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ON 3-COLORINGS OF DIRECT PRODUCTS OF GRAPHS

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Abstract

The k-independence number of a graph G, denoted as $\alpha_k(G)$, is the order of a largest induced k-colorable subgraph of G. In [S. Špacapan, *The k-independence number of direct products of graphs*, European J. Combin. 32 (2011) 1377–1383] the author conjectured that the direct product $G \times H$ of graphs G and H obeys the following bound

$$\alpha_k(G \times H) \leq \alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H),$$

and proved the conjecture for $k = 1$ and $k = 2$. If true for $k = 3$ the conjecture strenghtens the result of El-Zahar and Sauer who proved that any direct product of 4-chromatic graphs is 4-chromatic [M. El-Zahar and N. Sauer, *The chromatic number of the product of two* 4*-chromatic graphs is* 4, Combinatorica 5 (1985) 121–126]. In this paper we prove that the above bound is true for $k = 3$ provided that G and H are graphs that have complete tripartite subgraphs of orders $\alpha_3(G)$ and $\alpha_3(H)$, respectively.

Keywords: independence number, direct product, Hedetniemi's conjecture.

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