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THE QUEST FOR A CHARACTERIZATION OF HOM-PROPERTIES OF FINITE CHARACTER

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Abstract

A *graph property* is a set of (countable) graphs. A *homomorphism* from a graph G to a graph H is an edge-preserving map from the vertex set of G into the vertex set of H; if such a map exists, we write $G \to H$. Given any graph H, the *hom-property* $\to H$ is the set of H*-colourable graphs*, i.e., the set of all graphs G satisfying $G \to H$. A graph property $\mathcal{P}$ is *of finite character* if, whenever we have that $F \in \mathcal{P}$ for every finite induced subgraph F of a graph G, then we have that $G \in \mathcal{P}$ too. We explore some of the relationships of the property attribute of being of finite character to other property attributes such as being *finitely-induced-hereditary*, being *finitely determined*, and being *axiomatizable*. We study the hom-properties of finite character, and prove some necessary and some sufficient conditions on H for $\to H$ to be of finite character. A notable (but known) sufficient condition is that H is a finite graph, and our new model-theoretic proof of this compactness result extends from hom-properties to all axiomatizable properties. In our quest to find an intrinsic characterization of those H for which $\to H$ is of finite character, we find an example of an infinite connected graph with no finite core and chromatic number 3 but with hom-property not of finite character.

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Keywords: (countable) graph, homomorphism (of graphs), property of graphs, hom-property, (finitely-)induced-hereditary property, finitely determined property, (weakly) finite character, axiomatizable property, compactness theorems, core, connectedness, chromatic number, clique number, independence number, dominating set.

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