INDUCED ACYCLIC TOURNAMENTS IN RANDOM DIGRAPHS: SHARP CONCENTRATION, THRESHOLDS AND ALGORITHMS\(^1\)

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**Abstract**

Given a simple directed graph \(D = (V, A)\), let the size of the largest induced acyclic tournament be denoted by \(\text{mat}(D)\). Let \(D \in \mathcal{D}(n, p)\) (with \(p = p(n)\)) be a random instance, obtained by randomly orienting each edge of a random graph drawn from \(\mathcal{G}(n, 2p)\). We show that \(\text{mat}(D)\) is asymptotically almost surely (a.a.s.) one of only 2 possible values, namely either \(b^*\) or \(b^* + 1\), where \(b^* = \lfloor 2(\log r n) + 0.5 \rfloor\) and \(r = p^{-1}\).

It is also shown that if, asymptotically, \(2(\log r n) + 1\) is not within a distance of \(\omega(n)/\ln n\) (for any sufficiently slow \(\omega(n) \to \infty\)) from an integer, then \(\text{mat}(D)\) is \([2(\log r n) + 1]\) a.a.s. As a consequence, it is shown that \(\text{mat}(D)\) is 1-point concentrated for all \(n\) belonging to a subset of positive integers of density 1 if \(p\) is independent of \(n\). It is also shown that there are functions \(p = p(n)\) for which \(\text{mat}(D)\) is provably not concentrated in a single value. We also establish thresholds (on \(p\)) for the existence of induced acyclic tournaments of size \(i\) which are sharp for \(i = i(n) \to \infty\).

We also analyze a polynomial time heuristic and show that it produces a solution whose size is at least \(\log r n + \Theta(\sqrt{\log r n})\). Our results are valid as long as \(p \geq 1/n\). All of these results also carry over (with some slight changes) to a related model which allows 2-cycles.

**Keywords:** random digraphs, tournaments, concentration, thresholds, algorithms.

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1. Appendix

1.1. \textit{mat}(D) versus $\omega(G)$

The following lemma relates the probabilities in the two models $D(n, p)$ and $G(n, p)$ for having, respectively, tournaments and cliques of specific sizes. Its proof is similar to the proof of an analogous relationship involving $mas(D)$ and $\alpha(G)$ (maximum size of an independent set in $G$) established in [23].

\begin{lemma}
For any positive integer $b$, for a random digraph $D \in D(n, p)$, \\
\Pr[\text{mat}(D) \geq b] \geq \Pr[\omega(G) \geq b],
\end{lemma}

where $G \in G(n, p)$.

\textbf{Proof.} Given a linear ordering $\sigma$ of vertices of $D$ and a subset $A$ of size $b$, we say that $D[A]$ is consistent with $\sigma$ if for every $\sigma_i, \sigma_j \in A$ with $i < j$, $D[A]$ has the arc $(\sigma_i, \sigma_j)$.


Let \( \tau \) denote an arbitrary but fixed ordering of \( V \). Once we fix \( \tau \), the spanning subgraph of \( D \) formed by arcs of the form \((\tau(i), \tau(j)) \) \((i < j) \) is having the same distribution as \( G(n, p) \). Hence, for any \( A \), the event of \( D[A] \) being consistent with \( \tau \) is equivalent to the event of \( A \) inducing a clique in \( G(n, p) \). Hence,

\[
\Pr(\text{mat}(D) \geq b) = \Pr(\exists A, |A| = b, D[A] \text{ is an acyclic tournament})
\]

\[
= \Pr(\exists A, |A| = b, \exists \sigma, D[A] \text{ is consistent with } \sigma)
\]

\[
= \Pr(\exists \sigma, \exists A, |A| = b, D[A] \text{ is consistent with } \sigma)
\]

\[
\geq \Pr(\exists A, |A| = b, D[A] \text{ is consistent with } \tau)
\]

\[
= \Pr(\omega(G) \geq b).
\]

Hence it is natural that we have a bigger upper bound for \( \text{mat}(D) \) than we have for \( \omega(G) \).

**Note:** Recall that we first draw an undirected \( G \in \mathcal{G}(n, 2p) \) and then choose uniformly randomly an orientation of \( E(G) \). Hence, for any fixed \( A \subseteq V \) of size \( b \) with \( b = \omega(1) \),

\[
\Pr(D[A] \text{ is an acyclic tournament } | G[A] \text{ induces a clique }) = \frac{b!}{2^\binom{b}{2}} = o(1).
\]

However, there are so many cliques of size \( b \) in \( G \) that one of them manages to induce an acyclic tournament.

1.2. **Proof of Theorem ??**

We reduce the NP-complete Maximum Clique problem \( \text{MC}(G, k) \) to the \( \text{MAT}(D, k) \) problem as follows. Given an instance \((G = (V, E), k)\) of the first problem, compute an instance \( f(G) = (G', (V, A, k)) \) in polynomial time where

\[
A = \{(u, v) : uv \in E, u < v\}.
\]

Clearly, \( G' \) is a dag and it is easy to see that a set \( V' \subseteq V \) induces a clique in \( G \) if and only if \( V' \) induces an acyclic tournament in \( G' \). This establishes that \( \text{MAT}(D, k) \) is NP-hard even if \( D \) is restricted to be a dag.

The inapproximability of \( \text{MAT}(D) \) follows from the following observation. Note that the reduction \( G \rightarrow f(G) \) is an \( L \)-reduction in the sense of [20], since \( |f(G)| = |G| \) and \( \omega(G) = \text{mat}(G') \). Hence, any inapproximability result on maximum clique in undirected graphs (for example [12, 14]), implies a similar inapproximability for the \( \text{MAT}(D) \) problem.
1.3. Proof of Claim ??

Order the vertices of $U$ along a Hamilton path $P$ (if any exists) of $H$. An arc $(u, v) \in A$ is a forward arc if $u$ comes before $v$ in $P$ and is a backward arc otherwise. Since $H$ is acyclic, any arc $(v, u) \in A$ must be a forward arc, since otherwise the segment of $P$ from $u$ to $v$ along with $(v, u)$ forms a cycle in $H$.

Now if there is another Hamilton path $Q$ in $H$, $Q \neq P$, then walking along $P$, consider the first vertex $a$ where $Q$ differs from $P$. Then in the path $Q$, $a$ is visited immediately after some vertex $a'$ that comes after $a$ in $P$. But this implies that $(a', a)$ is a backward arc in $H$ contradicting the observation earlier that $H$ has no backward arc.

1.4. Remaining cases of Theorem ??

For $1/wn \leq p < 1/n$,

$$E[X(n, 4)] = \binom{n}{4} \cdot 4! \cdot p^{\binom{4}{2}} \leq n^4 p^6 \leq (1/n^2) = o(1).$$

Now, an acyclic tournament of size 2 is simply an edge which a.a.s. exists since:

$$\Pr[mat(D) < 2] = \Pr[D \text{ is the empty graph}] = (1 - 2p)^{\binom{2}{2}} \leq e^{-n(n-1)p} = o(1),$$

since $p \geq 1/wn \geq w/n^2$. Hence, when $1/wn \leq p \leq 1/n$, $mat(D) \in \{2, 3\}$, a.a.s.

For $wn^{-2} \leq p < 1/wn$,

$$E[X(n, 3)] = \binom{n}{3} \cdot 3! \cdot p^{\binom{3}{2}} \leq n^3 p^3 = o(1) \text{ since } np = o(1).$$

The proof for $mat(D) \geq 2$ is the same as in the previous case, since $n^2 p = \omega(1)$, and hence, at least one arc will exist, a.a.s. So when $w/n^2 \leq p \leq 1/wn$, $mat(D) = 2$, a.a.s.

For $(wn^2)^{-1} \leq p \leq w/n^2$, $E[X(n, 3)] = o(1)$, as in the previous case, and so $mat(D) = 1$ or 2, a.a.s. When $p < (wn^2)^{-1}$, $mat(D) = 1$ since $D$ a.a.s. has no directed edge.