

DESCRIBING NEIGHBORHOODS OF 5-VERTICES
IN 3-POLYTOPES WITH MINIMUM DEGREE 5
AND WITHOUT VERTICES OF
DEGREES FROM 7 TO 11¹

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Abstract

In 1940, Lebesgue proved that every 3-polytope contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

(6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11),
(5, 6, 7, 7, 8), (5, 6, 6, 7, 12), (5, 6, 6, 8, 10), (5, 6, 6, 6, 17),
(5, 5, 7, 7, 13), (5, 5, 7, 8, 10), (5, 5, 6, 7, 27),
(5, 5, 6, 6, ∞), (5, 5, 6, 8, 15), (5, 5, 6, 9, 11),
(5, 5, 5, 7, 41), (5, 5, 5, 8, 23), (5, 5, 5, 9, 17),
(5, 5, 5, 10, 14), (5, 5, 5, 11, 13).

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: (5, 5, 6, 6, ∞), (5, 6, 6, 6, 15), (6, 6, 6, 6, 6), where all parameters are tight.

Keywords: planar graph, structure properties, 3-polytope, neighborhood.

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