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# INDUCED ACYCLIC TOURNAMENTS IN RANDOM DIGRAPHS: SHARP CONCENTRATION, THRESHOLDS AND ALGORITHMS[1]

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**Abstract**

Given a simple directed graph $D = (V, A)$, let the size of the largest induced acyclic tournament be denoted by $mat(D)$. Let $D \in \mathcal{D}(n, p)$ (with $p = p(n)$) be a *random* instance, obtained by randomly orienting each edge of a random graph drawn from $\mathcal{G}(n, 2p)$. We show that $mat(D)$ is asymptotically almost surely (a.a.s.) one of only 2 possible values, namely either $b^*$ or $b^* + 1$, where $b^* = \lfloor 2(\log_r n) + 0.5 \rfloor$ and $r = p^{-1}$.

It is also shown that if, asymptotically, $2(\log_r n) + 1$ is not within a distance of $w(n)/(\ln n)$ (for any sufficiently slow $w(n) \to \infty$) from an integer, then $mat(D)$ is $\lfloor 2(\log_r n) + 1 \rfloor$ a.a.s. As a consequence, it is shown that $mat(D)$ is 1-point concentrated for all $n$ belonging to a subset of positive integers of density 1 if $p$ is independent of $n$. It is also shown that there are functions $p = p(n)$ for which $mat(D)$ is provably *not* concentrated in a single value. We also establish thresholds (on $p$) for the existence of induced acyclic tournaments of size $i$ which are sharp for $i = i(n) \to \infty$.

We also analyze a polynomial time heuristic and show that it produces a solution whose size is at least $\log_r n + \Theta(\sqrt{\log_r n})$. Our results are valid as long as $p \geq 1/n$. All of these results also carry over (with some slight changes) to a related model which allows 2-cycles.

**Keywords:** random digraphs, tournaments, concentration, thresholds, algorithms.

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# 1. Appendix

## 1.1. $mat(D)$ versus $\omega(G)$

The following lemma relates the probabilities in the two models $\mathcal{D}(n,p)$ and $\mathcal{G}(n,p)$ for having, respectively, tournaments and cliques of specific sizes. Its proof is similar to the proof of an analogous relationship involving $mas(D)$ and $\alpha(G)$ (maximum size of an independent set in $G$) established in [23].

**Lemma 1.1.** *For any positive integer $b$, for a random digraph $D \in \mathcal{D}(n,p)$,*

$$\mathbf{Pr}[mat(D) \geq b] \geq \mathbf{Pr}[\omega(G) \geq b],$$

*where $G \in \mathcal{G}(n,p)$.*

***Proof.*** Given a linear ordering $\sigma$ of vertices of $D$ and a subset $A$ of size $b$, we say that $D[A]$ is consistent with $\sigma$ if for every $\sigma_i, \sigma_j \in A$ with $i < j$, $D[A]$ has the arc $(\sigma_i, \sigma_j)$.

Let $\tau$ denote an arbitrary but fixed ordering of $V$. Once we fix $\tau$, the spanning subgraph of $D$ formed by arcs of the form $(\tau(i), \tau(j))$ $(i < j)$ is having the same distribution as $\mathcal{G}(n, p)$. Hence, for any $A$, the event of $D[A]$ being consistent with $\tau$ is equivalent to the event of $A$ inducing a clique in $\mathcal{G}(n, p)$. Hence,

$$\begin{aligned}
\mathbf{Pr}(\ mat(D) \geq b) &= \mathbf{Pr}(\ \exists A,\ |A| = b,\ D[A] \text{ is an acyclic tournament}) \\
&= \mathbf{Pr}(\ \exists A,\ |A| = b,\ \exists \sigma,\ D[A] \text{ is consistent with } \sigma) \\
&= \mathbf{Pr}(\ \exists \sigma,\ \exists A,\ |A| = b,\ D[A] \text{ is consistent with } \sigma) \\
&\geq \mathbf{Pr}(\ \exists A,\ |A| = b,\ D[A] \text{ is consistent with } \tau\ ) \\
&= \mathbf{Pr}(\ \omega(G) \geq b).
\end{aligned}$$

Hence it is natural that we have a bigger upper bound for $mat(D)$ than we have for $\omega(G)$. ■

**Note:** Recall that we first draw an undirected $G \in \mathcal{G}(n, 2p)$ and then choose uniformly randomly an orientation of $E(G)$. Hence, for any fixed $A \subseteq V$ of size $b$ with $b = \omega(1)$,

$$\mathbf{Pr}(\ D[A] \text{ is an acyclic tournament } \mid G[A] \text{ induces a clique }) = \frac{b!}{2^{\binom{b}{2}}} = o(1).$$

However, there are so many cliques of size $b$ in $G$ that one of them manages to induce an acyclic tournament.

### 1.2. Proof of Theorem ??

We reduce the NP-complete Maximum Clique problem MC$(G, k)$ to the MAT$(D$, $k)$ problem as follows. Given an instance $(G = (V, E), k)$ of the first problem, compute an instance $f(G) = (G' = (V, A), k)$ in polynomial time where

$$A = \{(u, v) : uv \in E, u < v\}.$$

Clearly, $G'$ is a dag and it is easy to see that a set $V' \subseteq V$ induces a clique in $G$ if and only if $V'$ induces an acyclic tournament in $G'$. This establishes that MAT$(D, k)$ is NP-hard even if $D$ is restricted to be a dag.

The inapproximability of MAT$(D)$ follows from the following observation. Note that the reduction $G \to f(G)$ is an $L$-reduction in the sense of [20], since $|f(G)| = |G|$ and $\omega(G) = mat(G')$. Hence, any inapproximability result on maximum clique in undirected graphs (for example [12, 14]), implies a similar inapproximability for the MAT$(D)$ problem.

### 1.3. Proof of Claim ??

Order the vertices of $U$ along a Hamilton path $P$ (if any exists) of $H$. An arc $(u, v) \in A$ is a forward arc if $u$ comes before $v$ in $P$ and is a backward arc otherwise. Since $H$ is acyclic, any arc $(v, u) \in A$ must be a forward arc, since otherwise the segment of $P$ from $u$ to $v$ along with $(v, u)$ forms a cycle in $H$.

Now if there is another Hamilton path $Q$ in $H$, $Q \neq P$, then walking along $P$, consider the first vertex $a$ where $Q$ differs from $P$. Then in the path $Q$, $a$ is visited immediately after some vertex $a'$ that comes after $a$ in $P$. But this implies that $(a', a)$ is a backward arc in $H$ contradicting the observation earlier that $H$ has no backward arc.

### 1.4. Remaining cases of Theorem ??

For $1/wn \leq p < 1/n$,

$$E[X(n,4)] = \binom{n}{4} \cdot 4! \cdot p^{\binom{4}{2}} \leq n^4 p^6 \ \leq \ (1/n^2) = o(1).$$

Now, an acyclic tournament of size 2 is simply an edge which a.a.s. exists since:

$$\mathbf{Pr}[mat(D) < 2] = \mathbf{Pr}[D \text{ is the empty graph}] = (1-2p)^{\binom{n}{2}} \leq e^{-n(n-1)p} = o(1),$$

since $p \geq 1/wn \geq w/n^2$. Hence, when $1/wn \leq p \leq 1/n$, $mat(D) \in \{2, 3\}$, a.a.s.

For $wn^{-2} \leq p < 1/wn$,

$$E[X(n,3)] = \binom{n}{3} \cdot 3! \cdot p^{\binom{3}{2}} \leq n^3 p^3 = o(1) \text{ since } np = o(1).$$

The proof for $mat(D) \geq 2$ is the same as in the previous case, since $n^2 p = \omega(1)$, and hence, at least one arc will exist, a.a.s. So when $w/n^2 \leq p \leq 1/wn$, $mat(D) = 2$, a.a.s.

For $(wn^2)^{-1} \leq p \leq w/n^2$, $E[X(n,3)] = o(1)$, as in the previous case, and so $mat(D) = 1$ or 2, a.a.s. When $p < (wn^2)^{-1}$, $mat(D) = 1$ since $D$ a.a.s. has no directed edge.

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