RAINBOW DISCONNECTION IN GRAPHS

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Abstract

Let \( G \) be a nontrivial connected, edge-colored graph. An edge-cut \( R \) of \( G \) is called a rainbow cut if no two edges in \( R \) are colored the same. An edge-coloring of \( G \) is a rainbow disconnection coloring if for every two distinct vertices \( u \) and \( v \) of \( G \), there exists a rainbow cut in \( G \), where \( u \) and \( v \) belong to different components of \( G - R \). We introduce and study the rainbow disconnection number \( \text{rd}(G) \) of \( G \), which is defined as the minimum number of colors required of a rainbow disconnection coloring of \( G \). It is shown that the rainbow disconnection number of a nontrivial connected graph \( G \) equals the maximum rainbow disconnection number among the blocks of \( G \). It is also shown that for a nontrivial connected graph \( G \) of order \( n \), \( \text{rd}(G) = n - 1 \) if and only if \( G \) contains at least two vertices of degree \( n - 1 \). The rainbow disconnection numbers of all grids \( P_m \square P_n \) are determined. Furthermore, it is shown for integers \( k \) and \( n \) with \( 1 \leq k \leq n - 1 \) that the minimum
size of a connected graph of order $n$ having rainbow disconnection number $k$ is $n + k - 2$. Other results and a conjecture are also presented.

**Keywords:** edge coloring, rainbow connection, rainbow disconnection.

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### 1. Introduction

An *edge-coloring* of a graph $G$ is a function $c : E(G) \rightarrow [k] = \{1, 2, \ldots, k\}$ for some positive integer $k$ where adjacent edges may be assigned the same color. A graph with an edge-coloring is an *edge-colored graph*. If no two adjacent edges of $G$ are colored the same, then $c$ is a *proper edge-coloring*. The minimum number of colors required of a proper edge-coloring of $G$ is the *chromatic index* of $G$, denoted by $\chi'(G)$. The minimum and maximum degrees of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. By a famous 1964 theorem of Vizing [7],

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph $G$.

A set $R$ of edges in a connected edge-colored graph $G$ is a *rainbow set* if no two edges in $R$ are colored the same. A path $P$ in $G$ is a *rainbow path* if no two edges in $P$ are colored the same. The graph $G$ is *rainbow-connected* if every two vertices of $G$ are connected by a rainbow path. An edge-coloring of $G$ with this property is called a *rainbow coloring*. The minimum number of colors needed in a rainbow coloring of $G$ is the *rainbow connection number* of $G$, denoted by $rc(G)$.

Rainbow connection was introduced [1] in 2006. For more details on rainbow connection, see the book [6] and the survey paper[5].

The object of this paper is to introduce a concept that is somewhat reverse to rainbow connection and to present some results dealing with this new concept.

### 2. An Introduction to Rainbow Disconnection

An *edge-cut* of a nontrivial connected graph $G$ is a set $R$ of edges of $G$ such that $G - R$ is disconnected. The minimum number of edges in an edge-cut of $G$ is its *edge-connectivity* $\lambda(G)$. We then have the well-known inequality $\lambda(G) \leq \delta(G)$.

For two distinct vertices $u$ and $v$ of $G$, let $\lambda(u, v)$ denote the minimum number of edges in an edge-cut $R$ of $G$ such that $u$ and $v$ lie in different components of $G - R$. The following result of Elias, Feinstein and Shannon [2] and Ford and Fulkerson [3] presents an alternate interpretation of $\lambda(u, v)$.

**Theorem 2.1.** For every two vertices $u$ and $v$ in a graph $G$, $\lambda(u, v)$ is the maximum number of pairwise edge-disjoint $u - v$ paths in $G$. 

The upper edge-connectivity $\lambda^+(G)$ is defined by

$$\lambda^+(G) = \max\{\lambda(u, v) : u, v \in V(G)\}.$$ 

Consider, for example, the graph $K_n + v$ obtained from the complete graph $K_n$, one vertex of which is attached to a single leaf $v$. For this graph, $\lambda(K_n + v) = 1$ while $\lambda^+(K_n + v) = n - 1$. Thus, $\lambda(G)$ denotes the global minimum edge-connectivity of a graph, while $\lambda^+(G)$ denotes the local maximum edge-connectivity of a graph.

A set $R$ of edges in a nontrivial connected, edge-colored graph $G$ is a **rainbow cut** of $G$ if $R$ is both a rainbow set and an edge-cut. A rainbow cut $R$ is said to **separate** two vertices $u$ and $v$ of $G$ if $u$ and $v$ belong to different components of $G - R$. Any such rainbow cut in $G$ is called a **$u−v$ rainbow cut** in $G$. An edge-coloring of $G$ is a **rainbow disconnection coloring** if for every two distinct vertices $u$ and $v$ of $G$, there exists a $u−v$ rainbow cut in $G$. The **rainbow disconnection number** $\text{rd}(G)$ of $G$ is the minimum number of colors required of a rainbow disconnection coloring of $G$. A rainbow disconnection coloring with $\text{rd}(G)$ colors is called an **rd-coloring** of $G$. We now present bounds for the rainbow disconnection number of a graph.

**Proposition 2.2.** If $G$ is a nontrivial connected graph, then

$$\lambda(G) \leq \lambda^+(G) \leq \text{rd}(G) \leq \chi'(G) \leq \Delta(G) + 1.$$ 

**Proof.** First, by Vizing’s theorem, $\chi'(G) \leq \Delta(G) + 1$. Now, let there be given a proper edge-coloring of $G$ using $\chi'(G)$ colors. Then, for each vertex $x$ of $G$, the set $E_x$ of edges incident with $x$ is a rainbow set and $|E_x| = \deg x \leq \Delta(G) \leq \chi'(G)$. Furthermore, $E_x$ is a rainbow cut in $G$ and so $\text{rd}(G) \leq \chi'(G)$.

Next, let there be given an rd-coloring of $G$. Let $u$ and $v$ be two vertices of $G$ such that $\lambda^+(G) = \lambda(u, v)$ and let $R$ be a $u−v$ rainbow cut with $|R| = \lambda(u, v)$. Then $|R| \leq \text{rd}(G)$. Thus, $\lambda(G) \leq \lambda^+(G) = |R| \leq \text{rd}(G)$. 

We now present examples of two classes of connected graphs $G$ for which $\lambda(G) = \text{rd}(G)$, namely cycles and wheels.

**Proposition 2.3.** If $C_n$ is a cycle of order $n \geq 3$, then $\text{rd}(C_n) = 2$.

**Proof.** Since $\lambda(C_n) = 2$, it follows by Proposition 2.2 that $\text{rd}(C_n) \geq 2$. To show that $\text{rd}(C_n) \leq 2$, let $c$ be an edge-coloring of $C_n$ that assigns the color 1 to exactly $n - 1$ edges of $C_n$ and the color 2 to the remaining edge $e$ of $C_n$. Let $u$ and $v$ be two vertices of $C_n$. There are two $u−v$ paths $P$ and $Q$ in $C_n$, exactly one of which contains the edge $e$, say $e \in E(P)$. Then any set $\{e, f\}$, where $f \in E(Q)$, is a $u−v$ rainbow cut. Thus, $c$ is a rainbow disconnection coloring of $C_n$ using two colors. Hence, $\text{rd}(C_n) = 2$. 


Proposition 2.4. If $W_n = C_n \lor K_1$ is the wheel of order $n + 1 \geq 4$, then $rd(W_n) = 3$.

Proof. Since $\lambda(W_n) = 3$, it follows by Proposition 2.2 that $rd(W_n) \geq 3$. It remains to show that there is a rainbow disconnection coloring of $W_n$ using only the colors 1, 2, 3. Suppose that $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and that $v$ is the center of $W_n$. Define an edge-coloring $c : E(W_n) \rightarrow \{1, 2, 3\}$ of $W_n$ as follows. First, let $c$ be a proper edge-coloring of $C_n$ using the colors 1, 2, 3. For each integer $i$ with $1 \leq i \leq n$, let $a_i \in \{1, 2, 3\} - \{c(v_{i-1}v_i), c(v_iv_{i+1})\}$ where each subscript is expressed as an integer 1, 2, \ldots, $n$ modulo $n$, and let $c(v_iv_i) = a_i$. Thus, the set $E_v$ of the three edges incident with $v_i$ is a rainbow set for $1 \leq i \leq n$. Let $x$ and $y$ be two distinct vertices of $W_n$. Then at least one of $x$ and $y$ belongs to $C_n$, say $x \in V(C_n)$. Since $E_x$ separates $x$ and $y$, it follows that $c$ is a rainbow disconnection coloring of $W_n$ using three colors. Hence, $rd(W_n) = 3$.

Since $\chi'(C_n) = 3$ when $n \geq 3$ is odd and $\chi'(W_n) = n$ for each integer $n \geq 3$, it follows that $rd(G) < \chi'(G)$ if $G$ is an odd cycle or if $G$ is a wheel of order at least 4. Wheels therefore illustrate that there are graphs $G$ for which $\chi'(G) - rd(G)$ can be arbitrarily large. We now give an example of a graph $G$ for which $\chi'(G) < rd(G) = \chi'(G)$.

Proposition 2.5. The rainbow disconnection number of the Petersen graph is 4.

Proof. Let $P$ denote the Petersen graph where $V(P) = \{v_1, v_2, \ldots, v_{10}\}$. Since $\lambda(P) = 3$ and $\chi'(P) = 4$, it follows by Proposition 2.2 that $rd(P) = 3$ or $rd(P) = 4$. Assume, to the contrary, that $rd(P) = 3$ and let there be given a rainbow disconnection 3-coloring of $P$. Now, let $u$ and $v$ be two vertices of $P$ and let $R$ be a $u - v$ rainbow cut. Hence, $|R| \leq 3$ and $P - R$ is disconnected, where $u$ and $v$ belong to different components of $P - R$. Let $U$ be the vertex set of the component of $P - R$ containing $u$, where $|U| = k$. We may assume that $1 \leq k \leq 5$. First, suppose that $1 \leq k \leq 4$. Since the girth of $P$ is 5, the subgraph $P[U]$ induced by $U$ contains $k - 1$ edges. Therefore, $|R| = 3k - (2k - 2) = k + 2$, where then $3 \leq k + 2 \leq 6$. If $k = 5$, then $P[U]$ contains at most five edges and so $|R| \geq 5$, which is impossible. Since $rd(P) = 3$, it follows that $|R| \leq 3$ and so $k = 1$. Hence, the only possible $u - v$ rainbow cut is the set of the three edges incident with $u$ (or with $v$).

Let the colors assigned to the edges of $P$ be red, blue and green. Since $\chi'(P) = 4$, there is at least one vertex of $P$ that is incident with two edges of the same color. We claim, in fact, that there are at least two such vertices. Let $E_R$, $E_B$ and $E_G$ denote the sets of edges of $P$ colored red, blue and green, respectively, and let $P_R$, $P_B$ and $P_G$ be the spanning subgraphs of $P$ with edge sets $E_R$, $E_B$ and $E_G$. We may assume that $|E_R| \geq |E_B| \geq |E_G|$ and so $|E_R| \geq 5$. If $|E_R| \geq 7$, then $\sum_{i=1}^{10} \deg_{P_R} v_i \geq 14$. Since $\deg_{P_R} v_i \leq 3$ for each $i$ with $1 \leq i \leq 10$, at least
two vertices are incident with two red edges, verifying the claim. If \(|E_R| = 6\), then \(\sum_{i=1}^{10} \deg_{P_R} v_i = 12\). Then either (i) at least two vertices are incident with two red edges or (ii) there is a vertex, say \(v_{10}\), incident with three red edges and each of \(v_1, v_2, \ldots, v_9\) is incident with exactly one red edge. If (ii) occurs, then either \(|E_B| = 6\) or \(|E_B| = 5\) and so \(\sum_{i=1}^{9} \deg_{P_B} v_i \geq 10\), which implies that at least one of the vertices \(v_1, v_2, \ldots, v_9\) is incident with two blue edges, again verifying the claim.

The only remaining possibility is therefore \(|E_R| = |E_B| = |E_G| = 5\). If \(E_R\) is an independent set of five edges, then \(P - E_R\) is a 2-regular graph. Since the girth of \(P\) is 5 and \(P\) is not Hamiltonian, it follows that \(P - E_R\) consists of two vertex-disjoint 5-cycles. Thus, there is a vertex of \(P\) in each cycle incident with two blue edges or with two green edges, verifying the claim. Hence, none of \(E_R, E_B\) or \(E_G\) is an independent set. This implies that for each of these colors, there is a vertex of \(P\) incident with two edges of this color, verifying the claim in general.

Thus, \(P\) contains two vertices \(u\) and \(v\), each of which is incident with two edges of the same color. Since the only \(u - v\) rainbow cut is the set of edges incident with \(u\) or \(v\), this is a contradiction.

The following two results are useful.

**Proposition 2.6.** If \(H\) is a connected subgraph of a graph \(G\), then \(\text{rd}(H) \leq \text{rd}(G)\).

**Proof.** Let \(c\) be an rd-coloring of \(G\) and let \(u\) and \(v\) are two vertices of \(G\). Suppose that \(R\) is a \(u - v\) rainbow cut. Then \(R \cap E(H)\) is a \(u - v\) rainbow cut in \(H\). Hence, \(c\) restricted to \(H\) is a rainbow disconnection coloring of \(H\). Thus, \(\text{rd}(H) \leq \text{rd}(G)\).

A block of a graph is a maximal connected graph of \(G\) containing no cut-vertices. The block decomposition of \(G\) is the set of blocks of \(G\).

**Proposition 2.7.** Let \(G\) be a nontrivial connected graph, and let \(B\) be a block of \(G\) such that \(\text{rd}(B)\) is maximum among all blocks of \(G\). Then \(\text{rd}(G) = \text{rd}(B)\).

**Proof.** Let \(G\) be a nontrivial connected graph. Let \(\{B_1, B_2, \ldots, B_t\}\) be a block decomposition of \(G\), and let \(k = \max\{\text{rd}(B_i) \mid 1 \leq i \leq t\}\). If \(G\) has no cut-vertex, then \(G = B_1\) and the result follows. Hence, we may assume that \(G\) has at least one cutvertex. By Proposition 2.6, \(k \leq \text{rd}(G)\).

Let \(c_i\) be an rd-coloring of \(B_i\). We define the edge-coloring \(c : E(G) \to [k]\) of \(G\) by \(c(e) = c_i(e)\) if \(e \in E(B_i)\).

Let \(x, y \in V(G)\). If there exists a block, say \(B_i\), that contains both \(x\) and \(y\), then any \(x - y\) rainbow cut in \(B_i\) is an \(x - y\) rainbow cut in \(G\). Hence, we can assume that no block of \(G\) contains both \(x\) and \(y\), and that \(x \in B_i\) and \(y \in B_j\),
where \( i \neq j \). Now every \( x - y \) path contains a cut-vertex, say \( v \), of \( G \) in \( B_i \) and a cutvertex, say \( w \), of \( G \) in \( B_j \). Note that \( v \) could equal \( w \). If \( x \neq v \), then any \( x - v \) rainbow cut of \( B_i \) is an \( x - y \) rainbow cut in \( G \). Similarly, if \( y \neq w \), then any \( y - w \) rainbow cut of \( B_j \) is an \( x - y \) rainbow cut in \( G \). Thus, we may assume that \( x = v \) and \( y = w \). It follows that \( v \neq w \). Consider the \( x - y \) path 

\[ P = (x = v_1, v_2, \ldots, v_p = y) \]

Since \( x \) and \( y \) are cutvertices in different blocks and no block contains both \( x \) and \( y \), \( P \) contains a cut-vertex \( z \) of \( G \) in \( B_i \), that is, \( z = v_k \) for some \( k \) \((2 \leq k \leq p - 1)\). Then any \( x - z \) rainbow cut of \( B_i \) is an \( x - y \) rainbow cut of \( G \). Hence, \( \text{rd}(G) \leq k \), and so \( \text{rd}(G) = k \).

As a consequence of Proposition 2.7, the study of rainbow disconnection numbers can be restricted to 2-connected graphs. We now present several corollaries of Proposition 2.7.

**Corollary 2.8.** Let \( G \) and \( H \) be any two nontrivial connected graphs, and let \( GvH \) be a graph formed by identifying a vertex in \( G \) with a vertex in \( H \). Then \( \text{rd}(GvH) = \max\{\text{rd}(G), \text{rd}(H)\} \).

**Corollary 2.9.** Let \( G \) and \( H \) be any two nontrivial connected graphs, and let \( GuvH \) be a graph formed by adding an edge between any vertex \( u \) in \( G \) and any vertex \( v \) in \( H \). Then \( \text{rd}(GuvH) = \max\{\text{rd}(G), \text{rd}(H)\} \).

**Corollary 2.10.** Let \( G \) be a nontrivial connected graph and \( G' \) the graph obtained by attaching a pendant edge \( uv \) to some vertex \( u \) of \( G \). Then \( \text{rd}(G') = \text{rd}(G) \).

The corona \( G \circ K_1 \) is the graph obtained from \( G \) by attaching a leaf to each vertex of \( G \). Thus, if \( G \) has order \( n \), then the corona \( G \circ K_1 \) has order \( 2n \) and has precisely \( n \) leaves.

**Corollary 2.11.** If \( G \) is a nontrivial connected graph, then \( \text{rd}(G \circ K_1) = \text{rd}(G) \).

**Corollary 2.12.** Let \( G \) be a nontrivial connected graph, let \( T \) be a nontrivial tree and let \( u \) and \( v \) be vertices of \( G \) and \( T \), respectively. If \( H \) is the graph obtained from \( G \) and \( T \) by identifying \( u \) and \( v \), then \( \text{rd}(H) = \text{rd}(G) \).

A unicyclic graph is a connected graph with exactly one cycle.

**Corollary 2.13.** If \( G \) is a unicyclic graph \( G \), then \( \text{rd}(G) = 2 \).

### 3. Graphs with Prescribed Order and Rainbow Disconnection Number

In this section, we characterize all those nontrivial connected graphs of order \( n \) with rainbow disconnection number \( k \) for each \( k \in \{1, 2, n - 1\} \). The result for graphs having rainbow disconnection number 1 follows directly from Propositions 2.6 and 2.7.
Proposition 3.1. Let $G$ be a nontrivial connected graph. Then $\text{rd}(G) = 1$ if and only if $G$ is a tree.

Next, we characterize all nontrivial connected graphs of order $n$ having rainbow disconnection number 2. By Proposition 3.1, such a graph must contain a cycle. An ear of a graph $G$ is a maximal path whose internal vertices have degree 2 in $G$. An ear decomposition of a graph is a decomposition $H_0, H_1, \ldots, H_k$ such that $H_0$ is a cycle in $G$ and $H_i$ is an ear of the subgraph of $G$ with edge set $E(H_0) \cup E(H_1) \cup \cdots \cup E(H_i)$ for each integer $i$ with $1 \leq i \leq k$. Whitney [8] proved the following result in 1932.

Theorem 3.2. A graph $G$ is 2-connected if and only if $G$ has an ear decomposition. Furthermore, every cycle is the initial cycle in some ear decomposition of $G$.

The following is a consequence of Theorem 3.2.

Lemma 3.3. A 2-connected graph $G$ is a cycle if and only if for every two vertices $u$ and $v$ of $G$, there are exactly two internally disjoint $u - v$ paths in $G$.

Also, by Theorem 3.2, if $G$ is a 2-connected, non-Hamiltonian graph, then $G$ contains a theta subgraph (a subgraph consisting of two vertices connected by three internally disjoint paths of length 2 or more).

Theorem 3.4. Let $G$ be a nontrivial connected graph. Then $\text{rd}(G) = 2$ if and only if each block of $G$ is either $K_2$ or a cycle and at least one block of $G$ is a cycle.

Proof. If $G$ a nontrivial connected graph, each block of which is either $K_2$ or a cycle and at least one block of $G$ is a cycle, then Propositions 2.3 and 2.7 imply that $\text{rd}(G) = 2$.

We now verify the converse. Assume, to the contrary, that there is a connected graph $G$ with $\text{rd}(G) = 2$ that does not have the property that each block of $G$ is either $K_2$ or a cycle and at least one block of $G$ is a cycle. First, not all blocks can be $K_2$, for otherwise, $G$ is a tree and so $\text{rd}(G) = 1$ by Proposition 3.1. Hence, $G$ contains a block that is neither $K_2$ nor a cycle. By Lemma 3.3, there exist two distinct vertices $u$ and $v$ of $G$ for which $G$ contains at least three internally disjoint $u - v$ paths $P_1, P_2$ and $P_3$. Thus, any $u - v$ rainbow cut $R$ must contain at least one edge from each of $P_1, P_2$ and $P_3$ and so $|R| \geq 3$, which is impossible.

We now consider those graphs that are, in a sense, opposite to trees.

Proposition 3.5. For each integer $n \geq 4$, $\text{rd}(K_n) = n - 1$. 

Proof. Suppose first that \( n \geq 4 \) is even. Then \( \lambda(K_n) = \chi'(K_n) = n - 1 \). It then follows by Proposition 2.2 that \( \text{rd}(K_n) = n - 1 \). Next, suppose that \( n \geq 5 \) is odd. Then \( n - 1 = \lambda(K_n) \leq \text{rd}(K_n) \leq \chi'(K_n) = n \) by Proposition 2.2. To show that \( \text{rd}(K_n) = n - 1 \), it remains to show that there is a rainbow disconnection coloring of \( K_n \) using \( n - 1 \) colors. Let \( x \in V(K_n) \). Then \( K_n - x = K_{n-1} \). Since \( n - 1 \) is even, it follows that \( \chi'(K_{n-1}) = n - 2 \). Thus, there is a proper edge-coloring \( c_0 \) of \( K_{n-1} \) using the colors \( 1, 2, \ldots, n - 2 \). We now extend \( c_0 \) to an edge-coloring \( c \) of \( K_n \) by assigning the color \( n - 1 \) to each edge of \( K_n \) that is incident with \( x \). We show that \( c \) is a rainbow disconnection coloring of \( K_n \). Let \( u \) and \( v \) be two vertices of \( K_n \), where say \( u \neq v \). Then the set \( E_u \) of edges incident with \( u \) is a \( u - v \) rainbow cut. Thus, \( c \) is a rainbow disconnection coloring of \( K_n \) and so \( \text{rd}(K_n) \leq n - 1 \) and so \( \text{rd}(K_n) = n - 1 \).

By Propositions 2.2, 2.6 and 3.5, if \( G \) is a nontrivial connected graph of order \( n \), then

\[
1 \leq \text{rd}(G) \leq n - 1.
\]

Furthermore, \( \text{rd}(G) = 1 \) if and only if \( G \) is a tree by Proposition 3.1. We have seen that the complete graphs \( K_n \) of order \( n \geq 2 \) have rainbow disconnection number \( n - 1 \). We now characterize all nontrivial connected graphs of order \( n \) having rainbow disconnection number \( n - 1 \).

Theorem 3.6. Let \( G \) be a nontrivial connected graph of order \( n \). Then \( \text{rd}(G) = n - 1 \) if and only if \( G \) contains at least two vertices of degree \( n - 1 \).

Proof. First, suppose that \( G \) is a nontrivial connected graph of order \( n \) containing at least two vertices of degree \( n - 1 \). Since \( \text{rd}(G) \leq n - 1 \) by (1), it remains to show that \( \text{rd}(G) \geq n - 1 \). Let \( u, v \in V(G) \) such that \( \deg u = \deg v = n - 1 \). Among all sets of edges that separate \( u \) and \( v \) in \( G \), let \( S \) be one of minimum size. We show that \( |S| \geq n - 1 \). Let \( U \) be a component of \( G - S \) that contains \( u \) and let \( W = V(G) - U \). Thus, \( v \in W \) and \( S = [U, W] \) consists of those edges in \( G - S \) joining a vertex of \( U \) and a vertex of \( W \). Suppose that \( |U| = k \) for some integer \( k \) with \( 1 \leq k \leq n - 1 \) and then \( |W| = n - k \). The vertex \( u \) is adjacent to each of the \( n - k \) vertices of \( W \) and each of the remaining \( k - 1 \) vertices in \( U \) is adjacent to at least one vertex in \( W \). Hence, \( |S| \geq n - k + (k - 1) = n - 1 \). This implies that every \( u - v \) rainbow cut contains at least \( n - 1 \) edges of \( G \) and so \( \text{rd}(G) \geq n - 1 \).

For the converse, suppose that \( G \) is a nontrivial connected graph of order \( n \) having at most one vertex of degree \( n - 1 \). We show that \( \text{rd}(G) \leq n - 2 \). We consider two cases.

Case 1. Exactly one vertex \( v \) of \( G \) has degree \( n - 1 \). Let \( H = G - v \). Thus, \( \Delta(H) \leq n - 3 \). Since \( \chi'(H) \leq \Delta(H) + 1 = n - 2 \), there is a proper edge-coloring
of $H$ using $n - 2$ colors. We now define an edge-coloring $c : E(G) \to [n - 2]$ of $G$.
First, let $c$ be a proper $(n - 2)$-edge-coloring of $H$. For each vertex $x \in V(H)$, since $\deg_H x \leq n - 3$, there is $a_x \in [n - 2]$ such that $a_x$ is not assigned to any edge incident with $x$. Define $c(vx) = a_x$. Thus, the set $E_x$ of edges incident with $x$ is a rainbow set for each $x \in V(H)$. Let $u$ and $w$ be two distinct vertices of $G$. Then at least one of $u$ and $w$ belongs to $H$, say $u \in V(H)$. Since $E_u$ separates $u$ and $w$, it follows that $c$ is a rainbow disconnection coloring of $G$ using $n - 2$ colors. Hence, $\text{rd}(G) \leq n - 2$.

Case 2. No vertex of $G$ has degree $n - 1$. Therefore $\Delta(G) \leq n - 2$. If $\Delta(G) \leq n - 3$, then $\text{rd}(G) \leq \chi'(G) \leq n - 2$ by Proposition 2.2. Thus, we may assume that $\Delta(G) = n - 2$. Suppose first that $G$ is not $(n - 2)$-regular. We claim that $G$ is a connected spanning subgraph of some graph $G^*$ of order $n$ having exactly one vertex of degree $n - 1$. Let $u$ be a vertex of degree $k \leq n - 3$ in $G$. Let $N(u)$ be the neighborhood of $u$ and $W = V(G) - N[u]$, where $N[u] = N(u) \cup \{u\}$ is the closed neighborhood of $u$. Then $|N(u)| = k$ and $|W| = n - k - 1 \geq 2$. If $W$ contains a vertex $v$ of degree $n - 2$ in $G$, then $v$ is the only vertex of degree $n - 1$ in $G^* = G + w$. If no vertex in $W$ has degree $n - 2$ in $G$, then let $G^*$ be the graph obtained from $G$ by joining $u$ to each vertex in $W$. In this case, $u$ is the only vertex of degree $n - 1$ in $G^*$. It then follows by Case 1 that $\text{rd}(G^*) \leq n - 2$. Since $G$ is a connected spanning subgraph of $G^*$, it follows by Proposition 2.6 that $\text{rd}(G) \leq \text{rd}(G^*) \leq n - 2$. Finally, suppose that $G$ is $(n - 2)$-regular. Thus, $G$ is 1-factorable and so $\chi'(G) = \Delta(G) = n - 2$. Therefore, $\text{rd}(G) \leq \chi'(G) = n - 2$ by Proposition 2.2.

4. Rainbow Disconnection in Grids and Prisms

We now determine the rainbow disconnection numbers of graphs belonging to one of two well-known classes formed by Cartesian products. The Cartesian product $G \square H$ of two vertex-disjoint graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, where $(u, v)$ is adjacent to $(w, x)$ in $G \square H$ if and only if either $u = w$ and $vx \in E(H)$ or $uw \in E(G)$ and $v = x$. We consider the $m \times n$ grid graph $G_{m,n} = P_m \square P_n$, which consists of $m$ horizontal paths $P_n$ and $n$ vertical paths $P_m$.

Theorem 4.1. The rainbow disconnection numbers of the grid graphs $G_{m,n}$ are as follows:

(i) for all $n \geq 2$, $\text{rd}(G_{1,n}) = \text{rd}(P_n) = 1$,
(ii) for all $n \geq 3$, $\text{rd}(G_{2,n}) = 3$,
(iii) for all $n \geq 4$, $\text{rd}(G_{3,n}) = 3$,
(iv) for all $4 \leq m \leq n$, $\text{rd}(G_{m,n}) = 4$. 

Proof. (i) That \( \text{rd}(G_{1,n}) = \text{rd}(P_n) = 1 \) for \( n \geq 2 \) is a consequence of Proposition 3.1.

For the remainder of the proof, we consider the vertices of \( G_{m,n} \) as a matrix, letting \( x_{i,j} \) denote the vertex in row \( i \) and column \( j \), where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

(ii) For the graph \( G_{2,n} \), \( n \geq 3 \), \( \Delta(G_{2,n}) = 3 \). First, we define an edge-coloring \( c \) of \( G_{2,n} \). It is convenient to use the elements of the set \( \mathbb{Z}_3 \) of integer modulo 3 as colors here. Define the edge-coloring \( c : E(G_{2,n}) \rightarrow \mathbb{Z}_3 \) by

\[
\begin{align*}
&\ast \ c(x_{i,j},x_{i,j+1}) = i + j + 1 \text{ for } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq n - 1; \\
&\ast \ c(x_{1,j},x_{2,j}) = j \text{ for } 1 \leq j \leq n.
\end{align*}
\]

Next, we show that \( c \) is a rainbow disconnection coloring of \( G_{2,n} \). Let \( u \) and \( v \) be any two vertices of \( G_{2,n} \). If \( u \) and \( v \) belong to two different columns, then there exist two parallel edges joining vertices in the same two columns whose removal separates \( u \) and \( v \). Each such set of two edges is a \( u - v \) rainbow cut. Next, suppose that \( u \) and \( v \) belong to the same column. Then, without loss of generality, \( u \) belongs to the first row and \( v \) belongs to the second row. Then \( u \) and \( v \) both have degree 2 or both have degree 3. Therefore, the edges incident with \( u \) form a rainbow cut, and so, \( \text{rd}(G_{2,n}) \leq 3 \).

On the other hand, \( \lambda(u, v) = 2 \) if \( u \) and \( v \) are two vertices of \( G_{2,n} \) belonging to the same row, or different rows and columns or are two vertices of degree 2 belonging to the same column; while \( \lambda(u, v) = 3 \) if \( u \) and \( v \) are (adjacent) vertices of degree 3 belonging to the same column. It then follows by Proposition 2.2 that \( 3 = \lambda^+(G_{2,n}) \leq \text{rd}(G_{2,n}) \), and so \( \text{rd}(G_{2,n}) = 3 \).

(iii) As with \( G_{2,n} \), we define an edge-coloring \( c \) of \( G_{3,n} \). Again we use the elements of the set \( \mathbb{Z}_3 \) of integer modulo 3 as colors here. Define the edge-coloring \( c : E(G_{3,n}) \rightarrow \mathbb{Z}_3 \) by

\[
\begin{align*}
&\ast \ c(x_{i,j},x_{i,j+1}) = i + j + 1 \text{ for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n - 1; \\
&\ast \ c(x_{1,j},x_{2,j}) = j \text{ for } 1 \leq j \leq n; \\
&\ast \ c(x_{2,j},x_{3,j}) = j + 2 \text{ for } 1 \leq j \leq n.
\end{align*}
\]

Next, we show that \( c \) is a rainbow disconnection coloring of \( G_{3,n} \). Let \( u \) and \( v \) be any two vertices of \( G_{3,n} \). If \( u \) and \( v \) belong to two different columns, then there exist three parallel edges joining vertices in the same two columns whose removal separates \( u \) and \( v \). Each such set of three edges is a \( u - v \) rainbow cut. Next, suppose that \( u \) and \( v \) belong to the same column. Then at least one of \( u \) and \( v \) belongs to the top or bottom row, say \( u \) is such a vertex, which has degree 2 or 3. Then the edges incident with \( u \) is a \( u - v \) rainbow cut. Thus, \( \text{rd}(G_{3,n}) \leq 3 \).

On the other hand, for any two adjacent vertices \( u \) and \( v \) of degree 4 in \( G_{3,n} \) (necessarily in the middle row), \( \lambda^+(u, v) = 3 \). Thus, by Proposition 2.2, \( 3 \leq \lambda^+(G_{3,n}) \leq \text{rd}(G_{3,n}) \leq 3 \) and so \( \text{rd}(G_{3,n}) = 3 \).
(iv) Finally, we consider $G_{m,n}$ for $4 \leq m \leq n$. Since there are four pairwise edge-disjoint $u-v$ paths in $G_{m,n}$ for every two vertices $u$ and $v$ of degree 4, it follows by Theorem 2.1 that $\lambda(u,v) = 4$. For any other pair $u, v$ of vertices of $G_{m,n}$, it follows that $\lambda(u,v) \leq 3$. By Proposition 2.2 then, $4 = \lambda^+(G_{m,n}) \leq \text{rd}(G_{m,n})$. On the other hand, since $G_{m,n}$ is bipartite, $\chi(G_{m,n}) = \Delta(G_{m,n}) = 4$. Again, by Proposition 2.2, $\text{rd}(G_{m,n}) \leq 4$ and so $\text{rd}(G_{4,n}) = 4$. \hfill □

Next we determine the rainbow disconnection number of prisms, namely those graphs of the type $G \square K_2$ for some graph $G$.

**Proposition 4.2.** If $G$ is a nontrivial connected graph, then

$$\text{rd}(G \square K_2) = \Delta(G) + 1.$$ 

**Proof.** Let $G$ and $G'$ be the two copies of $G$ in the prism $G \square K_2$, and for each $v \in V(G)$, let $v'$ be its corresponding vertex in $G'$. We first show that $G \square K_2$ has a proper edge-coloring using $\Delta(G \square K_2) = \Delta(G) + 1$ colors, that is, $\chi(G \square K_2) = \Delta(G) + 1$. Let $C$ be a proper edge-coloring of $G$ using $\chi(G)$ colors. Color the edges of $G$ and $G'$ using $C$, that is, $G$ and $G'$ have an identical edge-coloring $C$. By Vizing’s Theorem, $\Delta(G) \leq \chi(G) \leq \Delta(G) + 1$. First assume that $\chi(G) = \Delta(G)$. Then assigning the color $\Delta(G) + 1$ to each edge $vv'$ for every $v \in V(G)$ gives a proper edge-coloring of $G \square K_2$ with $\Delta(G) + 1$ colors. Next assume that $\chi(G) = \Delta(G) + 1$. Then for each $v \in V(G)$, at least one of the $\Delta(G) + 1$ colors is missing from the colors of the edges incident to $v$. Let $c_v$ be one such missing color. Note that $c_v$ is also missing from the colors of the edges incident to $v'$ in $G'$ because $G$ and $G'$ have the identical colorings. Hence, assigning $c_v$ to $vv'$ for each $v \in V(G)$ yields a proper edge-coloring of $G \square K_2$ having $\Delta(G) + 1$ colors. By Proposition 2.2, it follows that $\text{rd}(G \square K_2) \leq \Delta(G) + 1$.

To establish the lower bound, let $u$ be a vertex of $G$ with $\deg u = \Delta(G) = \Delta$. In $G \square K_2$, there exist $\Delta + 1$ edge-disjoint $u-u'$ paths, one of which is the edge $uu'$ and the remaining $\Delta$ of which have the form $(u, w, w', u')$, where $w \in V(G)$ and $w'$ is the corresponding vertex of $w$ in $G'$. It again follows by Proposition 2.2 that $\text{rd}(G \square K_2) \geq \lambda^+(G \square K_2) \geq \Delta(G) + 1$. \hfill □

Complementary products were introduced in [4] as a generalization of Cartesian products. We consider a subfamily of complementary products, namely, complementary prisms. For a graph $G = (V, E)$, the **complementary prism**, denoted $G\overline{G}$, is formed from the disjoint union of $G$ and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of $G$ and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to $v$. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $vv'$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen graph. In particular, the Petersen graph is the complementary prism $C_5 \overline{C}_5$. For another example of a complementary prism, the corona $K_n \circ K_1$ is the complementary prism $K_{n, \overline{K}_n}$. 
We refer to the complementary prism $GG$ as a copy of $G$ and a copy of $G$ with a perfect matching between corresponding vertices. For a set $S \subseteq V(G)$, let $S$ denote the corresponding set of vertices in $V(G)$. We note that $GG$ is isomorphic to $\overline{GG}$.

Since $\Delta(G) = \max\{\Delta(G), \Delta(\overline{G})\} + 1$, Proposition 2.2 implies that $\text{rd}(G) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$. This bound is sharp for the Petersen graph $P = C_5 \overline{C}_5$.

Our next result shows that for graphs $G$ of order $n$ with sufficiently large girth, $\text{rd}(G)$ is strictly greater than the maximum degree of $G$.

**Proposition 4.3.** If $G$ is a graph of order $n$, maximum degree $\Delta(G) < n - 1$, and girth at least five, then

$$\Delta(G) + 1 \leq \text{rd}(G).$$

**Proof.** Consider a vertex $u$ in $G$ such that $\deg_G u = \Delta(G)$. Let $A = N_G(u)$ and $B = V - N_G[u]$. Thus, in $G$, $N(\overline{u}) = B \cup \{u\}$. Note that since $n - 1 > \Delta(G)$, it follows that $B \neq \emptyset$.

We claim there are $\Delta(G) + 1$ edge-disjoint $u$-$\overline{B}$ paths, where $\overline{B} \subseteq \overline{B}$. To see this note that one such path is $(u, \overline{u}, \overline{B})$. Next consider the $u$-$\overline{B}$ paths containing a vertex $a \in A$. If $a$ is not adjacent to $b$ in $G$, then $\overline{\overline{a}}$ is adjacent to $\overline{b}$ in $\overline{G}$ and $(u, a, \overline{\overline{a}}, \overline{b})$ is a $u$-$\overline{B}$ path. If $ab \in E(G)$, then $(u, a, b, \overline{b})$ is a $u$-$\overline{B}$ path. Moreover, since $g(G) \geq 5$, at most one vertex in $A$ is adjacent to $b$, else a 4-cycle is formed. In any case, the collection of these $|A| + 1 = \Delta(G) + 1$ paths are edge-disjoint. Hence, by Proposition 2.2, it follows that $\text{rd}(G) \geq \lambda^+(G) \geq \Delta(G) + 1$. 

For an example of a complementary prism attaining the lower bound, let $G$ be the graph formed from a 5-cycle by attaching a leaf $x$ to a vertex $v$ of the cycle. Then, $\Delta(G) = 3$. We show that $\text{rd}(G) = 4$. First note that the Petersen graph $P$ is a proper subgraph of $G\overline{G}$, and by Propositions 2.5 and 3.6, $\text{rd}(G) \geq \text{rd}(P) = 4$. Furthermore, there is a proper edge-coloring $c$ of $P$ using four colors such that three colors are used to color $C_5$ and $C_5$ and the fourth color is used on the matching edges. Thus, we may assume, without loss of generality, that $v$ is incident to the edges colored 1 and 2 in $G$ and that $v\overline{u}$ is assigned color 4.

We extend $c$ to a rainbow disconnection coloring of $G\overline{G}$ as follows: let $c(x\overline{x}) = 3$, $c(x\overline{\overline{u}}) = 4$, and $c(\overline{u}\overline{v})$ be the color missing from the edges incident to $\overline{u}$ for each $\overline{u}$ adjacent to $u$ in $\overline{G}$. Consider two arbitrary vertices of $G\overline{G}$. At least one of the vertices, say $u$, is not $\overline{u}$. Thus, the edges incident with $u$ are a rainbow cut separating the two vertices. Since every such vertex $u$ has degree at most four, $\text{rd}(G) \leq 4$, and so, $\text{rd}(G\overline{G}) = 4$. 


5. Extremal Problems

In this section, we investigate the following problem:

For a given pair $k, n$ of positive integers with $k \leq n - 1$, what are the minimum possible size and maximum possible size of a connected graph $G$ of order $n$ such that $\text{rd}(G) = k$?

We have seen in Proposition 3.1 that the only connected graphs of order $n$ having rainbow disconnection number 1 are the trees of order $n$. That is, the connected graphs of order $n$ having rainbow disconnection number 1 have size $n - 1$. We have also seen in Theorem 3.4 that the minimum size of a connected graph of order $n \geq 3$ having rainbow disconnection number 2 is $n$. Furthermore, we have seen in Theorem 3.6 that the minimum size of a connected graph of order $n \geq 2$ having rainbow disconnection number $n - 1$ is $2n - 3$. In fact, these are special cases of a more general result. In order to show this, we first present a lemma.

**Lemma 5.1.** Let $H$ be a connected graph of order $n$ that is not complete and let $x$ and $y$ be two nonadjacent vertices of $H$. Then $\text{rd}(H + xy) \leq \text{rd}(H) + 1$.

**Proof.** Suppose that $\text{rd}(H) = k$ for some positive integer $k$ and let $c_0$ be a rainbow disconnection coloring of $H$ using the colors $1, 2, \ldots, k$. Extend the coloring $c_0$ to the edge-coloring $c$ of $H + xy$ by assigning the color $k + 1$ to the edge $xy$. Let $u$ and $v$ be two vertices of $H$ and let $R$ be a $u - v$ rainbow cut in $H$. Then $R \cup \{xy\}$ is a $u - v$ rainbow cut in $H + xy$. Hence, $c$ is a rainbow disconnection $(k + 1)$-coloring of $H + xy$. Therefore, $\text{rd}(H + xy) \leq k + 1 = \text{rd}(H) + 1$. ■

**Theorem 5.2.** For integers $k$ and $n$ with $1 \leq k \leq n - 1$, the minimum size of a connected graph of order $n$ having rainbow disconnection number $k$ is $n + k - 2$.

**Proof.** By Proposition 3.5, the result is true for $k = n - 1$. Hence, we may assume that $1 \leq k \leq n - 2$. First, we show that if the size of a connected graph $G$ of order $n$ is $n + k - 2$, then $\text{rd}(G) \leq k$. We proceed by induction on $k$. We have seen that the result is true for $k = 1, 2$ by Proposition 3.1 and Theorem 3.4. Suppose that if the size of a connected graph $H$ of order $n$ is $n + k - 2$ for some integer $k$ with $2 \leq k \leq n - 3$, then $\text{rd}(H) \leq k$. Let $G$ be a connected graph of order $n$ and size $n + (k + 1) - 2 = n + k - 1$. We show that $\text{rd}(G) \leq k + 1$. Since $G$ is not a tree, there is an edge $e$ such that $H = G - e$ is a connected spanning subgraph of $G$. Since the size of $H$ is $n + k - 2$, it follows by induction hypothesis that $\text{rd}(H) \leq k$. Hence, $\text{rd}(G) = \text{rd}(H + e) \leq k + 1$ by Lemma 5.1. Therefore, the minimum possible size for a connected graph $G$ of order $n$ to have $\text{rd}(G) = k$ is $n + k - 2$.

It remains to show that for each pair $k, n$ of integers with $1 \leq k \leq n - 1$ there is a connected graph $G$ of order $n$ and size $n + k - 2$ such that $\text{rd}(G) = k$. Since this
is true for $k = 1, 2, n - 1$, we now assume that $3 \leq k \leq n - 2$. Let $H = K_{2,k}$ with
partite set $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \ldots, w_k\}$. Now, let $G$ be the graph of
order $n$ and size $n + k - 2$ obtained from $H$ by subdividing the edge $u_1w_1$ a total
of $n - k - 2$ times, producing the path $P = (u_1, v_1, v_2, \ldots, v_{n-k-2}, w_1)$ in $G$. Since
$\chi'(H) = k$, there is a proper edge-coloring $c_H$ of $H$ using the colors $1, 2, \ldots, k$.
We may assume that $c(u_1w_1) = 1$ and $c(u_2w_1) = 2$. Next, we extend the coloring
$c_H$ to a proper edge-coloring $c_G$ of $G$ using the colors $1, 2, \ldots, k$ by defining
$c_G(u_1v_1) = 1$ and alternating the colors of the edges of $P$ with 3 and 1 thereafter.
Hence, $\chi'(G) = k$ and so $\text{rd}(G) \leq \chi'(G) = k$ by Proposition 2.2. Furthermore,
since $\lambda(u_1, u_2) = k$ and $\lambda(x, y) = 2$ for all other pairs $x, y$ of vertices of $G$, it
follows that $\lambda^+(G) = k$. Again, by Proposition 2.2, $\text{rd}(G) \geq \lambda^+(G) = k$ and so
$\text{rd}(G) = k$.

For given integers $k$ and $n$ with $1 \leq k \leq n - 1$, we have determined the
minimum size of a connected graph $G$ of order $n$ with $\text{rd}(G) = k$. So, this brings
up the question of determining the maximum size of a connected graph $G$ of
order $n$ with $\text{rd}(G) = k$. Of course, we know this size when $k = 1$; it is $n - 1$.
Also, we know this size when $k = n - 1$; it is $\binom{n}{2}$. For odd integers $n$, we have the
following conjecture.

**Conjecture 5.3.** Let $k$ and $n$ be integers with $1 \leq k \leq n - 1$ and $n \geq 5$ is odd.
Then the maximum size of a connected graph $G$ of order $n$ with $\text{rd}(G) = k$ is
$\binom{k+1}{2}(n-1)$.

Notice that when $k = 1$, then $\frac{(k+1)(n-1)}{2} = n - 1$ and when $k = n - 1$, then
$\frac{(k+1)(n-1)}{2} = \binom{n}{2}$. Also, when $k = 2$, then $\frac{(k+1)(n-1)}{2} = \frac{3n-2}{2}$.
This is the size of the so-called friendship graph $\left(\frac{k+1}{2}\right)K_2 \vee K_1$ of order $n$ (every two vertices has a
unique friend). Since each block of a friendship graph is a triangle, it follows by
Theorem 3.4 that each such graph has rainbow disconnection number 2.

For given integers $k$ and $n$ with $1 \leq k \leq n - 1$ and $n \geq 5$ is odd, let $H_k$ be a $(k - 1)$-regular graph of order $n - 1$. Since $n - 1$ is even, such graphs $H_k$ exist.
Now, let $G_k = H_k \vee K_1$ be the join of $H_k$ and $K_1$. Thus, $G_k$ is a connected graph
of order $n$ having one vertex of degree $n - 1$ and $n - 1$ vertices of degree $k$. The
size $m$ of $G_k$ satisfies the equation:

$$2m = (n - 1) + (n - 1)k = (k + 1)(n - 1)$$

and so $m = \frac{(k+1)(n-1)}{2}$. The graph $H_k$ can be selected so that it is 1-factorable
and so $\chi'(H_k) = k - 1$. If a proper $(k - 1)$-edge-coloring of $H_k$ is given using the
colors $1, 2, \ldots, k - 1$, and every edge incident with the vertex of $G_k$ of degree $n - 1$
is assigned the color $k$, then the edges incident with each vertex of degree $k$ are
properly colored with $k$ colors. For any two vertices $u$ and $v$ of $G_k$, at least one of
u and v has degree k in $G_k$, say $\deg_{G_k} u = k$. Then the set of edges incident with u is a $u - v$ rainbow cut in H. Since this is a rainbow disconnection $k$-coloring of G, it follows that $\text{rd}(G_k) \leq k$. It is reasonable to conjecture that $\text{rd}(G_k) = k$.

We would still be left with the question of whether every graph H of order $n$ and size $\frac{(k+1)(n-1)}{2} + 1$ must have $\text{rd}(H) > k$. Certainly, every such graph H must contain at least two vertices whose degrees exceed $k$.

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