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RAINBOW DISCONNECTION IN GRAPHS

Gary Chartrand[1], Stephen Devereaux[2], Teresa W. Haynes[3],

Stephen T. Hedetniemi[4] and Ping Zhang[1]

[1]*Department of Mathematics*
Western Michigan University
Kalamazoo, MI 49008-5248 USA

[2]*Department of Mathematics*
Cornerstone University
Grand Rapids, MI 49525 USA

[3]*Department of Mathematics and Statistics*
East Tennessee State University
Johnson City, TN 37614-0002 USA

[4]*School of Computing*
Clemson University
Clemson, SC 29634 USA

e-mail: gary.chartrand@wmich.edu
stephen.devereaux@cornerstone.edu
haynes@etsu.edu
hedet@clemson.edu
ping.zhang@wmich.edu

Abstract

Let G be a nontrivial connected, edge-colored graph. An edge-cut R of G is called a rainbow cut if no two edges in R are colored the same. An edge-coloring of G is a rainbow disconnection coloring if for every two distinct vertices u and v of G, there exists a rainbow cut in G, where u and v belong to different components of $G - R$. We introduce and study the rainbow disconnection number $\text{rd}(G)$ of G, which is defined as the minimum number of colors required of a rainbow disconnection coloring of G. It is shown that the rainbow disconnection number of a nontrivial connected graph G equals the maximum rainbow disconnection number among the blocks of G. It is also shown that for a nontrivial connected graph G of order n, $\text{rd}(G) = n-1$ if and only if G contains at least two vertices of degree $n-1$. The rainbow disconnection numbers of all grids $P_m \square P_n$ are determined. Furthermore, it is shown for integers k and n with $1 \le k \le n-1$ that the minimum

size of a connected graph of order n having rainbow disconnection number k is $n+k-2$. Other results and a conjecture are also presented.
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1. Introduction

An *edge-coloring* of a graph G is a function $c: E(G) \to [k] = \{1, 2, \ldots, k\}$ for some positive integer k where adjacent edges may be assigned the same color. A graph with an edge-coloring is an *edge-colored graph.* If no two adjacent edges of G are colored the same, then c is a *proper edge-coloring.* The minimum number of colors required of a proper edge-coloring of G is the *chromatic index* of G, denoted by $\chi'(G)$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. By a famous 1964 theorem of Vizing [7],

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph G.

A set R of edges in a connected edge-colored graph G is a *rainbow set* if no two edges in R are colored the same. A path P in G is a *rainbow path* if no two edges in P are colored the same. The graph G is *rainbow-connected* if every two vertices of G are connected by a rainbow path. An edge-coloring of G with this property is called a *rainbow coloring.* The minimum number of colors needed in a rainbow coloring of G is the *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$. Rainbow connection was introduced [1] in 2006. For more details on rainbow connection, see the book [6] and the survey paper[5].

The object of this paper is to introduce a concept that is somewhat reverse to rainbow connection and to present some results dealing with this new concept.

2. An Introduction to Rainbow Disconnection

An *edge-cut* of a nontrivial connected graph G is a set R of edges of G such that $G - R$ is disconnected. The minimum number of edges in an edge-cut of G is its *edge-connectivity* $\lambda(G)$. We then have the well-known inequality $\lambda(G) \leq \delta(G)$. For two distinct vertices u and v of G, let $\lambda(u, v)$ denote the minimum number of edges in an edge-cut R of G such that u and v lie in different components of $G - R$. The following result of Elias, Feinstein and Shannon [2] and Ford and Fulkerson [3] presents an alternate interpretation of $\lambda(u, v)$.

Theorem 2.1. *For every two vertices u and v in a graph G, $\lambda(u, v)$ is the maximum number of pairwise edge-disjoint $u - v$ paths in G.*

The *upper edge-connectivity* $\lambda^+(G)$ is defined by

$$\lambda^+(G) = \max\{\lambda(u,v) : \ u,v \in V(G)\}.$$

Consider, for example, the graph $K_n + v$ obtained from the complete graph K_n, one vertex of which is attached to a single leaf v. For this graph, $\lambda(K_n + v) = 1$ while $\lambda^+(K_n + v) = n - 1$. Thus, $\lambda(G)$ denotes the global minimum edge-connectivity of a graph, while $\lambda^+(G)$ denotes the local maximum edge-connectivity of a graph.

A set R of edges in a nontrivial connected, edge-colored graph G is a *rainbow cut* of G if R is both a rainbow set and an edge-cut. A rainbow cut R is said to *separate* two vertices u and v of G if u and v belong to different components of $G - R$. Any such rainbow cut in G is called a $u - v$ *rainbow cut* in G. An edge-coloring of G is a *rainbow disconnection coloring* if for every two distinct vertices u and v of G, there exists a $u - v$ rainbow cut in G. The *rainbow disconnection number* $\mathrm{rd}(G)$ of G is the minimum number of colors required of a rainbow disconnection coloring of G. A rainbow disconnection coloring with $\mathrm{rd}(G)$ colors is called an rd-*coloring* of G. We now present bounds for the rainbow disconnection number of a graph.

Proposition 2.2. *If G is a nontrivial connected graph, then*

$$\lambda(G) \leq \lambda^+(G) \leq \mathrm{rd}(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof. First, by Vizing's theorem, $\chi'(G) \leq \Delta(G) + 1$. Now, let there be given a proper edge-coloring of G using $\chi'(G)$ colors. Then, for each vertex x of G, the set E_x of edges incident with x is a rainbow set and $|E_x| = \deg x \leq \Delta(G) \leq \chi'(G)$. Furthermore, E_x is a rainbow cut in G and so $\mathrm{rd}(G) \leq \chi'(G)$.

Next, let there be given an rd-coloring of G. Let u and v be two vertices of G such that $\lambda^+(G) = \lambda(u,v)$ and let R be a $u - v$ rainbow cut with $|R| = \lambda(u,v)$. Then $|R| \leq \mathrm{rd}(G)$. Thus, $\lambda(G) \leq \lambda^+(G) = |R| \leq \mathrm{rd}(G)$. ■

We now present examples of two classes of connected graphs G for which $\lambda(G) = \mathrm{rd}(G)$, namely cycles and wheels.

Proposition 2.3. *If C_n is a cycle of order $n \geq 3$, then $\mathrm{rd}(C_n) = 2$.*

Proof. Since $\lambda(C_n) = 2$, it follows by Proposition 2.2 that $\mathrm{rd}(C_n) \geq 2$. To show that $\mathrm{rd}(C_n) \leq 2$, let c be an edge-coloring of C_n that assigns the color 1 to exactly $n - 1$ edges of C_n and the color 2 to the remaining edge e of C_n. Let u and v be two vertices of C_n. There are two $u - v$ paths P and Q in C_n, exactly one of which contains the edge e, say $e \in E(P)$. Then any set $\{e, f\}$, where $f \in E(Q)$, is a $u - v$ rainbow cut. Thus, c is a rainbow disconnection coloring of C_n using two colors. Hence, $\mathrm{rd}(C_n) = 2$. ■

Proposition 2.4. *If $W_n = C_n \vee K_1$ is the wheel of order $n+1 \geq 4$, then* $\mathrm{rd}(W_n) = 3$.

Proof. Since $\lambda(W_n) = 3$, it follows by Proposition 2.2 that $\mathrm{rd}(W_n) \geq 3$. It remains to show that there is a rainbow disconnection coloring of W_n using only the colors $1, 2, 3$. Suppose that $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and that v is the center of W_n. Define an edge-coloring $c : E(W_n) \to \{1, 2, 3\}$ of W_n as follows. First, let c be a proper edge-coloring of C_n using the colors $1, 2, 3$. For each integer i with $1 \leq i \leq n$, let $a_i \in \{1, 2, 3\} - \{c(v_{i-1}v_i), c(v_iv_{i+1})\}$ where each subscript is expressed as an integer $1, 2, \ldots, n$ modulo n, and let $c(vv_i) = a_i$. Thus, the set E_{v_i} of the three edges incident with v_i is a rainbow set for $1 \leq i \leq n$. Let x and y be two distinct vertices of W_n. Then at least one of x and y belongs to C_n, say $x \in V(C_n)$. Since E_x separates x and y, it follows that c is a rainbow disconnection coloring of W_n using three colors. Hence, $\mathrm{rd}(W_n) = 3$. ■

Since $\chi'(C_n) = 3$ when $n \geq 3$ is odd and $\chi'(W_n) = n$ for each integer $n \geq 3$, it follows that $\mathrm{rd}(G) < \chi'(G)$ if G is an odd cycle or if G is a wheel of order at least 4. Wheels therefore illustrate that there are graphs G for which $\chi'(G) - \mathrm{rd}(G)$ can be arbitrarily large. We now give an example of a graph G for which $\lambda^+(G) < \mathrm{rd}(G) = \chi'(G)$.

Proposition 2.5. *The rainbow disconnection number of the Petersen graph is* 4.

Proof. Let P denote the Petersen graph where $V(P) = \{v_1, v_2, \ldots, v_{10}\}$. Since $\lambda(P) = 3$ and $\chi'(P) = 4$, it follows by Proposition 2.2 that $\mathrm{rd}(P) = 3$ or $\mathrm{rd}(P) = 4$. Assume, to the contrary, that $\mathrm{rd}(P) = 3$ and let there be given a rainbow disconnection 3-coloring of P. Now, let u and v be two vertices of P and let R be a $u - v$ rainbow cut. Hence, $|R| \leq 3$ and $P - R$ is disconnected, where u and v belong to different components of $P - R$. Let U be the vertex set of the component of $P - R$ containing u, where $|U| = k$. We may assume that $1 \leq k \leq 5$. First, suppose that $1 \leq k \leq 4$. Since the girth of P is 5, the subgraph $P[U]$ induced by U contains $k - 1$ edges. Therefore, $|R| = 3k - (2k - 2) = k + 2$, where then $3 \leq k + 2 \leq 6$. If $k = 5$, then $P[U]$ contains at most five edges and so $|R| \geq 5$, which is impossible. Since $\mathrm{rd}(P) = 3$, it follows that $|R| \leq 3$ and so $k = 1$. Hence, the only possible $u - v$ rainbow cut is the set of the three edges incident with u (or with v).

Let the colors assigned to the edges of P be red, blue and green. Since $\chi'(P) = 4$, there is at least one vertex of P that is incident with two edges of the same color. We claim, in fact, that there are at least two such vertices. Let E_R, E_B and E_G denote the sets of edges of P colored red, blue and green, respectively, and let P_R, P_B and P_G be the spanning subgraphs of P with edge sets E_R, E_B and E_G. We may assume that $|E_R| \geq |E_B| \geq |E_G|$ and so $|E_R| \geq 5$. If $|E_R| \geq 7$, then $\sum_{i=1}^{10} \deg_{P_R} v_i \geq 14$. Since $\deg_{P_R} v_i \leq 3$ for each i with $1 \leq i \leq 10$, at least two vertices are incident with two red edges, verifying the claim. If $|E_R| = 6$, then $\sum_{i=1}^{10} \deg_{P_R} v_i = 12$. Then either (i) at least two vertices are incident with two red edges or (ii) there is a vertex, say v_{10}, incident with three red edges and each of $v_1, v_2, \ldots, v_9$ is incident with exactly one red edge. If (ii) occurs, then either $|E_B| = 6$ or $|E_B| = 5$ and so $\sum_{i=1}^{9} \deg_{P_B} v_i \geq 10$, which implies that at least one of the vertices $v_1, v_2, \ldots, v_9$ is incident with two blue edges, again verifying the claim.

The only remaining possibility is therefore $|E_R| = |E_B| = |E_G| = 5$. If E_R is an independent set of five edges, then $P - E_R$ is a 2-regular graph. Since the girth of P is 5 and P is not Hamiltonian, it follows that $P - E_R$ consists of two vertex-disjoint 5-cycles. Thus, there is a vertex of P in each cycle incident with two blue edges or with two green edges, verifying the claim. Hence, none of E_R, E_B or E_G is an independent set. This implies that for each of these colors, there is a vertex of P incident with two edges of this color, verifying the claim in general.

Thus, P contains two vertices u and v, each of which is incident with two edges of the same color. Since the only $u - v$ rainbow cut is the set of edges incident with u or v, this is a contradiction. ■

The following two results are useful.

Proposition 2.6. *If H is a connected subgraph of a graph G, then* $\mathrm{rd}(H) \leq \mathrm{rd}(G)$.

Proof. Let c be an rd-coloring of G and let u and v are two vertices of G. Suppose that R is a $u - v$ rainbow cut. Then $R \cap E(H)$ is a $u - v$ rainbow cut in H. Hence, c restricted to H is a rainbow disconnection coloring of H. Thus, $\mathrm{rd}(H) \leq \mathrm{rd}(G)$. ■

A *block* of a graph is a maximal connected graph of G containing no cut-vertices. The *block decomposition* of G is the set of blocks of G.

Proposition 2.7. *Let G be a nontrivial connected graph, and let B be a block of G such that $rd(B)$ is maximum among all blocks of G. Then* $\mathrm{rd}(G) = \mathrm{rd}(B)$.

Proof. Let G be a nontrivial connected graph. Let $\{B_1, B_2, \ldots, B_t\}$ be a block decomposition of G, and let $k = \max\{\mathrm{rd}(B_i) \mid 1 \leq i \leq t\}$. If G has no cut-vertex, then $G = B_1$ and the result follows. Hence, we may assume that G has at least one cutvertex. By Proposition 2.6, $k \leq \mathrm{rd}(G)$.

Let c_i be an rd-coloring of B_i. We define the edge-coloring $c : E(G) \to [k]$ of G by $c(e) = c_i(e)$ if $e \in E(B_i)$.

Let $x, y \in V(G)$. If there exists a block, say B_i, that contains both x and y, then any $x - y$ rainbow cut in B_i is an $x - y$ rainbow cut in G. Hence, we can assume that no block of G contains both x and y, and that $x \in B_i$ and $y \in B_j$, where $i \neq j$. Now every $x - y$ path contains a cut-vertex, say v, of G in B_i and a cutvertex, say w, of G in B_j. Note that v could equal w. If $x \neq v$, then any $x - v$ rainbow cut of B_i is an $x - y$ rainbow cut in G. Similarly, if $y \neq w$, then any $y - w$ rainbow cut of B_j is an $x - y$ rainbow cut in G. Thus, we may assume that $x = v$ and $y = w$. It follows that $v \neq w$. Consider the $x - y$ path $P = (x = v_1, v_2, \ldots, v_p = y)$. Since x and y are cutvertices in different blocks and no block contains both x and y, P contains a cut-vertex z of G in B_i, that is, $z = v_k$ for some k $(2 \leq k \leq p - 1)$. Then any $x - z$ rainbow cut of B_i is an $x - y$ rainbow cut of G. Hence, $\mathrm{rd}(G) \leq k$, and so $\mathrm{rd}(G) = k$. ■

As a consequence of Proposition 2.7, the study of rainbow disconnection numbers can be restricted to 2-connected graphs. We now present several corollaries of Proposition 2.7.

Corollary 2.8. *Let G and H be any two nontrivial connected graphs, and let GvH be a graph formed by identifying a vertex in G with a vertex in H. Then* $\mathrm{rd}(GvH) = \max\{\mathrm{rd}(G), \mathrm{rd}(H)\}$.

Corollary 2.9. *Let G and H be any two nontrivial connected graphs, and let $GuvH$ be a graph formed by adding an edge between any vertex u in G and any vertex v in H. Then* $\mathrm{rd}(GuvH) = \max\{\mathrm{rd}(G), \mathrm{rd}(H)\}$.

Corollary 2.10. *Let G be a nontrivial connected graph and G' the graph obtained by attaching a pendant edge uv to some vertex u of G. Then* $\mathrm{rd}(G') = \mathrm{rd}(G)$.

The *corona* $G \circ K_1$ is the graph obtained from G by attaching a leaf to each vertex of G. Thus, if G has order n, then the corona $G \circ K_1$ has order $2n$ and has precisely n leaves.

Corollary 2.11. *If G is a nontrivial connected graph, then* $\mathrm{rd}(G \circ K_1) = \mathrm{rd}(G)$.

Corollary 2.12. *Let G be a nontrivial connected graph, let T be a nontrivial tree and let u and v be vertices of G and T, respectively. If H is the graph obtained from G and T by identifying u and v, then* $\mathrm{rd}(H) = \mathrm{rd}(G)$.

A *unicyclic graph* is a connected graph with exactly one cycle.

Corollary 2.13. *If G is a unicyclic graph G, then* $\mathrm{rd}(G) = 2$.

3. Graphs with Prescribed Order and Rainbow Disconnection Number

In this section, we characterize all those nontrivial connected graphs of order n with rainbow disconnection number k for each $k \in \{1, 2, n - 1\}$. The result for graphs having rainbow disconnection number 1 follows directly from Propositions 2.6 and 2.7.

Proposition 3.1. *Let G be a nontrivial connected graph. Then* $\mathrm{rd}(G) = 1$ *if and only if G is a tree.*

Next, we characterize all nontrivial connected graphs of order n having rainbow disconnection number 2. By Proposition 3.1, such a graph must contain a cycle. An *ear* of a graph G is a maximal path whose internal vertices have degree 2 in G. An *ear decomposition* of a graph is a decomposition $H_0, H_1, \ldots, H_k$ such that H_0 is a cycle in G and H_i is an ear of the subgraph of G with edge set $E(H_0) \cup E(H_1) \cup \cdots \cup E(H_i)$ for each integer i with $1 \leq i \leq k$. Whitney [8] proved the following result in 1932.

Theorem 3.2. *A graph G is 2-connected if and only if G has an ear decomposition. Furthermore, every cycle is the initial cycle in some ear decomposition of G.*

The following is a consequence of Theorem 3.2.

Lemma 3.3. *A 2-connected graph G is a cycle if and only if for every two vertices u and v of G, there are exactly two internally disjoint $u - v$ paths in G.*

Also, by Theorem 3.2, if G is a 2-connected, non-Hamiltonian graph, then G contains a theta subgraph (a subgraph consisting of two vertices connected by three internally disjoint paths of length 2 or more).

Theorem 3.4. *Let G be a nontrivial connected graph. Then* $\mathrm{rd}(G) = 2$ *if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Proof. If G a nontrivial connected graph, each block of which is either K_2 or a cycle and at least one block of G is a cycle, then Propositions 2.3 and 2.7 imply that $\mathrm{rd}(G) = 2$.

We now verify the converse. Assume, to the contrary, that there is a connected graph G with $\mathrm{rd}(G) = 2$ that does not have the property that each block of G is either K_2 or a cycle and at least one block of G is a cycle. First, not all blocks can be K_2, for otherwise, G is a tree and so $\mathrm{rd}(G) = 1$ by Proposition 3.1. Hence, G contains a block that is neither K_2 nor a cycle. By Lemma 3.3, there exist two distinct vertices u and v of G for which G contains at least three internally disjoint $u - v$ paths P_1, P_2 and P_3. Thus, any $u - v$ rainbow cut R must contain at least one edge from each of P_1, P_2 and P_3 and so $|R| \geq 3$, which is impossible. ■

We now consider those graphs that are, in a sense, opposite to trees.

Proposition 3.5. *For each integer $n \geq 4$,* $\mathrm{rd}(K_n) = n - 1$.

Proof. Suppose first that $n \geq 4$ is even. Then $\lambda(K_n) = \chi'(K_n) = n-1$. It then follows by Proposition 2.2 that $\text{rd}(K_n) = n-1$. Next, suppose that $n \geq 5$ is odd. Then $n-1 = \lambda(K_n) \leq \text{rd}(K_n) \leq \chi'(K_n) = n$ by Proposition 2.2. To show that $\text{rd}(K_n) = n-1$, it remains to show that there is a rainbow disconnection coloring of K_n using $n-1$ colors. Let $x \in V(K_n)$. Then $K_n - x = K_{n-1}$. Since $n-1$ is even, it follows that $\chi'(K_{n-1}) = n-2$. Thus, there is a proper edge-coloring c_0 of K_{n-1} using the colors $1, 2, \ldots, n-2$. We now extend c_0 to an edge-coloring c of K_n by assigning the color $n-1$ to each edge of K_n that is incident with x. We show that c is a rainbow disconnection coloring of K_n. Let u and v be two vertices of K_n, where say $u \neq x$. Then the set E_u of edges incident with u is a $u-v$ rainbow cut. Thus, c is a rainbow disconnection coloring of K_n and so $\text{rd}(K_n) \leq n-1$ and so $\text{rd}(K_n) = n-1$. ■

By Propositions 2.2, 2.6 and 3.5, if G is a nontrivial connected graph of order n, then

$$1 \leq \text{rd}(G) \leq n-1. \tag{1}$$

Furthermore, $\text{rd}(G) = 1$ if and only if G is a tree by Proposition 3.1. We have seen that the complete graphs K_n of order $n \geq 2$ have rainbow disconnection number $n-1$. We now characterize all nontrivial connected graphs of order n having rainbow disconnection number $n-1$.

Theorem 3.6. *Let G be a nontrivial connected graph of order n. Then* $\text{rd}(G) = n-1$ *if and only if G contains at least two vertices of degree $n-1$.*

Proof. First, suppose that G is a nontrivial connected graph of order n containing at least two vertices of degree $n-1$. Since $\text{rd}(G) \leq n-1$ by (1), it remains to show that $\text{rd}(G) \geq n-1$. Let $u, v \in V(G)$ such that $\deg u = \deg v = n-1$. Among all sets of edges that separate u and v in G, let S be one of minimum size. We show that $|S| \geq n-1$. Let U be a component of $G-S$ that contains u and let $W = V(G) - U$. Thus, $v \in W$ and $S = [U, W]$ consists of those edges in $G-S$ joining a vertex of U and a vertex of W. Suppose that $|U| = k$ for some integer k with $1 \leq k \leq n-1$ and then $|W| = n-k$. The vertex u is adjacent to each of the $n-k$ vertices of W and each of the remaining $k-1$ vertices in U is adjacent to at least one vertex in W. Hence, $|S| \geq n-k+(k-1) = n-1$. This implies that every $u-v$ rainbow cut contains at least $n-1$ edges of G and so $\text{rd}(G) \geq n-1$.

For the converse, suppose that G is a nontrivial connected graph of order n having at most one vertex of degree $n-1$. We show that $\text{rd}(G) \leq n-2$. We consider two cases.

Case 1. *Exactly one vertex v of G has degree $n-1$.* Let $H = G - v$. Thus, $\Delta(H) \leq n-3$. Since $\chi'(H) \leq \Delta(H)+1 = n-2$, there is a proper edge-coloring of H using $n-2$ colors. We now define an edge-coloring $c : E(G) \to [n-2]$ of G. First, let c be a proper $(n-2)$-edge-coloring of H. For each vertex $x \in V(H)$, since $\deg_H x \le n-3$, there is $a_x \in [n-2]$ such that a_x is not assigned to any edge incident with x. Define $c(vx) = a_x$. Thus, the set E_x of edges incident with x is a rainbow set for each $x \in V(H)$. Let u and w be two distinct vertices of G. Then at least one of u and w belongs to H, say $u \in V(H)$. Since E_u separates u and w, it follows that c is a rainbow disconnection coloring of G using $n-2$ colors. Hence, $\mathrm{rd}(G) \le n-2$.

Case 2. No vertex of G has degree $n-1$. Therefore $\Delta(G) \le n-2$. If $\Delta(G) \le n-3$, then $\mathrm{rd}(G) \le \chi'(G) \le n-2$ by Proposition 2.2. Thus, we may assume that $\Delta(G) = n-2$. Suppose first that G is not $(n-2)$-regular. We claim that G is a connected spanning subgraph of some graph G^* of order n having exactly one vertex of degree $n-1$. Let u be a vertex of degree $k \le n-3$ in G. Let $N(u)$ be the neighborhood of u and $W = V(G) - N[u]$, where $N[u] = N(u) \cup \{u\}$ is the closed neighborhood of u. Then $|N(u)| = k$ and $|W| = n-k-1 \ge 2$. If W contains a vertex v of degree $n-2$ in G, then v is the only vertex of degree $n-1$ in $G^* = G + uv$. If no vertex in W has degree $n-2$ in G, then let G^* be the graph obtained from G by joining u to each vertex in W. In this case, u is the only vertex of degree $n-1$ in G^*. It then follows by Case 1 that $\mathrm{rd}(G^*) \le n-2$. Since G is a connected spanning subgraph of G^*, it follows by Proposition 2.6 that $\mathrm{rd}(G) \le \mathrm{rd}(G^*) \le n-2$. Finally, suppose that G is $(n-2)$-regular. Thus, G is 1-factorable and so $\chi'(G) = \Delta(G) = n-2$. Therefore, $\mathrm{rd}(G) \le \chi'(G) = n-2$ by Proposition 2.2. ■

4. Rainbow Disconnection in Grids and Prisms

We now determine the rainbow disconnection numbers of graphs belonging to one of two well-known classes formed by Cartesian products. The *Cartesian product* $G \,\square\, H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (w, x) in $G \,\square\, H$ if and only if either $u = w$ and $vx \in E(H)$ or $uw \in E(G)$ and $v = x$. We consider the $m \times n$ grid graph $G_{m,n} = P_m \,\square\, P_n$, which consists of m horizontal paths P_n and n vertical paths P_m.

Theorem 4.1. *The rainbow disconnection numbers of the grid graphs $G_{m,n}$ are as follows:*

(i) *for all* $n \ge 2$, $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$,
(ii) *for all* $n \ge 3$, $\mathrm{rd}(G_{2,n}) = 3$,
(iii) *for all* $n \ge 4$, $\mathrm{rd}(G_{3,n}) = 3$,
(iv) *for all* $4 \le m \le n$, $\mathrm{rd}(G_{m,n}) = 4$.

Proof. (i) That $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$ for $n \geq 2$ is a consequence of Proposition 3.1.

For the remainder of the proof, we consider the vertices of $G_{m,n}$ as a matrix, letting $x_{i,j}$ denote the vertex in row i and column j, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

(ii) For the graph $G_{2,n}$, $n \geq 3$, $\Delta(G_{2,n}) = 3$. First, we define an edge-coloring c of $G_{2,n}$. It is convenient to use the elements of the set $\mathbf{Z}_3$ of integer modulo 3 as colors here. Define the edge-coloring $c : E(G_{2,n}) \to \mathbf{Z}_3$ by

- $\star$ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 2$ and $1 \leq j \leq n - 1$;
- $\star$ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n$.

Next, we show that c is a rainbow disconnection coloring of $G_{2,n}$. Let u and v be any two vertices of $G_{2,n}$. If u and v belong to two different columns, then there exist two parallel edges joining vertices in the same two columns whose removal separates u and v. Each such set of two edges is a $u - v$ rainbow cut. Next, suppose that u and v belong to the same column. Then, without loss of generality, u belongs to the first row and v belongs to the second row. Then u and v both have degree 2 or both have degree 3. Therefore, the edges incident with u form a rainbow cut, and so, $\mathrm{rd}(G_{2,n}) \leq 3$.

On the other hand, $\lambda(u, v) = 2$ if u and v are two vertices of $G_{2,n}$ belonging to the same row, or different rows and columns or are two vertices of degree 2 belonging to the same column; while $\lambda(u, v) = 3$ if u and v are (adjacent) vertices of degree 3 belonging to the same column. It then follows by Proposition 2.2 that $3 = \lambda^+(G_{2,n}) \leq \mathrm{rd}(G_{2,n})$, and so $\mathrm{rd}(G_{2,n}) = 3$.

(iii) As with $G_{2,n}$, we define an edge-coloring c of $G_{3,n}$. Again we use the elements of the set $\mathbf{Z}_3$ of integer modulo 3 as colors here. Define the edge-coloring $c : E(G_{3,n}) \to \mathbf{Z}_3$ by

- $\star$ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 3$ and $1 \leq j \leq n - 1$;
- $\star$ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n$;
- $\star$ $c(x_{2,j}x_{3,j}) = j + 2$ for $1 \leq j \leq n$.

Next, we show that c is a rainbow disconnection coloring of $G_{3,n}$. Let u and v be any two vertices of $G_{3,n}$. If u and v belong to two different columns, then there exist three parallel edges joining vertices in the same two columns whose removal separates u and v. Each such set of three edges is a $u - v$ rainbow cut. Next, suppose that u and v belong to the same column. Then at least one of u and v belongs to the top or bottom row, say u is such a vertex, which has degree 2 or 3. Then the edges incident with u is a $u - v$ rainbow cut. Thus, $\mathrm{rd}(G_{3,n}) \leq 3$.

On the other hand, for any two adjacent vertices u and v of degree 4 in $G_{3,n}$ (necessarily in the middle row), $\lambda^+(u, v) = 3$. Thus, by Proposition 2.2, $3 \leq \lambda^+(G_{3,n}) \leq \mathrm{rd}(G_{3,n}) \leq 3$ and so $\mathrm{rd}(G_{3,n}) = 3$.

(iv) Finally, we consider $G_{m,n}$ for $4 \leq m \leq n$. Since there are four pairwise edge-disjoint $u - v$ paths in $G_{m,n}$ for every two vertices u and v of degree 4, it follows by Theorem 2.1 that $\lambda(u, v) = 4$. For any other pair u, v of vertices of $G_{m,n}$, it follows that $\lambda(u, v) \leq 3$. By Proposition 2.2 then, $4 = \lambda^+(G_{m,n}) \leq \text{rd}(G_{m,n})$. On the other hand, since $G_{m,n}$ is bipartite, $\chi'(G_{m,n}) = \Delta(G_{m,n}) = 4$. Again, by Proposition 2.2, $\text{rd}(G_{m,n}) \leq 4$ and so $\text{rd}(G_{4,n}) = 4$. ■

Next we determine the rainbow disconnection number of prisms, namely those graphs of the type $G \,\square\, K_2$ for some graph G.

Proposition 4.2. *If G is a nontrivial connected graph, then*

$$\text{rd}(G \,\square\, K_2) = \Delta(G) + 1.$$

Proof. Let G and G' be the two copies of G in the prism $G \,\square\, K_2$, and for each $v \in V(G)$, let v' be its corresponding vertex in G'. We first show that $G \,\square K_2$ has a proper edge-coloring using $\Delta(G \,\square\, K_2) = \Delta(G) + 1$ colors, that is, $\chi'(G \,\square\, K_2) = \Delta(G)+1$. Let C be a proper edge-coloring of G using $\chi'(G)$ colors. Color the edges of G and G' using C, that is, G and G' have an identical edge-coloring C. By Vizing's Theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. First assume that $\chi'(G) = \Delta(G)$. Then assigning the color $\Delta(G) + 1$ to each edge vv' for every $v \in V(G)$ gives a proper edge-coloring of $G \,\square\, K_2$ with $\Delta(G)+1$ colors. Next assume that $\chi'(G) = \Delta(G) + 1$. Then for each $v \in V(G)$, at least one of the $\Delta(G) + 1$ colors is missing from the colors of the edges incident to v. Let c_v be one such missing color. Note that c_v is also missing from the colors of the edges incident to v' in G' because G and G' have the identical colorings. Hence, assigning c_v to vv' for each $v \in V(G)$ yields a proper edge-coloring of $G \,\square\, K_2$ having $\Delta(G) + 1$ colors. By Proposition 2.2, it follows that $\text{rd}(G \,\square\, K_2) \leq \Delta(G) + 1$.

To establish the lower bound, let u be a vertex of G with $\deg u = \Delta(G) = \Delta$. In $G \,\square\, K_2$, there exist $\Delta + 1$ edge-disjoint $u - u'$ paths, one of which is the edge uu' and the remaining Δ of which have the form (u, w, w', u'), where $w \in V(G)$ and w' is the corresponding vertex of w in G'. It again follows by Proposition 2.2 that $\text{rd}(G \,\square\, K_2) \geq \lambda^+(G \,\square\, K_2) \geq \Delta(G) + 1$. ■

Complementary products were introduced in [4] as a generalization of Cartesian products. We consider a subfamily of complementary products, namely, complementary prisms. For a graph $G = (V, E)$, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen graph. In particular, the Petersen graph is the complementary prism $C_5\overline{C}_5$. For another example of a complementary prism, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$.

We refer to the complementary prism $G\overline{G}$ as a copy of G and a copy of $\overline{G}$ with a perfect matching between corresponding vertices. For a set $S \subseteq V(G)$, let $\overline{S}$ denote the corresponding set of vertices in $V(\overline{G})$. We note that $G\overline{G}$ is isomorphic to $\overline{G}G$.

Since $\Delta(G\overline{G}) = \max\{\Delta(G), \Delta(\overline{G})\}+1$, Proposition 2.2 implies that $\text{rd}(G\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$. This bound is sharp for the Petersen graph $P = C_5\overline{C}_5$ since by Proposition 2.5, $\text{rd}(P) = 4 = \Delta(C_5) + 2$. On the other hand, for the complementary prisms $K_n\overline{K}_n$, Corollary 2.11 and Proposition 3.5 imply that $\text{rd}(K_n\overline{K}_n) = \text{rd}(K_n) = n - 1 = \Delta(K_n) < \max\{\Delta(K_n), \Delta(\overline{K}_n\} + 2 = n + 1$. Our next result shows that for graphs G with sufficiently large girth, $\text{rd}(G\overline{G})$ is strictly greater than the maximum degree of G.

Proposition 4.3. *If G is a graph of order n, maximum degree $\Delta(G) < n - 1$, and girth at least five, then*

$$\Delta(G) + 1 \leq \text{rd}(G\overline{G}).$$

Proof. Consider a vertex u in G such that $\text{deg}_\text{G}\text{u} = \Delta(\text{G})$. Let $A = N_G(u)$ and $B = V - N_G[u]$. Thus, in $G\overline{G}$, $N(\overline{u}) = \overline{B} \cup \{u\}$. Note that since $n - 1 > \Delta(G)$, it follows that $\overline{B} \neq \emptyset$.

We claim there are $\Delta(G) + 1$ edge-disjoint u-$\bar{b}$ paths, where $\bar{b} \in \overline{B}$. To see this note that one such path is $(u, \overline{u}, \bar{b})$. Next consider the u-$\bar{b}$ paths containing a vertex $a \in A$. If a is not adjacent to b in G, then $\overline{a}$ is adjacent to $\bar{b}$ in $\overline{G}$ and $(u, a, \overline{a}, \bar{b})$ is a u-$\bar{b}$ path. If $ab \in E(G)$, then $(u, a, b, \bar{b})$ is a u-$\bar{b}$ path. Moreover, since $g(G) \geq 5$, at most one vertex in A is adjacent to b, else a 4-cycle is formed. In any case, the collection of these $|A| + 1 = \Delta(G) + 1$ paths are edge-disjoint. Hence, by Proposition 2.2, it follows that $\text{rd}(G\overline{G}) \geq \lambda^+(G\overline{G}) \geq \Delta(G) + 1$. ■

For an example of a complementary prism attaining the lower bound, let G be the graph formed from a 5-cycle by attaching a leaf x to a vertex v of the cycle. Then, $\Delta(G) = 3$. We show that $\text{rd}(G\overline{G}) = 4$. First note that the Petersen graph P is a proper subgraph of $G\overline{G}$, and by Propositions 2.5 and 2.6, $\text{rd}(G\overline{G}) \geq \text{rd}(P) = 4$. Furthermore, there is a proper edge-coloring c of P using four colors such that three colors are used to color C_5 and $\overline{C}_5$ and the fourth color is used on the matching edges. Thus, we may assume, without loss of generality, that v is incident to the edges colored 1 and 2 in G and that $v\overline{v}$ is assigned color 4. We extend c to a rainbow disconnection coloring of $G\overline{G}$ as follows: let $c(vx) = 3$, $c(x\overline{x}) = 4$, and $c(\overline{xu})$ be the color missing from the edges incident to $\overline{u}$ for each $\overline{u}$ adjacent to $\overline{x}$ in $\overline{G}$. Consider two arbitrary vertices of $G\overline{G}$. At least one of the vertices, say u, is not $\overline{x}$. Thus, the edges incident with u are a rainbow cut separating the two vertices. Since every such vertex u has degree at most four, $\text{rd}(G\overline{G}) \leq 4$, and so, $\text{rd}(G\overline{G}) = 4$.

5. Extremal Problems

In this section, we investigate the following problem:

> For a given pair k, n of positive integers with $k \leq n-1$, what are the minimum possible size and maximum possible size of a connected graph G of order n such that the rainbow disconnection number of G is k?

We have seen in Proposition 3.1 that the only connected graphs of order n having rainbow disconnection number 1 are the trees of order n. That is, the connected graphs of order n having rainbow disconnection number 1 have size $n-1$. We have also seen in Theorem 3.4 that the minimum size of a connected graph of order $n \geq 3$ having rainbow disconnection number 2 is n. Furthermore, we have seen in Theorem 3.6 that the minimum size of a connected graph of order $n \geq 2$ having rainbow disconnection number $n-1$ is $2n-3$. In fact, these are special cases of a more general result. In order to show this, we first present a lemma.

Lemma 5.1. *Let H be a connected graph of order n that is not complete and let x and y be two nonadjacent vertices of H. Then* $\text{rd}(H + xy) \leq \text{rd}(H) + 1$.

Proof. Suppose that $\text{rd}(H) = k$ for some positive integer k and let c_0 be a rainbow disconnection coloring of H using the colors $1, 2, \ldots, k$. Extend the coloring c_0 to the edge-coloring c of $H + xy$ by assigning the color $k+1$ to the edge xy. Let u and v be two vertices of H and let R be a $u-v$ rainbow cut in H. Then $R \cup \{xy\}$ is a $u-v$ rainbow cut in $H+xy$. Hence, c is a rainbow disconnection $(k+1)$-coloring of $H + xy$. Therefore, $\text{rd}(H + xy) \leq k + 1 = \text{rd}(H) + 1$. ■

Theorem 5.2. *For integers k and n with $1 \leq k \leq n-1$, the minimum size of a connected graph of order n having rainbow disconnection number k is $n + k - 2$.*

Proof. By Proposition 3.5, the result is true for $k = n-1$. Hence, we may assume that $1 \leq k \leq n-2$. First, we show that if the size of a connected graph G of order n is $n + k - 2$, then $\text{rd}(G) \leq k$. We proceed by induction on k. We have seen that the result is true for $k = 1, 2$ by Proposition 3.1 and Theorem 3.4. Suppose that if the size of a connected graph H of order n is $n + k - 2$ for some integer k with $2 \leq k \leq n-3$, then $\text{rd}(H) \leq k$. Let G be a connected graph of order n and size $n + (k+1) - 2 = n + k - 1$. We show that $\text{rd}(G) \leq k+1$. Since G is not a tree, there is an edge e such that $H = G - e$ is a connected spanning subgraph of G. Since the size of H is $n+k-2$, it follows by induction hypothesis that $\text{rd}(H) \leq k$. Hence, $\text{rd}(G) = \text{rc}(H + e) \leq k + 1$ by Lemma 5.1. Therefore, the minimum possible size for a connected graph G of order n to have $\text{rd}(G) = k$ is $n + k - 2$.

It remains to show that for each pair k, n of integers with $1 \leq k \leq n-1$ there is a connected graph G of order n and size $n+k-2$ such that $\text{rd}(G) = k$. Since this is true for $k = 1, 2, n-1$, we now assume that $3 \le k \le n-2$. Let $H = K_{2,k}$ with partite set $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_k\}$. Now, let G be the graph of order n and size $n+k-2$ obtained from H by subdividing the edge u_1w_1 a total of $n-k-2$ times, producing the path $P = (u_1, v_1, v_2, \ldots, v_{n-k-2}, w_1)$ in G. Since $\chi'(H) = k$, there is a proper edge-coloring c_H of H using the colors $1, 2, \ldots, k$. We may assume that $c(u_1w_1) = 1$ and $c(u_2w_1) = 2$. Next, we extend the coloring c_H to a proper edge-coloring c_G of G using the colors $1, 2, \ldots, k$ by defining $c_G(u_1v_1) = 1$ and alternating the colors of the edges of P with 3 and 1 thereafter. Hence, $\chi'(G) = k$ and so $\text{rd}(G) \le \chi'(G) = k$ by Proposition 2.2. Furthermore, since $\lambda(u_1, u_2) = k$ and $\lambda(x, y) = 2$ for all other pairs x, y of vertices of G, it follows that $\lambda^+(G) = k$. Again, by Proposition 2.2, $\text{rd}(G) \ge \lambda^+(G) = k$ and so $\text{rd}(G) = k$. ■

For given integers k and n with $1 \le k \le n-1$, we have determined the minimum size of a connected graph G of order n with $\text{rd}(G) = k$. So, this brings up the question of determining the maximum size of a connected graph G of order n with $\text{rd}(G) = k$. Of course, we know this size when $k = 1$; it is $n-1$. Also, we know this size when $k = n-1$; it is $\binom{n}{2}$. For odd integers n, we have the following conjecture.

Conjecture 5.3. *Let k and n be integers with $1 \le k \le n-1$ and $n \ge 5$ is odd. Then the maximum size of a connected graph G of order n with $\text{rd}(G) = k$ is $\frac{(k+1)(n-1)}{2}$.*

Notice that when $k = 1$, then $\frac{(k+1)(n-1)}{2} = n-1$ and when $k = n-1$, then $\frac{(k+1)(n-1)}{2} = \binom{n}{2}$. Also, when $k = 2$, then $\frac{(k+1)(n-1)}{2} = \frac{3n-3}{2}$. This is the size of the so-called *friendship graph* $\left(\frac{k-1}{2}\right) K_2 \vee K_1$ of order n (every two vertices has a unique friend). Since each block of a friendship graph is a triangle, it follows by Theorem 3.4 that each such graph has rainbow disconnection number 2.

For given integers k and n with $1 \le k \le n-1$ and $n \ge 5$ is odd, let H_k be a $(k-1)$-regular graph of order $n-1$. Since $n-1$ is even, such graphs H_k exist. Now, let $G_k = H_k \vee K_1$ be the join of H_k and K_1. Thus, G_k is a connected graph of order n having one vertex of degree $n-1$ and $n-1$ vertices of degree k. The size m of G_k satisfies the equation:

$$2m = (n-1) + (n-1)k = (k+1)(n-1)$$

and so $m = \frac{(k+1)(n-1)}{2}$. The graph H_k can be selected so that it is 1-factorable and so $\chi'(H_k) = k-1$. If a proper $(k-1)$-edge-coloring of H_k is given using the colors $1, 2, \ldots, k-1$, and every edge incident with the vertex of G_k of degree $n-1$ is assigned the color k, then the edges incident with each vertex of degree k are properly colored with k colors. For any two vertices u and v of G_k, at least one of

u and v has degree k in G_k, say $\deg_{G_k} u = k$. Then the set of edges incident with u is a $u - v$ rainbow cut in H. Since this is a rainbow disconnection k-coloring of G, it follows that $\text{rd}(G_k) \leq k$. It is reasonable to conjecture that $\text{rd}(G_k) = k$.

We would still be left with the question of whether every graph H of order n and size $\frac{(k+1)(n-1)}{2} + 1$ must have $\text{rd}(H) > k$. Certainly, every such graph H must contain at least two vertices whose degrees exceed k.

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