ and $S = \{P_1, P_2\}$.

In each one of those cases, we are going to determine the order of magnitude, and, with one exception, the correct asymptotic value of the maximum possible value of $\lambda_S(G)$ in terms of the maximum degree of G.

First, we consider $S = \{P_2\}$, i.e., when the only path of length two considered is $a \to b \leftarrow c$. We have the following asymptotically sharp result.

Theorem 3. *Let G be an oriented graph such that $d_+(v) \leq k$ and $d_-(v) \leq k$ for all $v \in V(G)$, where $d_+(v)$ and $d_-(v)$ denote, respectively, the out- and the in-degree of a vertex v. Then $\lambda_{\{P_2\}}(G) \leq k^2 + O(k)$, and there is a family of graphs that matches this upper bound asymptotically.*

Proof. We color G greedily with the colors $\{0,\dots,k^2+5k\}$: given a vertex v, each of its at most $2k$ neighbors forbid at most 3 colors for v. Among the second neighbors, only the at most $k(k-1)=k^2-k$ vertices that are joined by a P_2 to v forbid colors for v, at most one new color per vertex. In total, at most $3\cdot 2k+k^2-k=k^2+5k$ colors are forbidden for v.

As for the sharpness of the bound, the same construction as in the undirected case works. Let $G=(A,B,E)$ be the oriented bipartite incidence graph of a projective plane with point set A, line set B, $|A|=|B|=k^2+k+1$, and all the edges pointing from A to B. Both the in- and out-degrees of G are bounded by k and there is a P_2 joining every pair of vertices in A. Therefore, at least k^2+k+1 different colors are needed in any valid labeling of G. ■

In the case $S=\{P_2,P_3\}$, we have the following result, which does not yield an asymptotic sharp bound, but a factor 2 for the ratio between the upper and lower estimates.

Theorem 4. *Let G be an oriented graph such that $d_+(v)\le k$ and $d_-(v)\le k$ for all $v\in V(G)$. Then $\lambda_{\{P_2,P_3\}}(G)\le 2k^2+O(k)$. On the other hand, there is a family of graphs G with $d_+(v)=(1+o(1))k$, $d_-(v)=(1+o(1))k$ for every $v\in V(G)$ and $\lambda_{\{P_2,P_3\}}(G)\ge k^2+O(k)$.*

The proof of the upper bound in Theorem 4 is obtained in a similar way as in Theorem 3, i.e., coloring the graph greedily, bounding the number of forbidden colors for a given vertex using the sizes of its first and second neighboorhoods. The lower bound comes from the very same construction as in Theorem 3. We omit the details.

Finally, in the case $S=\{P_1,P_2\}$, we have a different upper bound and an asymptotically sharp construction, as stated in the following theorem.

Theorem 5. *Let G be an oriented graph such that $d_+(v)\le k$ and $d_-(v)\le k$ for all $v\in V(G)$. Then $\lambda_{\{P_1,P_2\}}(G)\le 3k^2+O(k)$. Furthermore, there is a family of graphs that matches this bound asymptotically.*

Proof. Again, we apply the greedy algorithm as in the proof of Theorem 3 to get the upper bound.

On the other hand, consider the following construction: if $H=(A,B,E)$ is the bipartite incidence graph of a projective plane with point set A, line set B and $|A|=|B|=k^2+k+1$ with all edges oriented from A to B, let $H'=(A',B',E')$ and $H''=(A'',B'',E'')$ be two copies of H with edges oriented from A' to B' and A'' to B'', respectively. For a vertex $p\in V(H)$, we denote by p' (respectively, p'') its copy in H' (respectively, H''), and we call p,p',p'' twin vertices. We construct an oriented graph G as follows. The vertex set of G is $V(G)=V(H)\cup V(H')\cup V(H'')$. The edge set of G is $E(G)=E(H)\cup E(H')\cup E(H'')\cup\{(l,p'),(l',p''),(l'',p): l\in$

$B, p \in A$ and $(p,l) \in E(H)\} \cup \{(p,p'),(p',p''),(p'',p) : p \in A\}$. In the graph G, all degrees are bounded by k. Moreover, given two vertices p,q from $A \cup A' \cup A''$, either they are joined by a P_2 (in case both vertices come from the same set), by a P_1 (if they are in different sets and are not twin vertices) or by an edge (if they are twin vertices). This shows that a valid labeling of G must use at least $3(k^2+k+1)$ colors. ■

4. Open Problems

There is a big list of problems to investigate about the labelings defined in the present note. Virtually every question studied for the undirected or the classical directed $L(2,1)$-labelings can be asked in the newly introduced settings. This list includes determining the exact value of the parameters for specific classes of graphs and finding relations between $\lambda_S(G)$ and other graph parameters, as it was done, for instance, with the path covering number [9]. Moreover, it would be interesting to determine the correct asymptotic values in the cases $S = \{P_1\}$ and $S = \{P_2, P_3\}$. In particular, we conjecture that the construction in Theorem 2 can be improved to match the upper bound asymptotically.

Conjecture 6. *There is an oriented graph G for which each in-degree and out-degree is bounded by $(1+o(1))k$ and $\overrightarrow{\lambda}_{2,1}(G) \geq 2k^2 + O(k)$.*

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