HEREDITARY EQUALITY OF DOMINATION AND
EXPONENTIAL DOMINATION IN SUBCUBIC GRAPHS

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Abstract

Let $\gamma(G)$ and $\gamma_e(G)$ denote the domination number and exponential
domination number of graph $G$, respectively. Henning et al., in [Hereditary
equality of domination and exponential domination, Discuss. Math. Graph Theory 38
(2018) 275–285] gave a conjecture: There is a finite set $\mathcal{F}$ of graphs such that a
graph $G$ satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $G$ if and only if $G$
is $\mathcal{F}$-free. In this paper, we study the conjecture for subcubic graphs. We characterize
the class $\mathcal{F}$ by minimal forbidden induced subgraphs and prove that the conjecture
holds for subcubic graphs.

Keywords: dominating set, exponential dominating set, subcubic graphs.

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1. Introduction

Graph theory terminology not presented here can be found in [3]. Let $G$ be a
simple and undirected graph. The vertex set and the edge set of $G$ are denoted by $V(G)$
and $E(G)$, respectively. The degree, neighborhood and closed neighborhood
of a vertex $v$ in the graph $G$ are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph $G$ is clear from context, we simply write $d(v)$, $N(v)$
and $N[v]$, respectively. The minimum degree and maximum degree of the graph $G$
are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $S \subseteq V(G)$; $N(S) = \bigcup_{v \in S} N(v)$
and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. The distance $dist_G(X,Y)$ between two sets $X$ and $Y$ of vertices in $G$ is the minimum length of a path in $G$ between a vertex in $X$ and a vertex in $Y$. If no such path exists, then let $dist_G(X,Y) = \infty$. Let $P_n$, $C_n$ and $K_n$ denote the path, cycle and complete graph with order $n$, respectively. Let $l(G)$ denote the maximum length of an induced cycle in $G$. If $\Delta(G) \leq 3$, then $G$ is called a subcubic graph.

A set $D \subseteq V$ in a graph $G$ is called a dominating set if every vertex outside $D$ is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [3] and [4].

Let $D$ be a set of vertices of a graph $G$. For two vertices $u$ and $v$ of $G$, let $dist_{(G,D)}(u,v)$ be the minimum length of a path $P$ in $G$ between $u$ and $v$ such that $D$ contains exactly one endvertex of $P$ but no internal vertex of $P$. If no such path exists, then let $dist_{(G,D)}(u,v) = \infty$. Note that, if $u$ and $v$ are distinct vertices in $D$, then $dist_{(G,D)}(u,u) = 0$ and $dist_{(G,D)}(u,v) = \infty$. For a vertex $u$ of $G$, let $\omega_{(G,D)}(u) = \sum_{v \in D} \left( \frac{1}{2} \right)^{dist_{(G,D)}(u,v)-1}$, where $\left( \frac{1}{2} \right)^{\infty} = 0$.

Dankelmann et al. [2] define a set $D$ to be an exponential dominating set of $G$ if $\omega_{(G,D)}(u) \geq 1$ for every vertex $u$ of $G$, and the exponential domination number $\gamma_e(G)$ of $G$ as the minimum size of an exponential dominating set of $G$. Note that $\omega_{(G,D)}(u) \geq 2$ for $u \in D$, and that $\omega_{(G,D)}(u) \geq 1$ for every vertex $u$ that has a neighbor in $D$, which implies $\gamma_e(G) \leq \gamma(G)$.

Bessy et al. [1] show that computing the exponential domination number is APX-hard for subcubic graphs. It is not even known how to decide efficiently for a given tree $T$ whether its exponential domination number $\gamma_e(T)$ equals its domination number $\gamma(T)$. The difficulty to decide whether $\gamma_e(G) = \gamma(G)$ for a given graph $G$ motivates the study of the hereditary class $\mathcal{G}$ of graphs that satisfy this equality, that is, $\mathcal{G}$ is the set of those graphs $G$ such that $\gamma_e(H) = \gamma(H)$ for every induced subgraph $H$ of $G$.

Henning et al. [5] proved the following results.

**Proposition 1** [5]. If $G$ is a $\{B,D,K_4,K_{2,3},P_2 \square P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $G$ if and only if $G$ is $\{P_7,C_7,F_1,F_2,F_3,F_4,F_5\}$-free.

**Proposition 2** [5]. If $T$ is a tree, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $T$ if and only if $T$ is $\{P_7,F_1\}$-free.

Furthermore, they gave the following conjecture.

**Conjecture 1** [5]. There is a finite set $\mathcal{G}$ of graphs such that graph $G$ satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $G$ if and only if $G$ is $\mathcal{G}$-free.
In this paper, we study the conjecture for subcubic graphs. We characterize the class $\mathcal{F}$ by minimal forbidden induced subgraphs. Our main result is the following.

**Figure 1.** The graphs $K_{2,3}$, $P_2 \Box P_3$, $B$ and $D$.

**Figure 2.** The graphs $F_1$, $F_2$, $F_3$, $F_4$ and $F_5$.

**Theorem 1.** Let $G$ be a subcubic graph. Then $\gamma(H) = \gamma_e(H)$ for every induced subgraph $H$ of $G$ if and only if $G$ is $\mathcal{F}$-free, where $\mathcal{F} = \{P_7, C_7, F_1, F_2, F_3, F_6, F_7, F_8, F_9, F_{10}, F_{11}\}$.

**Figure 3.** The graphs $F_6, \ldots, F_{11}$. 
2. Proof of Theorem 1

Proof. Since $\gamma(H) > \gamma_e(H)$ for every graph $H$ in $\mathcal{F}$, necessity follows. In order to prove sufficiency, suppose that $G$ is an $\mathcal{F}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of $G$, we have $\gamma(H) = \gamma_e(H)$ for every proper induced subgraph $H$ of $G$. Clearly, $G$ is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$. Since $G$ is $\{P_4, C_7\}$-free, either $G$ is a tree or $G$ is a subcubic graph with $3 \leq l(G) \leq 6$.

By Proposition 2, $G$ is not a tree. Then $G$ is a connected subcubic graph with $3 \leq l(G) \leq 6$. Let $C : x_1x_2x_3 \cdots x_{l(G)x_1}$ be a longest induced cycle of $G$. Let $R = V(G) \setminus V(C)$.

Case 1. $l(G) = 6$. Assume some vertex $z$ has distance 2 from a vertex on $V(C)$ in $G$ and $xyz$ is a path in $G$. If $y$ is adjacent to $x_2$, then $G\{x_1, x_2, x_3, x_4, y, z\} = F_6$, which is a contradiction. If $y$ is adjacent to $x_3$, then $G\{x_1, x_3, x_4, x_5, y, z\} = F_5$, which is a contradiction. By symmetry, we can assume without loss of generality that $y$ is adjacent to neither $x_5$ nor $x_6$. Then $G\{x_1, x_2, x_5, x_6, y, z\} = F_3$, which is a contradiction. So every vertex in $R$ has distance one from one vertex on $V(C)$. Since $G$ is $F_1$-free, every vertex in $R$ has at least two neighbors on $C$. Since $G$ is a subcubic graph and $\gamma(G) \geq 3$, $2 \leq |R| \leq 3$.

Case 1.1. $|R| = 3$. Say $R = \{u, v, w\}$. Then every vertex in $R$ is adjacent to exactly two vertices on $C$. Suppose that there exists one vertex in $R$ that is adjacent to two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G\{x_1, x_2, x_3, x_5, x_6, u\} = F_1$, which is a contradiction. Hence every vertex in $R$ is adjacent to two vertices on $C$ with distance at most two. Since $G$ is subcubic and the three vertices in $R$ can not all be adjacent to two vertices on $C$, there exists a vertex in $R$ that is adjacent to two adjacent vertices on $C$. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_2$. Assume that $x_3$ is adjacent to $v$. Then $v$ is adjacent to either $x_4$ or $x_5$.

If $v$ is adjacent to $x_4$, then $w$ is adjacent to $x_5$ and $x_6$. If $vw \notin E(G)$, then $G\{x_1, x_2, x_3, x_4, x_5, v, w\} = F_{10}$, which is a contradiction. If $vw \in E(G)$, then $G\{x_1, x_2, x_3, x_4, x_6, u, v\} = F_{10}$, which is a contradiction.

If $v$ is adjacent to $x_5$, then $w$ is adjacent to $x_4$ and $x_6$. If $vw \in E(G)$, then $G\{x_1, x_4, x_5, x_6, u, v, w\} = F_8$, which is a contradiction. If $vw \notin E(G)$, then $G\{x_1, x_2, x_5, x_6, v, w\} = F_1$, which is a contradiction.

Case 1.2. $|R| = 2$. Say $R = \{u, v\}$. Suppose that there exists one vertex in $R$ such that it is adjacent to exactly two vertices on $C$ with distance three. Without loss of generality, we can assume that $u$ is adjacent to $x_1$ and $x_4$. Then $G\{x_1, x_2, x_3, x_5, x_6, u\} = F_1$, which is a contradiction. Hence, we can assume that every vertex in $R$ is not adjacent to exactly two vertices on $C$ with distance
three. So there exists one vertex, say \( u \in R \), such that \( u \) is adjacent to two vertices on \( C \) with distance at most two.

Suppose that \( u \) is adjacent to \( x_1 \) and \( x_2 \). If \( v \) is adjacent to \( x_i \), where \( i \in \{4, 5\} \), then \( \{x_1, x_4\} \) or \( \{x_2, x_5\} \) is a dominating set of \( G \) and \( \gamma(G) \leq 2 \), which is a contradiction. So \( v \) is adjacent to exactly two vertices \( x_3 \) and \( x_6 \) on \( C \) with distance three, which is a contradiction.

Suppose that \( u \) is adjacent to \( x_1 \) and \( x_3 \). If \( v \) is adjacent to \( x_i \), where \( i \in \{4, 6\} \), then \( \{x_1, x_4\} \) or \( \{x_3, x_6\} \) is a dominating set of \( G \) and \( \gamma(G) \leq 2 \), which is a contradiction. So \( v \) is adjacent to exactly two vertices \( x_2 \) and \( x_5 \) on \( C \) with distance three, which is a contradiction.

**Case 2.** \( l(G) = 5 \). Assume some vertex \( z \) has distance 2 from \( V(C) \) in \( G \) and \( x_1yz \) is a path in \( G \). If \( y \) is adjacent to \( x_2 \), then \( G[\{x_1, x_2, x_3, x_5, y, z\}] = F_5 \), which is a contradiction. If \( y \) is adjacent to \( x_3 \), then \( G[\{x_2, x_3, x_4, x_5, y, z\}] = F_1 \), which is a contradiction. By symmetry, \( y \) has exactly one neighbor \( x_1 \) on \( C \). Then \( G[\{x_1, x_2, x_3, x_5, y, z\}] = F_1 \), which is a contradiction. So every vertex in \( R \) has distance one from one vertex on \( V(C) \). Since \( G \) is a subcubic graph and \( \gamma(G) \geq 3, 2 \leq |R| \leq 5 \).

**Case 2.1.** \(|R| = 5\). Say \( R = \{y_i | x_i y_i \in E(G), i = 1, 2, \ldots, 5\} \). If \( y_1 y_2 \notin E(G) \), then \( G[\{x_1, x_2, x_4, x_5, y_1, y_2\}] = F_1 \), which is a contradiction. Hence, \( y_1 y_2 \in E(G) \). Similarly, \( y_i y_{i+1} \in E(G) \) for \( i = 1, 2, 3, 4 \). Then \( G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2 \), which is a contradiction.

**Case 2.2.** \(|R| = 4\). Say \( R = \{y_i | x_i y_i \in E(G), i = 1, 2, 3, 4\} \). If \( y_1 y_2 \notin E(G) \), then \( G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1 \), which is a contradiction. If \( y_3 y_4 \notin E(G) \), then \( G[\{x_1, x_2, x_3, x_4, y_3, y_4\}] = F_1 \), which is a contradiction. Hence, \( y_1 y_2 \in E(G) \) and \( y_3 y_4 \in E(G) \). Since \( x_5 \) is adjacent to at most one vertex in \( \{y_1, y_2, y_3, y_4\} \), either \( G[V(C) \cup \{y_1, y_2\}] = F_3 \) or \( G[V(C) \cup \{y_3, y_4\}] = F_3 \), which is a contradiction.

**Case 2.3.** \(|R| = 3\). Let \( G' \) be a graph with \( V(G') = V(C) \cup \{y_1, y_2, y_3\} \) and \( E(G') = E(C) \cup \{x_1 y_1, x_2 y_2, x_3 y_3, y_1 y_2\} \). Suppose that \( G' \) is a subgraph of \( G \). If \( y_1 x_5 \in E(G) \), then \( \{x_3, y_1\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( y_1 x_5 \notin E(G) \). It follows that \( y_1 \) is adjacent to at most one vertex in \( \{x_4, y_3\} \).

Suppose that \( y_1 x_4 \in E(G) \). If \( y_2 x_5 \in E(G) \), then \( G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3 \), which is a contradiction. If \( y_3 x_5 \in E(G) \), then \( G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3 \) or \( G[V(C) \cup \{y_1, y_2, y_3\}] = F_1 \), which is a contradiction. If \( d_G(x_5) = 2 \), then \( G[\{x_2, x_3, x_4, x_5, y_2, y_3\}] = F_1 \) or \( G[V(C) \cup \{y_2, y_3\}] = F_3 \), which is a contradiction. Hence, \( y_1 x_4 \notin E(G) \).

Suppose that \( y_1 y_3 \in E(G) \). If \( y_3 x_4 \in E(G) \), then \( G[\{x_2, x_3, x_4, x_5, y_1, y_3\}] = F_6 \), which is a contradiction. If \( y_3 x_5 \in E(G) \), then \( G[V(C) \cup \{y_1, y_3\}] = F_3 \) or \( G[V(C) \cup \{y_1, y_2, y_3\}] = F_1 \), which is a contradiction. If \( y_3 x_4 \notin E(G) \) and
Suppose that $y_2 \in V(G)$. If $y_2 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, x_5, y_1, y_2\}] = C_6$ and $l(G) \geq 6$, which is a contradiction. Hence, $y_1 y_3 \notin E(G)$.

So $d_G(y_1) = 2$. Since $G$ is $F_3$-free, $y_2 x_4 \in E(G)$ or $y_2 x_5 \in E(G)$. If $y_2 x_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_5, y_2, y_3\}] = F_1$ or $G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3$, which is a contradiction. If $y_2 x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, we can assume that no subgraph in $G$ is isomorphic to $G'$.

By symmetry, we discuss it in the following cases.

**Case 2.3.1.** $R = \{y_i | x_i y_i \in E(G), i = 1, 2, 3\}$. If $E(G[\{y_1, y_2, y_3\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_5, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_3\}]) \neq \emptyset$. Since no subgraph in $G$ is isomorphic to $G'$, $y_1 y_2, y_2 y_3 \notin E(G)$ and $y_1 y_3 \in E(G)$. Since no subgraph in $G$ is isomorphic to $G'$, $y_1 x_4 \notin E(G)$. If $y_2 x_4 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_2 x_4 \in E(G)$ and $y_1 x_3 \notin E(G)$.

Suppose that $y_2 x_3 \in E(G)$. If $E(G[\{y_1, y_2, y_4\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_{10}$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_4\}]) \neq \emptyset$. If $y_1 y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_4\}] = C_6$, which is a contradiction. If $y_1 y_4 \in E(G)$ or $y_2 y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_7$, which is a contradiction. Hence, $y_2 x_3 \notin E(G)$.

So $y_1 x_3 \notin E(G)$, $y_2 x_3 \notin E(G)$ and $y_1 y_2 \in E(G)$. Since no subgraph in $G$ is isomorphic to $G'$, $y_4 x_3 \notin E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2$ or $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_3$, which is a contradiction.

**Case 2.4.** $|R| = 2$. Say $R = \{y_1, y_2\}$ and $y_1 x_1 \in E(G)$. If $y_2 x_i \in E(G)$ for $i \in \{3, 4\}$, then $\{x_1, x_i\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_2 x_3 \notin E(G)$ and $y_2 x_4 \notin E(G)$. Without loss of generality, we can assume that $y_2 x_2 \in E(G)$. If $y_1 x_i \in E(G)$ for $i \in \{4, 5\}$, then $\{x_2, x_i\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_1 x_4 \notin E(G)$ and $y_1 x_5 \notin E(G)$.

If $y_1 x_3 \notin E(G)$ and $y_1 y_2 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_1 x_3 \in E(G)$ or $y_1 y_2 \in E(G)$.

Suppose that $y_1 x_3 \in E(G)$. If $y_2 x_5 \in E(G)$, then $\{x_3, x_5\}$ is a dominating set of $G$, which is a contradiction. Hence, $y_2 x_5 \notin E(G)$. If $y_1 y_2 \notin E(G)$, then $G[\{x_2, x_3, x_4, x_5, y_1, y_2\}] = F_1$, which is a contradiction. If $y_1 y_2 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, $y_1 x_3 \in E(G)$ and $y_1 y_2 \in E(G)$. If $y_2 x_5 \notin E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_3$, which is a contradiction. If $y_2 x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction.
Case 3. \( l(G) = 4 \). Assume some vertex \( t \) has distance 3 from one vertex on \( V(C) \) in \( G \) and \( x_1yzt \) is a path in \( G \). If \( y \) is adjacent to \( x_2 \), then \( G[V(C) \cup \{y,z,t\}] = F_7 \), which is a contradiction. If \( y \) is adjacent to \( x_3 \), then \( G[V(C) \cup \{y,z,t\}] = F_8 \), which is a contradiction. If \( y \) is not adjacent to \( x_i \) for \( i = 2,3,4 \), then \( G[V(C) \cup \{y,z,t\}] = F_2 \), which is a contradiction. So every vertex in \( R \) has distance at most two from a vertex on \( V(C) \). If \( |N(V(C)) \cap R| = 1 \), say \( x_1y_1 \in E(G) \), then \( \{y_1,x_3\} \) is a dominating set of \( G \), which is a contradiction. Hence, \( 2 \leq |N(V(C)) \cap R| \leq 4 \).

Case 3.1. \( |N(V(C)) \cap R| = 4 \). Say \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1,2,3,4\} \). If \( y_1y_3 \in E(G) \), then \( G[\{x_1,x_2,x_3,y_1,y_3\}] = C_5 \), which is a contradiction with \( l(G) = 4 \). By symmetry, \( y_1y_3 \notin E(G) \) and \( y_2y_4 \notin E(G) \).

If \( y_1y_2 \notin E(G) \) and \( y_2y_3 \notin E(G) \), then \( G[\{x_1,x_2,x_3,y_1,y_2,y_3\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \) or \( y_2y_3 \in E(G) \). Without loss of generality, we can assume that \( y_1y_2 \in E(G) \). If \( y_2y_3 \in E(G) \), then \( G[\{x_1,x_3,x_4,y_1,y_2,y_3\}] = C_6 \), which is a contradiction. If \( y_1y_4 \in E(G) \), then \( G[\{x_2,x_3,x_4,y_1,y_2,y_3\}] = C_6 \), which is a contradiction. Hence, \( y_2y_3 \notin E(G) \) and \( y_1y_4 \notin E(G) \). If \( y_3y_4 \notin E(G) \), then \( G[\{x_1,x_2,x_4,y_2,y_3,y_4\}] = F_2 \), which is a contradiction. If \( y_3y_4 \notin E(G) \), then \( G[\{x_2,x_3,y_2,y_3,y_4\}] = F_1 \), which is a contradiction.

Case 3.2. \( |N(V(C)) \cap R| = 3 \). Say \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1,2,3\} \). If \( y_1y_3 \in E(G) \), then \( G[\{x_1,x_2,x_3,y_1,y_3\}] = C_5 \), which is a contradiction. Hence, \( y_1y_3 \notin E(G) \). If \( y_1y_2 \notin E(G) \) and \( y_2y_3 \notin E(G) \), then \( G[\{x_1,x_2,x_3,y_1,y_2,y_3\}] = F_1 \), which is a contradiction. Hence, \( y_1y_2 \in E(G) \) or \( y_2y_3 \in E(G) \).

Without loss of generality, we can assume that \( y_1y_2 \in E(G) \). If \( y_1x_4 \in E(G) \), then \( G[\{x_2,x_3,y_2,y_4\}] = C_5 \), which is a contradiction.

Suppose that \( y_2y_3 \in E(G) \). If \( y_3x_4 \in E(G) \), then \( G[\{x_1,x_4,y_1,y_2,y_3\}] = C_5 \), which is a contradiction. Hence, \( y_1x_4 \notin E(G) \) and \( y_3x_4 \notin E(G) \). Then \( G[\{x_1,x_3,x_4,y_1,y_2,y_3\}] = C_6 \), which is a contradiction. Hence, \( y_2y_3 \notin E(G) \).

Suppose that \( N(y_3) \setminus (V(C) \cup \{y_1,y_2,y_3\}) \neq \emptyset \), say \( t \in N(y_3) \setminus (V(C) \cup \{y_1,y_2,y_3\}) \). Since \( l(G) = 4 \), \( y_3t \notin E(G) \). Then \( G[\{x_1,x_2,x_3,y_1,y_2,y_3,t\}] = F_2 \), which is a contradiction. Hence \( N(y_3) \setminus (V(C) \cup \{y_1,y_2,y_3\}) = \emptyset \).

Suppose that \( N(y_2) \setminus (V(C) \cup N[y_1]) \neq \emptyset \), say \( t \in N(y_2) \setminus (V(C) \cup N[y_1]) \). Since \( l(G) = 4 \), \( y_2t \notin E(G) \). Then \( G[\{x_1,x_2,x_3,y_2,y_3,t\}] = F_1 \), which is a contradiction. Hence, \( N(y_2) \setminus (V(C) \cup N[y_1]) = \emptyset \). Then \( \{y_1,x_3\} \) is a dominating set of \( G \), which is a contradiction.

Case 3.3. \( |N(V(C)) \cap R| = 2 \).

Case 3.3.1. \( N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1,2\} \). Since \( \{x_1,x_2\} \) is not a dominating set of \( G \), \( V(G) \setminus (V(C) \cup \{y_1,y_2\}) \neq \emptyset \). Say \( t_1 \in N(y_1) \setminus (V(C) \cup \{y_2\}) \).

Suppose that \( y_1y_2 \in E(G) \). If \( N(y_2) \setminus (V(C) \cup \{y_1,t_1\}) = \emptyset \), then \( \{y_1,x_3\} \) is a dominating set of \( G \), which is a contradiction. Hence, we can assume that \( t_2 = N(y_2) \setminus \{x_2,y_1\} \). If \( t_1t_2 \notin E(G) \), then \( G[\{x_2,x_3,y_2,t_1,t_2\}] = F_1 \), which
is a contradiction. If $t_1t_2 \in E(G)$, then $G[x_2, x_3, x_4, y_1, y_2, t_1, t_2] = F_2$, which is a contradiction. Hence, we can assume that $y_1y_2 \notin E(G)$.

Suppose that $N(y_2) \setminus V(C) = \emptyset$. Since $G[x_1, x_2, x_4, y_1, y_2, t_1] = F_1$, $x_4y_1 \in E(G)$ or $x_4y_2 \in E(G)$. If $x_4y_1 \in E(G)$, then $\{y_1, x_2\}$ is a dominating set of $G$, which is a contradiction. Hence, $x_4y_2 \in E(G)$. If $x_3y_1 \in E(G)$ or $x_3y_2 \in E(G)$, then $\{y_1, y_2\}$ is a dominating set of $G$, which is a contradiction. Hence, $x_3y_1 \notin E(G)$ and $x_3y_2 \notin E(G)$. Then $G[x_1, x_2, x_3, y_1, y_2, t_1] = F_5$, which is a contradiction. Hence, $N(y_2) \setminus V(C) \neq \emptyset$. Say $t_2 \in N(y_2) \setminus V(C)$. Since $G$ is $F_1$-free, $\{x_3, x_4\} \subseteq N(\{y_1, y_2\})$. Then $\{y_1, y_2\}$ is a dominating set of $G$, which is a contradiction.

Case 3.3.2. $N(C) = \{y_i \mid x_iy_i \in E(G), i = 1, 3\}$. Since $l = 4$, $y_1y_3 \notin E(G)$. Since $\{x_1, x_3\}$ is not a dominating set of $G$, $V(G) \setminus (V(C) \cup \{y_1, y_2\}) \neq \emptyset$. Say $t_1 \in N(y_1) \setminus V(C)$. If $N(y_2) \setminus V(C) = \emptyset$, then $\{y_1, x_3\}$ is a dominating set of $G$, which is a contradiction. Hence, we can assume that $t_3 \in N(y_3) \setminus V(C)$. Then $G[x_1, x_2, x_3, y_1, y_3, t_1, t_3] = P_7$, which is a contradiction.

Case 4. $l(G) = 3$. Since $G$ is $P_7$-free, every vertex in $R$ has distance at most 4 from one vertex on $V(C)$. Assume vertex $y_4$ has distance 4 from one vertex on $V(C)$ in $G$ and $x_1y_1y_2y_3y_1$ is a path in $G$. Since $\{x_1, y_3\}$ is not a dominating set of $G$, there is a vertex $u$ at distance 2 from $\{x_1, y_3\}$ in $G$. If $ux_1 \in E(G)$ for $i \in \{2, 3\}$, then $G[u, x_1, x_1, y_1, y_2, y_3, y_4] = P_7$, which is a contradiction. Suppose that $u$ is adjacent to $y_1$. If $uy_2 \notin E(G)$, then $G[u, x_1, x_2, y_1, y_2, y_3] = F_1$, which is a contradiction. If $uy_2 \in E(G)$, then $G[u, x_1, x_2, y_1, y_2, y_3, y_4] = F_{10}$, which is a contradiction. Suppose that $u$ is adjacent to $y_1$. If $uy_2 \notin E(G)$, then $G[u, x_1, x_2, y_1, y_2, y_3] = F_6$, which is a contradiction. Since $l = 3$, $\{x_2, x_3, y_1\} \subseteq N(v) = \emptyset$. So $G[u, v, x_1, x_2, y_1, y_2, y_3] = P_7$, which is a contradiction. Hence, we can assume that every vertex in $R$ has distance at most 3 from one vertex on $V(C)$.

Case 4.1. $|N(V(C)) \cap R| = 3$. Say $|N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1, 2, 3\}$. Since $l = 3$, $E(G[\{y_1, y_2, y_3\}]) = \emptyset$. Then $G[x_1, x_2, x_3, y_1, y_2, y_3] = F_6$, which is a contradiction.

Case 4.2. $|N(V(C)) \cap R| = 2$. Say $|N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G), i = 1, 2\}$. Since $l = 3$, $y_1y_2 \notin E(G)$. Suppose that there exists an induced path $x_1y_1u_1v_1$. Since $G$ is $P_7$-free, $N(y_2) \setminus V(C) = \emptyset$.

Suppose that there exists a vertex $u$ such that $u \notin N(y_1) \setminus \{x_1, u_1\}$. If $u_1u \notin E(G)$, then $G[u, x_1, x_2, y_1, u_1, v_1] = F_1$, which is a contradiction. If $u_1u \in E(G)$, then $\{u_1, x_2\}$ is a dominating set of $G$, which is a contradiction. If $N(y_1) \setminus \{x_1, u_1\} = \emptyset$, then $\{u_1, x_2\}$ is a dominating set of $G$, which is a contradiction. Hence, we can assume that every vertex in $V(G) \setminus (V(C) \cup \{y_1, y_2\})$ is adjacent to exactly one vertex in $\{y_1, y_2\}$. 
If $N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) = \emptyset$, then $\{x_i, y_j\}$ is a dominating set of $G$, where $i, j \in \{1, 2\}$ and $j \neq i$, which is a contradiction. Suppose that $N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) \neq \emptyset$ for $i \in \{1, 2\}$. If $x_3$ is not adjacent to $y_1$ and $y_2$, then $G[\{x_1, x_2, x_3, y_1, s_1, y_2, s_2\}] = F_{10}$, where $s_i \in N(y_i)$, which is a contradiction. If $x_3$ is adjacent to $y_1$ or $y_2$, then $\{y_1, y_2\}$ is a dominating set of $G$, which is a contradiction.

\textbf{Case 4.3.} $|N(V(C)) \cap R| = 1$. Say $y_1x_1 \in E(G)$. Since $\{x_1, y_1\}$ is not a dominating set of $G$, there is a vertex $u$ at distance 2 from $y_1$ in $G$. Without loss of generality, we can assume that $y_1vu$ be a induced path. If there exists a vertex $t$ such that $y_1t \in E(G)$. If $tv \notin E(G)$, then $G[\{u, v, x_1, x_2, y_1, t\}] = F_1$, which is a contradiction. Suppose that $tv \in E(G)$. If $N(t) \setminus \{y_1, v\} = \emptyset$, say $s \in N(t) \setminus \{y_1, v\}$, then $G[\{t, s, u, v, u, x_1, y_1\}] = F_6$, which is a contradiction. If $N(t) \setminus \{y_1, v\} = \emptyset$ or $d_G(y_1) = 2$, then $\{v, x_1\}$ is a dominating set of $G$, which is a contradiction. \hfill \blacksquare

3. Remark

Henning \textit{et al.} also gave the following conjecture.

\textbf{Conjecture 2} [5]. The set $\mathcal{F}$ in Conjecture 1 can be chosen such that $\gamma(F) = 3$ and $\gamma_e(F) = 2$ for every graph $F$ in $\mathcal{F}$.

It is obvious that the conjecture holds for subcubic graphs.

\textbf{References}


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