Discussiones Mathematicae
Graph Theory xx (xxxx) 1–9
doi:10.7151/dmgt.2237

HEREDITARY EQUALITY OF DOMINATION AND EXPONENTIAL DOMINATION IN SUBCUBIC GRAPHS

Xue-Gang Chen, Yu-Feng Wang

and

Xiao-Fei Wu

Department of Mathematics
North China Electric Power University
Beijing 102206, China

e-mail: gxcxdm@163.com

Abstract

Let $\gamma(G)$ and $\gamma_e(G)$ denote the domination number and exponential domination number of graph G, respectively. Henning *et al.*, in [*Hereditary equality of domination and exponential domination*, Discuss. Math. Graph Theory 38 (2018) 275–285] gave a conjecture: There is a finite set $\mathscr{F}$ of graphs such that a graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\mathscr{F}$-free. In this paper, we study the conjecture for subcubic graphs. We characterize the class $\mathscr{F}$ by minimal forbidden induced subgraphs and prove that the conjecture holds for subcubic graphs.

Keywords: dominating set, exponential dominating set, subcubic graphs.

2010 Mathematics Subject Classification: 05C69, 05C35.

1. Introduction

Graph theory terminology not presented here can be found in [3]. Let G be a simple and undirected graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $S \subseteq V(G)$; $N(S) = \bigcup_{v \in S} N(v)$

and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. The distance $dist_G(X, Y)$ between two sets X and Y of vertices in G is the minimum length of a path in G between a vertex in X and a vertex in Y. If no such path exists, then let $dist_G(X, Y) = \infty$. Let P_n, C_n and K_n denote the path, cycle and complete graph with order n, respectively. Let $l(G)$ denote the maximum length of an induced cycle in G. If $\Delta(G) \leq 3$, then G is called a *subcubic graph*.

A set $D \subseteq V$ in a graph G is called a *dominating set* if every vertex outside D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [3] and [4].

Let D be a set of vertices of a graph G. For two vertices u and v of G, let $dist_{(G,D)}(u, v)$ be the minimum length of a path P in G between u and v such that D contains exactly one endvertex of P but no internal vertex of P. If no such path exists, then let $dist_{(G,D)}(u, v) = \infty$. Note that, if u and v are distinct vertices in D, then $dist_{(G,D)}(u, u) = 0$ and $dist_{(G,D)}(u, v) = \infty$. For a vertex u of G, let $\omega_{(G,D)}(u) = \sum_{v \in D} \left(\frac{1}{2}\right)^{dist_{(G,D)}(u,v)-1}$, where $\left(\frac{1}{2}\right)^{\infty} = 0$.

Dankelmann *et al.* [2] define a set D to be an *exponential dominating set* of G if $\omega_{(G,D)}(u) \geq 1$ for every vertex u of G, and the *exponential domination number* $\gamma_e(G)$ of G as the minimum size of an exponential dominating set of G. Note that $\omega_{(G,D)}(u) \geq 2$ for $u \in D$, and that $\omega_{(G,D)}(u) \geq 1$ for every vertex u that has a neighbor in D, which implies $\gamma_e(G) \leq \gamma(G)$.

Bessy *et al.* [1] show that computing the exponential domination number is APX-hard for subcubic graphs. It is not even known how to decide efficiently for a given tree T whether its exponential domination number $\gamma_e(T)$ equals its domination number $\gamma(T)$. The difficulty to decide whether $\gamma_e(G) = \gamma(G)$ for a given graph G motivates the study of the hereditary class $\mathscr{G}$ of graphs that satisfy this equality, that is, $\mathscr{G}$ is the set of those graphs G such that $\gamma_e(H) = \gamma(H)$ for every induced subgraph H of G.

Henning *et al.* [5] proved the following results.

Proposition 1 [5]**.** *If G is a $\{B, D, K_4, K_{2,3}, P_2 \square P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1, F_2, F_3, F_4, F_5\}$-free.*

Proposition 2 [5]**.** *If T is a tree, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of T if and only if T is $\{P_7, F_1\}$-free.*

Furthermore, they gave the following conjecture.

Conjecture 1 [5]**.** *There is a finite set $\mathscr{F}$ of graphs such that graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\mathscr{F}$-free.*

In this paper, we study the conjecture for subcubic graphs. We characterize the class $\mathscr{F}$ by minimal forbidden induced subgraphs. Our main result is the following.

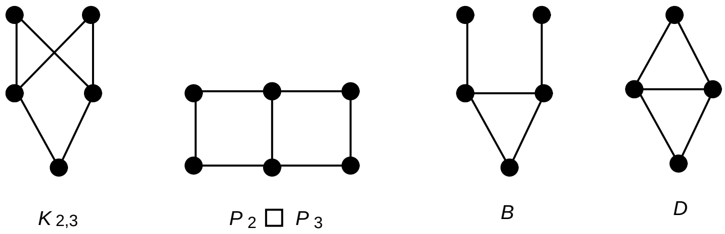

Figure 1. The graphs $K_{2,3}$, $P_2 \Box P_3$, B and D.

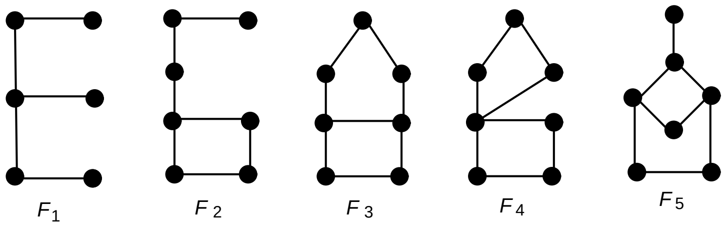

Figure 2. The graphs F_1, F_2, F_3, F_4 and F_5.

Theorem 1. *Let G be a subcubic graph. Then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\mathscr{F}$-free, where $\mathscr{F} = \{P_7, C_7, F_1, F_2, F_3, F_6, F_7, F_8, F_9, F_{10}, F_{11}\}$.*

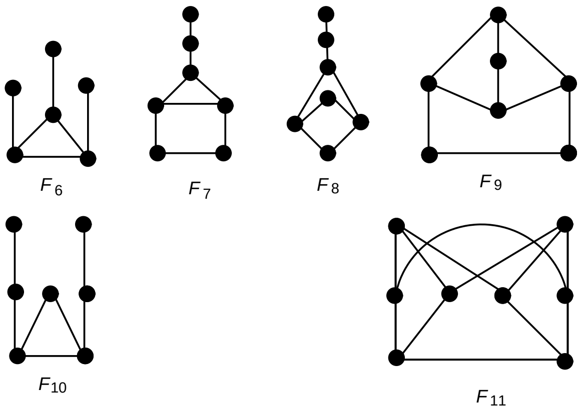

Figure 3. The graphs $F_6, \ldots, F_{11}$.

2. Proof of Theorem 1

Proof. Since $\gamma(H) > \gamma_e(H)$ for every graph H in $\mathscr{F}$, necessity follows. In order to prove sufficiency, suppose that G is an $\mathscr{F}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of G, we have $\gamma(H) = \gamma_e(H)$ for every proper induced subgraph H of G. Clearly, G is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$. Since G is $\{P_7, C_7\}$-free, either G is a tree or G is a subcubic graph with $3 \leq l(G) \leq 6$.

By Propostion 2, G is not a tree. Then G is a connected subcubic graph with $3 \leq l(G) \leq 6$. Let $C : x_1x_2x_3 \cdots x_{l(G)}x_1$ be a longest induced cycle of G. Let $R = V(G) \setminus V(C)$.

Case 1. $l(G) = 6$. Assume some vertex z has distance 2 from a vertex on $V(C)$ in G and x_1yz is a path in G. If y is adjacent to x_2, then $G[\{x_1, x_2, x_3, x_6, y, z\}] = F_6$, which is a contradiction. If y is adjacent to x_3, then $G[\{x_1, x_3, x_4, x_6, y, z\}] = F_1$, which is a contradiction. By symmetry, we can assume without loss of generality that y is adjacent to neither x_5 nor x_6. Then $G[\{x_1, x_2, x_5, x_6, y, z\}] = F_1$, which is a contradiction. So every vertex in R has distance one from one vertex on $V(C)$. Since G is F_1-free, every vertex in R has at least two neighbors on C. Since G is a subcubic graph and $\gamma(G) \geq 3$, $2 \leq |R| \leq 3$.

Case 1.1. $|R| = 3$. Say $R = \{u, v, w\}$. Then every vertex in R is adjacent to exactly two vertices on C. Suppose that there exists one vertex in R that is adjacent to two vertices on C with distance three. Without loss of generality, we can assume that u is adjacent to x_1 and x_4. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence every vertex in R is adjacent to two vertices on C with distance at most two. Since G is subcubic and the three vertices in R can not all be adjacent to two vertices on C, there exists a vertex in R that is adjacent to two adjacent vertices on C. Without loss of generality, we can assume that u is adjacent to x_1 and x_2. Assume that x_3 is adjacent to v. Then v is adjacent to either x_4 or x_5.

If v is adjacent to x_4, then w is adjacent to x_5 and x_6. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, x_5, v, w\}] = F_{10}$, which is a contradiction. If $vw \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, x_6, u, w\}] = F_{10}$, which is a contradiction.

If v is adjacent to x_5, then w is adjacent to x_4 and x_6. If $vw \in E(G)$, then $G[\{x_1, x_4, x_5, x_6, u, v, w\}] = F_8$, which is a contradiction. If $vw \notin E(G)$, then $G[\{x_1, x_2, x_5, x_6, v, w\}] = F_1$, which is a contradiction.

Case 1.2. $|R| = 2$. Say $R = \{u, v\}$. Suppose that there exists one vertex in R such that it is adjacent to exactly two vertices on C with distance three. Without loss of generality, we can assume that u is adjacent to x_1 and x_4. Then $G[\{x_1, x_2, x_3, x_5, x_6, u\}] = F_1$, which is a contradiction. Hence, we can assume that every vertex in R is not adjacent to exactly two vertices on C with distance three. So there exists one vertex, say $u \in R$, such that u is adjacent to two vertices on C with distance at most two.

Suppose that u is adjacent to x_1 and x_2. If v is adjacent to x_i, where $i \in \{4,5\}$, then $\{x_1, x_4\}$ or $\{x_2, x_5\}$ is a dominating set of G and $\gamma(G) \leq 2$, which is a contradiction. So v is adjacent to exactly two vertices x_3 and x_6 on C with distance three, which is a contradiction.

Suppose that u is adjacent to x_1 and x_3. If v is adjacent to x_i, where $i \in \{4,6\}$, then $\{x_1, x_4\}$ or $\{x_3, x_6\}$ is a dominating set of G and $\gamma(G) \leq 2$, which is a contradiction. So v is adjacent to exactly two vertices x_2 and x_5 on C with distance three, which is a contradiction.

Case 2. $l(G) = 5$. Assume some vertex z has distance 2 from $V(C)$ in G and x_1yz is a path in G. If y is adjacent to x_2, then $G[\{x_1, x_2, x_3, x_5, y, z\}] = F_6$, which is a contradiction. If y is adjacent to x_3, then $G[\{x_2, x_3, x_4, x_5, y, z\}] = F_1$, which is a contradiction. By symmetry, y has exactly one neighbor x_1 on C. Then $G[\{x_1, x_2, x_3, x_5, y, z\}] = F_1$, which is a contradiction. So every vertex in R has distance one from one vertex on $V(C)$. Since G is a subcubic graph and $\gamma(G) \geq 3$, $2 \leq |R| \leq 5$.

Case 2.1. $|R| = 5$. Say $R = \{y_i \mid x_i y_i \in E(G), i = 1, 2, \ldots, 5\}$. If $y_1y_2 \notin E(G)$, then $G[\{x_1, x_2, x_4, x_5, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_1y_2 \in E(G)$. Similarly, $y_iy_{i+1} \in E(G)$ for $i = 1, 2, 3, 4$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2$, which is a contradiction.

Case 2.2. $|R| = 4$. Say $R = \{y_i \mid x_i y_i \in E(G), i = 1, 2, 3, 4\}$. If $y_1y_2 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. If $y_3y_4 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_3, y_4\}] = F_1$, which is a contradiction. Hence, $y_1y_2 \in E(G)$ and $y_3y_4 \in E(G)$. Since x_5 is adjacent to at most one vertex in $\{y_1, y_2, y_3, y_4\}$, either $G[V(C) \cup \{y_1, y_2\}] = F_3$ or $G[V(C) \cup \{y_3, y_4\}] = F_3$, which is a contradiction.

Case 2.3. $|R| = 3$. Let G' be a graph with $V(G') = V(C) \cup \{y_1, y_2, y_3\}$ and $E(G') = E(C) \cup \{x_1y_1, x_2y_2, x_3y_3, y_1y_2\}$. Suppose that G' is a subgraph of G. If $y_1x_5 \in E(G)$, then $\{x_3, y_1\}$ is a dominating set of G, which is a contradiction. Hence, $y_1x_5 \notin E(G)$. It follows that y_1 is adjacent to at most one vertex in $\{x_4, y_3\}$.

Suppose that $y_1x_4 \in E(G)$. If $y_2x_5 \in E(G)$, then $G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_8$, which is a contradiction. If $y_3x_5 \in E(G)$, then $G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3$ or $G[V(C) \cup \{y_1, y_2, y_3\}] = F_{11}$, which is a contradiction. If $d_G(x_5) = 2$, then $G[\{x_2, x_3, x_4, x_5, y_2, y_3\}] = F_1$ or $G[V(C) \cup \{y_2, y_3\}] = F_3$, which is a contradiction. Hence, $y_1x_4 \notin E(G)$.

Suppose that $y_1y_3 \in E(G)$. If $y_3x_4 \in E(G)$, then $G[\{x_2, x_3, x_4, x_5, y_1, y_3\}] = F_6$, which is a contradiction. If $y_3x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_3$ or $G[V(C) \cup \{y_1, y_2, y_3\}] = F_{11}$, which is a contradiction. If $y_3x_4 \notin E(G)$ and

$y_3x_5 \notin E(G)$, then $G[\{x_1, x_3, x_4, x_5, y_1, y_3\}] = C_6$ and $l(G) \geq 6$, which is a contradiction. Hence, $y_1y_3 \notin E(G)$.

So $d_G(y_1) = 2$. Since G is F_3-free, $y_2x_4 \in E(G)$ or $y_2x_5 \in E(G)$. If $y_2x_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_5, y_2, y_3\}] = F_1$ or $G[\{x_1, x_2, x_3, x_5, y_1, y_2, y_3\}] = F_3$, which is a contradiction. If $y_2x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, we can assume that no subgraph in G is isomorphic to G'.

By symmetry, we discuss it in the following cases.

Case 2.3.1. $R = \{y_i \,|\, x_iy_i \in E(G),\ i = 1, 2, 3\}$. If $E(G[\{y_1, y_2, y_3\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_3\}]) \neq \emptyset$. Since no subgraph in G is isomorphic to G', $y_1y_2, y_2y_3 \notin E(G)$ and $y_1y_3 \in E(G)$. Since no subgraph in G is isomorphic to G', $y_1x_4 \notin E(G)$. If $y_2x_4 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_2x_4 \in E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}] = F_3$, which is a contradiction.

Case 2.3.2. $R = \{y_i \,|\, x_iy_i \in E(G),\ i = 1, 2, 4\}$. If $E(G[\{y_1, y_2, x_3\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $E(G[\{y_1, y_2, x_3\}]) \neq \emptyset$. Suppose that $y_1x_3 \in E(G)$. Since G is F_1-free and no subgraph in G is isomorphic to G', $y_1y_2, y_1y_4 \notin E(G)$ and $y_2y_4 \in E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_3$, which is a contradiction. Hence, $y_1x_3 \notin E(G)$.

Suppose that $y_2x_3 \in E(G)$. If $E(G[\{y_1, y_2, y_4\}]) = \emptyset$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_{10}$, which is a contradiction. Hence, $E(G[\{y_1, y_2, y_4\}]) \neq \emptyset$. If $y_1y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_4\}] = C_6$, which is a contradiction. If $y_1y_2 \in E(G)$ or $y_2y_4 \in E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_7$, which is a contradiction. Hence, $y_2x_3 \notin E(G)$.

So $y_1x_3 \notin E(G)$, $y_2x_3 \notin E(G)$ and $y_1y_2 \in E(G)$. Since no subgraph in G is isomorphic to G', $y_4x_3 \notin E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_2$ or $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}] = F_3$, which is a contradiction.

Case 2.4. $|R| = 2$. Say $R = \{y_1, y_2\}$ and $y_1x_1 \in E(G)$. If $y_2x_i \in E(G)$ for $i \in \{3, 4\}$, then $\{x_1, x_i\}$ is a dominating set of G, which is a contradiction. Hence, $y_2x_3 \notin E(G)$ and $y_2x_4 \notin E(G)$. Without loss of generality, we can assume that $y_2x_2 \in E(G)$. If $y_1x_i \in E(G)$ for $i \in \{4, 5\}$, then $\{x_2, x_i\}$ is a dominating set of G, which is a contradiction. Hence, $y_1x_4 \notin E(G)$ and $y_1x_5 \notin E(G)$.

If $y_1x_3 \notin E(G)$ and $y_1y_2 \notin E(G)$, then $G[\{x_1, x_2, x_3, x_4, y_1, y_2\}] = F_1$, which is a contradiction. Hence, $y_1x_3 \in E(G)$ or $y_1y_2 \in E(G)$.

Suppose that $y_1x_3 \in E(G)$. If $y_2x_5 \in E(G)$, then $\{x_3, x_5\}$ is a dominating set of G, which is a contradiction. Hence, $y_2x_5 \notin E(G)$. If $y_1y_2 \notin E(G)$, then $G[\{x_2, x_3, x_4, x_5, y_1, y_2\}] = F_1$, which is a contradiction. If $y_1y_2 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction. Hence, $y_1x_3 \notin E(G)$ and $y_1y_2 \in E(G)$. If $y_2x_5 \notin E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_3$, which is a contradiction. If $y_2x_5 \in E(G)$, then $G[V(C) \cup \{y_1, y_2\}] = F_9$, which is a contradiction.

Case 3. $l(G) = 4$. Assume some vertex t has distance 3 from one vertex on $V(C)$ in G and x_1yzt is a path in G. If y is adjacent to x_2, then $G[V(C) \cup \{y, z, t\}] = F_7$, which is a contradiction. If y is adjacent to x_3, then $G[V(C) \cup \{y, z, t\}] = F_8$, which is a contradiction. If y is not adjacent to x_i for $i = 2, 3, 4$, then $G[V(C) \cup \{y, z, t\}] = F_2$, which is a contradiction. So every vertex in R has distance at most two from a vertex on $V(C)$. If $|N(V(C)) \cap R| = 1$, say $x_1y_1 \in E(G)$, then $\{y_1, x_3\}$ is a dominating set of G, which is a contradiction. Hence, $2 \le |N(V(C)) \cap R| \le 4$.

Case 3.1. $|N(V(C)) \cap R| = 4$. Say $N(V(C)) \cap R = \{y_i \,|\, x_iy_i \in E(G), i = 1, 2, 3, 4\}$. If $y_1y_3 \in E(G)$, then $G[\{x_1, x_2, x_3, y_1, y_3\}] = C_5$, which is a contradiction with $l(G) = 4$. By symmetry, $y_1y_3 \notin E(G)$ and $y_2y_4 \notin E(G)$.

If $y_1y_2 \notin E(G)$ and $y_2y_3 \notin E(G)$, then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $y_1y_2 \in E(G)$ or $y_2y_3 \in E(G)$. Without loss of generality, we can assume that $y_1y_2 \in E(G)$. If $y_2y_3 \in E(G)$, then $G[\{x_1, x_3, x_4, y_1, y_2, y_3\}] = C_6$, which is a contradiction. If $y_1y_4 \in E(G)$, then $G[\{x_2, x_3, x_4, y_1, y_2, y_4\}] = C_6$, which is a contradiction. Hence, $y_2y_3 \notin E(G)$ and $y_1y_4 \notin E(G)$. If $y_3y_4 \in E(G)$, then $G[\{x_1, x_3, x_4, y_1, y_2, y_3, y_4\}] = F_2$, which is a contradiction. If $y_3y_4 \notin E(G)$, then $G[\{x_2, x_3, x_4, y_2, y_3, y_4\}] = F_1$, which is a contradiction.

Case 3.2. $|N(V(C)) \cap R| = 3$. Say $N(V(C)) \cap R = \{y_i \,|\, x_iy_i \in E(G), i = 1, 2, 3\}$. If $y_1y_3 \in E(G)$, then $G[\{x_1, x_2, x_3, y_1, y_3\}] = C_5$, which is a contradiction. Hence $y_1y_3 \notin E(G)$. If $y_1y_2 \notin E(G)$ and $y_2y_3 \notin E(G)$, then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. Hence, $y_1y_2 \in E(G)$ or $y_2y_3 \in E(G)$. Without loss of generality, we can assume that $y_1y_2 \in E(G)$. If $y_1x_4 \in E(G)$, then $G[\{x_2, x_3, x_4, y_1, y_2\}] = C_5$, which is a contradiction.

Suppose that $y_2y_3 \in E(G)$. If $y_3x_4 \in E(G)$, then $G[\{x_1, x_4, y_1, y_2, y_3\}] = C_5$, which is a contradiction. Hence, $y_1x_4 \notin E(G)$ and $y_3x_4 \notin E(G)$. Then $G[\{x_1, x_3, x_4, y_1, y_2, y_3\}] = C_6$, which is a contradiction. Hence $y_2y_3 \notin E(G)$.

Suppose that $N(y_3) \setminus (V(C) \cup \{y_1, y_2, y_3\}) \neq \emptyset$, say $t \in N(y_3) \setminus (V(C) \cup \{y_1, y_2, y_3\})$. Since $l(G) = 4$, $y_1t, y_2t \notin E(G)$. Then $G[\{x_1, x_2, x_3, y_1, y_2, y_3, t\}] = F_2$, which is a contradiction. Hence $N(y_3) \setminus (V(C) \cup \{y_1, y_2, y_3\}) = \emptyset$.

Suppose that $N(y_2) \setminus (V(C) \cup N[y_1]) \neq \emptyset$, say $t \in N(y_2) \setminus (V(C) \cup N[y_1])$. Since $l(G) = 4$, $y_3t \notin E(G)$. Then $G[\{x_1, x_2, x_3, y_2, y_3, t\}] = F_1$, which is a contradiction. Hence, $N(y_2) \setminus (V(C) \cup N[y_1]) = \emptyset$. Then $\{y_1, x_3\}$ is a dominating set of G, which is a contradiction.

Case 3.3. $|N(V(C)) \cap R| = 2$.

Case 3.3.1. $N(V(C)) \cap R = \{y_i \,|\, x_iy_i \in E(G), i = 1, 2\}$. Since $\{x_1, x_2\}$ is not a dominating set of G, $V(G) \setminus (V(C) \cup \{y_1, y_2\}) \neq \emptyset$. Say $t_1 \in N(y_1) \setminus (V(C) \cup \{y_2\})$. Suppose that $y_1y_2 \in E(G)$. If $N(y_2) \setminus (V(C) \cup \{y_1, t_1\}) = \emptyset$, then $\{y_1, x_3\}$ is a dominating set of G, which is a contradiction. Hence, we can assume that $t_2 = N(y_2) \setminus \{x_2, y_1\}$. If $t_1t_2 \notin E(G)$, then $G[\{x_2, x_3, y_1, y_2, t_1, t_2\}] = F_1$, which is a contradiction. If $t_1t_2 \in E(G)$, then $G[\{x_2, x_3, x_4, y_1, y_2, t_1, t_2\}] = F_2$, which is a contradiction. Hence, we can assume that $y_1y_2 \notin E(G)$.

Suppose that $N(y_2) \setminus V(C) = \emptyset$. Since $G[\{x_1, x_2, x_4, y_1, y_2, t_1\}] = F_1$, $x_4y_1 \in E(G)$ or $x_4y_2 \in E(G)$. If $x_4y_1 \in E(G)$, then $\{y_1, x_2\}$ is a dominating set of G, which is a contradiction. Hence, $x_4y_2 \in E(G)$. If $x_3y_1 \in E(G)$ or $x_3y_2 \in E(G)$, then $\{y_1, y_2\}$ is a dominating set of G, which is a contradiction. Hence, $x_3y_1 \notin E(G)$ and $x_3y_2 \notin E(G)$. Then $G[\{x_1, x_2, x_3, x_4, y_1, y_2, t_1\}] = F_8$, which is a contradiction. Hence, $N(y_2) \setminus V(C) \neq \emptyset$. Say $t_2 \in N(y_2) \setminus V(C)$. Since G is F_1-free, $\{x_3, x_4\} \subseteq N(\{y_1, y_2\})$. Then $\{y_1, y_2\}$ is a dominating set of G, which is a contradiction.

Case 3.3.2. $N(C) = \{y_i \mid x_iy_i \in E(G),\ i = 1, 3\}$. Since $l = 4$, $y_1y_3 \notin E(G)$. Since $\{x_1, x_3\}$ is not a dominating set of G, $V(G) \setminus (V(C) \cup \{y_1, y_2\}) \neq \emptyset$. Say $t_1 \in N(y_1) \setminus V(C)$. If $N(y_3) \setminus V(C) = \emptyset$, then $\{y_1, x_3\}$ is a dominating set of G, which is a contradiction. Hence, we can assume that $t_3 \in N(y_3) \setminus V(C)$. Then $G[\{x_1, x_2, x_3, y_1, y_3, t_1, t_3\}] = P_7$, which is a contradiction.

Case 4. $l(G) = 3$. Since G is P_7-free, every vertex in R has distance at most 4 from one vertex on $V(C)$. Assume vertex y_4 has distance 4 from one vertex on $V(C)$ in G and $x_1y_1y_2y_3y_4$ is a path in G. Since $\{x_1, y_3\}$ is not a dominating set of G, there is a vertex u at distance 2 from $\{x_1, y_3\}$ in G. If $ux_i \in E(G)$ for $i \in \{2, 3\}$, then $G[\{u, x_i, x_1, y_1, y_2, y_3, y_4\}] = P_7$, which is a contradiction. Suppose that u is adjacent to y_1. If $uy_2 \notin E(G)$, then $G[\{u, x_1, x_2, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If $uy_2 \in E(G)$, then $G[\{u, x_1, x_2, y_1, y_2, y_3, y_4\}] = F_{10}$, which is a contradiction. Hence, $uy_1 \notin E(G)$. By a similar way, $uy_2 \notin E(G)$ and $uy_4 \notin E(G)$. Suppose that there exists a path y_3vu. If $vy_2 \in E(G)$, then $G[\{u, v, y_1, y_2, y_3, y_4\}] = F_6$, which is a contradiction. Since $l = 3$, $\{x_2, x_3, y_1\} \cap N(v) = \emptyset$. So $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = P_7$, which is a contradiction. Hence, we can assume that every vertex in R has distance at most 3 from one vertex on $V(C)$.

Case 4.1. $|N(V(C)) \cap R| = 3$. Say $|N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G),\ i = 1, 2, 3\}$. Since $l = 3$, $E(G[\{y_1, y_2, y_3\}]) = \emptyset$. Then $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_6$, which is a contradiction.

Case 4.2. $||N(V(C)) \cap R| = 2$. Say $|N(V(C)) \cap R = \{y_i \mid x_iy_i \in E(G),\ i = 1, 2\}$. Since $l = 3$, $y_1y_2 \notin E(G)$. Suppose that there exists an induced path $x_1y_1u_1v_1$. Since G is P_7-free, $N(y_2) \setminus V(C) = \emptyset$.

Suppose that there exists a vertex u such that $u \in N(y_1) \setminus \{x_1, u_1\}$. If $u_1u \notin E(G)$, then $G[\{u, x_1, x_2, y_1, u_1, v_1\}] = F_1$, which is a contradiction. If $u_1u \in E(G)$, then $\{u_1, x_2\}$ is a dominating set of G, which is a contradiction. If $N(y_1) \setminus \{x_1, u_1\} = \emptyset$, then $\{u_1, x_2\}$ is a dominating set of G, which is a contradiction. Hence, we can assume that every vertex in $V(G) \setminus (V(C) \cup \{y_1, y_2\})$ is adjacent to exactly one vertex in $\{y_1, y_2\}$.

If $N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) = \emptyset$, then $\{x_i, y_j\}$ is a dominating set of G, where $i, j \in \{1, 2\}$ and $j \neq i$, which is a contradiction. Suppose that $N(y_i) \cap (V(G) \setminus (V(C) \cup \{y_1, y_2\})) \neq \emptyset$ for $i \in \{1, 2\}$. If x_3 is not adjacent to y_1 and y_2, then $G[\{x_1, x_2, x_3, y_1, s_1, y_2, s_2\}] = F_{10}$, where $s_i \in N(y_i)$, which is a contradiction. If x_3 is adjacent to y_1 or y_2, then $\{y_1, y_2\}$ is a dominating set of G, which is a contradiction.

Case 4.3. $||N(V(C)) \cap R| = 1$. Say $y_1x_1 \in E(G)$. Since $\{x_1, y_1\}$ is not a dominating set of G, there is a vertex u at distance 2 from y_1 in G. Without loss of generality, we can assume that y_1vu be a induced path. If there exists a vertex t such that $y_1t \in E(G)$. If $tv \notin E(G)$, then $G[\{u, v, x_1, x_2, y_1, t\}] = F_1$, which is a contradiction. Suppose that $tv \in E(G)$. If $N(t) \setminus \{y_1, v\} \neq \emptyset$, say $s \in N(t) \setminus \{y_1, v\}$, then $G[\{t, s, u, v, u, x_1, y_1\}] = F_6$, which is a contradiction. If $N(t) \setminus \{y_1, v\} = \emptyset$ or $d_G(y_1) = 2$, then $\{v, x_1\}$ is a dominating set of G, which is a contradiction. ■

3. Remark

Henning *et al.* also gave the following conjecture.

Conjecture 2 [5]. *The set* $\mathscr{F}$ *in Conjecture* 1 *can be chosen such that* $\gamma(F) = 3$ *and* $\gamma_e(F) = 2$ *for every graph* F *in* $\mathscr{F}$.

It is obvious that the conjecture holds for subcubic graphs.

References

[1] S. Bessy, P. Ochem and D. Rautenbach, *Exponential domination in subcubic graphs*, Electron. J. Combin. **23** (2016) #P4.42.

[2] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi and H. Swart, *Domination with exponential decay*, Discrete Math. **309** (2009) 5877–5883. doi:10.1016/j.disc.2008.06.040

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[5] M.A. Henning, S. Jäger and D. Rautenbach, *Hereditary equality of domination and exponential domination*, Discuss. Math. Graph Theory **38** (2018) 275–285. doi:10.7151/dmgt.2006

Received 11 October 2018
Revised 13 May 2019
Accepted 13 May 2019