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A CONSTRUCTIVE CHARACTERIZATION OF VERTEX COVER ROMAN TREES

Abel Cabrera Martínez

Universitat Rovira i Virgili
Departament d'Enginyeria Informàtica i Matemàtiques
Av. Països Catalans 26, 43007 Tarragona, Spain

e-mail: abel.cabrera@urv.cat

Dorota Kuziak

Universidad de Cádiz
Departamento de Estadística e Investigación Operativa
Escuela Politécnica Superior de Algeciras
Av. Ramón Puyol s/n, 11202 Algeciras, Spain

e-mail: dorota.kuziak@uca.es

and

Ismael G. Yero

Universidad de Cádiz
Departamento de Matemáticas
Escuela Politécnica Superior de Algeciras
Av. Ramón Puyol s/n, 11202 Algeciras, Spain

e-mail: ismael.gonzalez@uca.es

Abstract

A Roman dominating function on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The Roman dominating function f is an outer-independent Roman dominating function on G if the set of vertices labeled with zero under f is an independent set. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight $w(f) = \sum_{v\in V(G)} f(v)$ of any outer-independent Roman dominating function f of G. A vertex cover of a graph G is a set of vertices that covers all the edges of G. The minimum cardinality of a vertex cover is denoted by $\alpha(G)$. A graph G is a vertex cover Roman graph if $\gamma_{oiR}(G) = 2\alpha(G)$. A constructive characterization of the vertex cover Roman trees is given in this article.

Keywords: Roman domination, outer-independent Roman domination, vertex cover, vertex independence, trees.

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1. Introduction

Throughout this work we consider $G = (V, E)$ as a simple graph of order $n = |V|$. That is, a graphs that is finite, undirected, and without loops or multiple edges. Given a vertex v of G, $N_G(v)$ represents the *open neighborhood* of v, i.e., the set of all neighbors of v in G, and the *degree* of v is $d(v) = |N_G(v)|$. If $S \subset V(G)$, then the *open neighborhood* of S is $N_G(S) = \bigcup_{v \in S} N_G(v)$. Whenever it is no confusion, we shall skip the subindex G in the notations above. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any two vertices u and v, the *distance* $d(u, v)$ between u and v is the length of a shortest $u - v$ path.

A *leaf vertex* of G is a vertex of degree one. A *support vertex* of G is a vertex adjacent to a leaf; a *weak support vertex* is a support vertex adjacent to exactly one leaf; a *strong support vertex* is a support vertex that is not a weak support; a *strong leaf vertex* is a leaf vertex adjacent to a strong support vertex; and a *semi-support vertex* is a vertex adjacent to a support vertex that is not a leaf. The set of leaves is denoted by $L(G)$; the set of support vertices is denoted by $S(G)$; the set of weak support vertices is denoted by $S_w(G)$; the set of strong support vertices is denoted by $S_s(G)$; the set of strong leaves is denoted by $L_s(G)$; and the set of semi-support vertices is denoted by $SS(G)$.

A set S of vertices is *independent* if S induces an edgeless graph. An independent set of maximum cardinality is a *maximum independent set* of G. The *independence number* of G is the cardinality of a maximum independent set of G and is denoted by $\beta(G)$. An independent set of cardinality $\beta(G)$ is called a $\beta(G)$-*set*. A *vertex cover* of G is a set of vertices S that covers all the edges, i.e., every edge is incident with a vertex of S. The minimum cardinality of a vertex cover is denoted by $\alpha(G)$. A vertex cover of cardinality $\alpha(G)$ is called an $\alpha(G)$-*set*.

A *dominating set* of a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. The literature on the subject of domination in graphs up to the year 1997 has been surveyed and detailed in the two books [4, 5].

A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Notice that f generates three sets V_0, V_1 and V_2 such that $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. In this sense, from now on we will write $f = (V_0, V_1, V_2)$ so as to refer to the Roman dominating function f. The *weight* of an RDF is the value $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u) = |V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. A vertex $v \in V_2$ is said to have a *private neighbor* if there exists a vertex $w \in N(v) \cap V_0$ for which $N(w) \cap (V_1 \cup V_2) = \{v\}$. Roman domination in graphs was formally defined by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [2] motivated, in part, by an article in *Scientific American* of Ian Stewart entitled "Defend the Roman Empire" [9].

Once the seminal article [2] appeared, the topic immediately attracted the attention of several researchers, which has made that Roman domination in graphs is nowadays very well studied. Clearly, Roman domination is strongly related to domination in graphs. Thus, a relatively straightforward relationship (see [2]) states that for any graph G, $\gamma(G) \le \gamma_R(G) \le 2\gamma(G)$. The particular case of graphs satisfying the equality $\gamma_R(G) = 2\gamma(G)$ motivated the definition of the so called *Roman graphs*, i.e., graphs G for which $\gamma_R(G) = 2\gamma(G)$. An open problem concerning characterizing all the Roman graphs still remains open although some contributions to this topic are already known. Perhaps, the most remarkable contribution in this direction appeared in [6], where all the Roman trees were characterized.

An RDF is an *outer-independent Roman dominating function* (OIRDF) on G if V_0 is an independent set. The *outer-independent Roman domination number* $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. An OIRDF with weight $\gamma_{oiR}(G)$ is called a $\gamma_{oiR}(G)$-function. The concepts above were introduced and studied in [1]. In such work was proved that for any graph G, $\alpha(G) + 1 \le \gamma_{oiR}(G) \le 2\alpha(G)$ and those graphs achieving the equality in the upper bound were called vertex cover Roman graphs (VC-Roman graphs for short). Hence, an open problem was then raised up. That was, characterizing all the VC-Roman graphs. In this sense, and following with the traditions of Roman trees and some other works in the same style, in this work, we give a characterization of VC-Roman trees.

In connection with this, we make the following remark commented by a referee of this work, and we cite exactly his/her words: "*By the definition of vertex cover sets, all the vertices outside the vertex cover set form an independent set too. Thus, if we add the property of vertex cover on a Roman dominating function, then it is natural to consider an outer-independent Roman dominating function. In* 1998, *the paper* Characterization of graphs with equal domination and covering number by *Randerath and Volkmann* (*see* [8]) *showed a related result.*

By combining the results of [6]*, I conjecture that this problem is already solved by results of the mentioned two papers*". In one direction this is true, but as we next show, the contrary direction is not true.

In [8], the graphs G of minimum degree one for which $\alpha(G) = \gamma(G)$ were characterized. Also, in [6], the trees T for which $\gamma_R(T) = 2\gamma(T)$ were characterized. A combination of both properties, for a tree T, means that $\gamma_R(T) = 2\alpha(T)$. Now, since $\gamma_R(G) \leq \gamma_{oiR}(G)$ and $\gamma_{oiR}(G) \leq 2\alpha(G)$ are satisfied for any graph G, we can deduce that $2\alpha(T) = \gamma_R(T) \leq \gamma_{oiR}(T) \leq 2\alpha(T)$. Thus, there must be equalities in the chain of inequalities above, and therefore $\gamma_{oiR}(T) = 2\alpha(T)$, or equivalently, T is a VC-Roman tree (notice that this is satisfied in general for any Roman graph of minimum degree one). Now, for the contrary, if we assume that a tree T is a VC-Roman tree ($\gamma_{oiR}(T) = 2\alpha(T)$), then this does not mean T is a Roman tree for which $\alpha(T) = \gamma(T)$. As an example, we can observe the VC-Roman tree T in Figure 1 for which $\alpha(T) = \gamma(T) = 3$, $\gamma_{oiR}(T) = 6$ and $\gamma_R(T) = 5$.

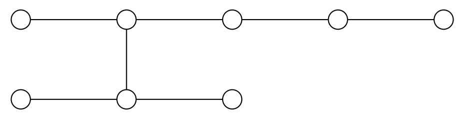

Figure 1. A VC-Roman tree T for which $\gamma_{oiR}(T) = 6$, $\alpha(T) = \gamma(T) = 3$ and $\gamma_R(T) = 5$.

Consequently, we observe that the trees belonging to the intersection family of the families given in [6] and [8] is a subfamily of the family of trees which we construct in our work.

2. Results

The next theoretical characterization for VC-Roman graph was given in [1]. However, such characterization lacks of usefulness, since it is precisely based on finding a $\gamma_{oiR}(G)$-function.

Proposition 1 [1]**.** *A graph G is a VC-Roman graph if and only if it has a $\gamma_{oiR}(G)$-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

The following well-known result, due to Gallai [3], states the relationship between the independence number and the vertex cover number of a graph.

Theorem 2 (Gallai, [3])**.** *A vertex set S of a graph G is independent if and only if the set $V(G) \setminus S$ is a vertex cover. Moreover, $\alpha(G) + \beta(G) = |V(G)|$.*

By the definition of VC-Roman graphs, Proposition 1 and Theorem 2, the next results follow.

Proposition 3. *Let G be a VC-Roman graph and let $f = (V_0, \emptyset, V_2)$ be a $\gamma_{oiR}(G)$-function. Then*

(i) *V_0 is a $\beta(G)$-set.*

(ii) *V_2 is an $\alpha(G)$-set.*

(iii) *Every vertex in V_2 has a private neighbor.*

Proof. First notice that $2|V_2| = \gamma_{oiR}(G) = 2\alpha(G)$ and so, $|V_2| = \alpha(G)$. Moreover, by Theorem 2, it follows $|V_0| = \beta(G)$, which completes the proof of (i) and (ii). On the other hand, let v be a vertex belonging to V_2. By definition, $N(v) \cap V_0 \neq \emptyset$. Now, suppose that v does not have a private neighbor. Hence, every vertex $w \in N(v) \cap V_0$ satisfies that $\{v\} \subsetneq N(w) \cap V_2$. Consider a function $f' = (V_0', V_1', V_2') = (V_0, \{v\}, V_2 \setminus \{v\})$. Since $V_0' = V_0$, we observe that f' is an OIRDF on G and satisfies that $w(f') < w(f)$, a contradiction. Thus, every vertex in V_2 has a private neighbor, and the proof of (iii) is complete. ∎

One consequence of the proposition above is the next theorem, which will further play an important role.

Theorem 4. *Let G be a VC-Roman graph and let $f = (V_0, \emptyset, V_2)$ be a $\gamma_{oiR}(G)$-function. Then $V(G) = L(G) \cup S(G) \cup SS(G)$, where $V_0 = L(G) \cup SS(G)$ and $V_2 = S(G)$.*

Proof. We first note that Proposition 3(iii) implies that every vertex belonging to V_2 is a support vertex of G since every vertex of V_2 has a private neighbor $x \in V_0$ which has no neighbor in V_0. Thus, x must be a leaf and so, $V_2 \subset S(G)$. Now, suppose that there exists a support vertex u belonging to V_0. As V_0 is an independent set, the leaf w adjacent to u belongs to V_2, but then w does not have a private neighbor, which is a contradiction with Proposition 3(iii). Thus, every support belongs to V_2 and therefore, $V_2 = S(G)$. Also, the independent set V_0 satisfies that $V_0 \subset N(V_2)$, which means that $V_0 = L(G) \cup SS(G)$ and, in consequence, that $V(G) = L(G) \cup S(G) \cup SS(G)$, which ends the proof. ∎

Corollary 5. *If G is a VC-Roman graph, then $\delta(G) = 1$.*

We next continue with some other extra properties of VC-Roman graphs which will further on be useful.

Proposition 6. *Let G be a VC-Roman graph with $SS(G) = \emptyset$. Then every support vertex is a strong support vertex.*

Proof. Let $f = (V_0, \emptyset, V_2)$ be a $\gamma_{oiR}(G)$-function. By Theorem 4 we have $f = (L(G), \emptyset, S(G))$. Suppose there exists a support vertex s satisfying $|N(s) \cap L(G)| = \{h\}$. Now, it is readily seen that $f' = ((L(G) \setminus \{h\}) \cup \{s\}, \{h\}, S(G) \setminus \{s\})$ is an OIDRF on G of weight less than $w(f)$, a contradiction, since f is a $\gamma_{oiR}(G)$-function. Thus, every support vertex is a strong support vertex. ∎

Proposition 7. *Let G be a graph containing a strong support vertex v. Then there exists a $\gamma_{oiR}(G)$-function f satisfying $f(v) = 2$.*

Proof. Let $v \in S_s(G)$ and let $h_1, h_2 \in N(v) \cap L(G)$. Let f be a $\gamma_{oiR}(G)$-function satisfying $f(v) \neq 2$. First, we note that $f(v) \neq 1$, (otherwise $f(v) = 1$ implies that $f(h_1) = f(h_2) = 1$ and the function g satisfying that for every $u \in V(G) \setminus \{v, h_1, h_2\}$, $g(u) = f(u)$, $g(v) = 2, g(h_1) = g(h_2) = 0$ is an OIRDF with weight less that f, which is a contradiction). Thus, it must happen $f(v) = 0$, which implies $f(h_1) = f(h_2) = 1$. Now, by considering a function as that g defined above, we can clearly note that g is an OIRDF with the same weight as f. Therefore, g is a $\gamma_{oiR}(G)$-function that satisfies the necessary requirements. ■

Corollary 8. *Let G be a graph containing a strong leaf vertex v. Then there exists a $\gamma_{oiR}(G)$-function f satisfying $f(v) = 0$.*

Observation 9. *Let T be a tree where $V(T) = L(T) \cup S(T) \cup SS(T)$ and $L(T) \cup SS(T)$ is an independent set. Then*

(i) *$L(T) \cup SS(T)$ is a $\beta(T)$-set.*

(ii) *$S(T)$ is an $\alpha(T)$-set.*

(iii) *$f = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF.*

In order to present our characterization we need the following definitions. A *near outer-independent Roman dominating function*, abbreviated near-OIRDF, of a graph G, relative to a vertex v, is a function $f = (V_0, V_1, V_2)$ satisfying the following.

(i) $v \in V_0$.

(ii) V_0 is an independent set.

(iii) Every vertex $u \in V_0 \setminus \{v\}$ is adjacent to at least one vertex in V_2.

The weight of a near-OIRDF of G relative to v is the value $f(V(G)) = \sum_{u \in V} f(u)$. The minimum weight of a near-OIRDF on G relative to v is called the *near outer-independent Roman domination number of G relative to v*, which we denote as $\gamma^n_{oiR}(G; v)$. Notice that, for every vertex v of G we have $\gamma_{oiR}(G) \leq \gamma^n_{oiR}(G; v) + 1$. We now define a vertex v to be a *near stable vertex* in G, if $\gamma_{oiR}(G) \leq \gamma^n_{oiR}(G; v)$. In this sense, the set of *near stable weak support vertices* of G is denoted by $S^{ns}_w(G)$. For example, every weak support of a path P_5 is a near stable weak support vertex. We remark that the terminology of "near" style parameters and "near stable" vertices with respect to a parameter is a very well known and commonly used technique in domination theory. In order to simply mention a recently published example where this was used, we can for instance refer to [7].

With all the tools presented till now, we are then able to begin with the characterization of the family of VC-Roman trees. To this end, we need the following operations F_1, F_2, F_3, F_4 and F_5 on a tree T (by *attaching* a path P to a vertex v of T we mean adding the path P and joining v to a vertex of P). Also, we assume that $|S(T)| \geq 1$, since the case $|S(T)| = 0$ (when T is a path P_2 and T is a VC-Roman tree) is straightforward.

Operation F_1. Attach a path P_1 to a vertex $v \in S(T)$.

Operation F_2. Attach a path P_2 to a vertex $v \in L_s(T)$.

Operation F_3. Attach a path P_3 to a vertex $v \in SS(T)$, by joining v to the support vertex of P_3.

Operation F_4. Attach a path P_3 to a vertex $v \in S_s(T) \cup S_w^{ns}(T)$, by joining v to the support vertex of P_3.

Operation F_5. Attach a path P_5 to a vertex $v \in S_s(T) \cup S_w^{ns}(T)$, by joining v to the semi-support vertex of P_5.

Let $\mathcal{F}$ be the family of trees defined as $\mathcal{F} = \{T \mid T = P_3$ or T is obtained from P_3 by a finite sequence of the operations F_1, F_2, F_3, F_4 or $F_5\}$. We first show that every tree of the family $\mathcal{F}$ is a VC-Roman tree.

Lemma 10. *If $T \in \mathcal{F}$, then T is a VC-Roman tree.*

Proof. We proceed by induction on the number $r(T)$ of operations required to construct the tree T. If $r(T) = 0$, then $T = P_3$ is a VC-Roman tree. This establishes the base case. Hence, we now assume that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{F}$ with $r(T') < k$ satisfies that T' is a VC-Roman tree.

Let $T \in \mathcal{F}$ be a tree with $r(T) = k$. Then T can be obtained from a tree $T' \in \mathcal{F}$ with $r(T') = k - 1$ by one of the operations F_1, F_2, F_3, F_4 or F_5. We shall prove that T is a VC-Roman tree. To this end, and using Theorem 4, we consider the $\gamma_{oiR}(T')$-function $f' = (L(T') \cup SS(T'), \emptyset, S(T'))$ (notice that such f' exists because T' is a VC-Roman tree). We consider the following situations.

Case 1. T is obtained from T' by operation F_1. Assume T is obtained from T' by adding the vertex u and the edge uv where $v \in S(T')$. Notice that u is a leaf of T. By using Observation 9 we see that the function $f = (L(T') \cup SS(T') \cup \{u\}, \emptyset, S(T')) = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF on T with weight $w(f) = \gamma_{oiR}(T')$. So, $\gamma_{oiR}(T) \leq w(f) = \gamma_{oiR}(T')$. Now, since $u \in L(T)$ and $v \in S(T')$, it follows $v \in S_s(T)$. So, by Proposition 7 there exists a $\gamma_{oiR}(T)$-function g such that $g(v) = 2$ and for every leaf h adjacent to v, $g(h) = 0$. Thus, $\gamma_{oiR}(T) = w(g) = g(V(T')) + g(u) = g(V(T'))$. Note also that g restricted to $V(T')$ is an OIRDF on T', which leads to $\gamma_{oiR}(T') \leq g(V(T')) = \gamma_{oiR}(T)$. Thus, we get $\gamma_{oiR}(T) = \gamma_{oiR}(T')$. On the other hand, it is easy to see that $\alpha(T) = \alpha(T')$ and by using the hypothesis $\gamma_{oiR}(T') = 2\alpha(T')$ (because T' is a VC-Roman tree), we deduce $\gamma_{oiR}(T) = 2\alpha(T)$ and T is a VC-Roman tree.

Case 2. T is obtained from T' by operation F_2. Assume T is obtained from T' by adding the path u_1u_2 and the edge u_1v where $v \in L_s(T')$. Notice that $u_1 \in S(T)$ and $u_2 \in L(T)$, and let $u \in S_s(T') \cap N(v)$. By using Observation 9, we see that the function $f = (L(T') \cup SS(T') \cup \{u_2\}, \emptyset, S(T') \cup \{u_1\}) = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF on T with weight $w(f) = \gamma_{oiR}(T') + 2$ and so, $\gamma_{oiR}(T) \leq w(f) = \gamma_{oiR}(T') + 2$. On the other hand, let g be a $\gamma_{oiR}(T)$-function such that the number of vertices labeled with one under g is minimum. Now consider the function g restricted to $V(T')$, say g'. Suppose g' is not an OIRDF on T'. Hence, this can only happen when $g'(v) = 0$ and $g'(u) \neq 2$. Thus, it must be $g(u_1) = 2$ and $g'(u) = 1$, which also leads to $g'(u') = 1$ for any leaf $u' \in N(u) \setminus \{v\}$ (note that at least one of such leaves exists because $u \in S_s(T')$). So, we can redefine g by making $g(u) = 2$ and $g(u') = 0$ and obtain a $\gamma_{oiR}(T)$-function with a smaller number of vertices labeled with one under g, which is a contradiction. Thus, g' is an OIRDF on T' and so, $g(V(T')) = g'(V(T')) = w(g') \geq \gamma_{oiR}(T')$. Moreover, we observe that $1 \leq g(u_1) + g(u_2) \leq 2$. If $g(u_1) + g(u_2) = 1$, then this can only occur when $g(u_1) = 0$ and $g(u_2) = 1$, which leads to $g(v) = 2$ and $g(u)$ can take any value. In such case, we can again redefine g by making $g(u) = 2$, $g(v) = g(u') = 0$, $g(u_1) = 2$ and $g(u_2) = 0$ and obtain a new function g'' which satisfies one of the following situations.

- g'' has weight smaller than g (if $g(u) \neq 0$ and u has only one leaf neighbor), and this is not possible.
- g'' has the same weight as g (if $g(u) = 0$ or u has more than one leaf neighbor), but a smaller number of vertices labeled with one under g'' than g, and this is a contradiction with the choice of g.

Thus, the only possibility is that $g(u_1) + g(u_2) = 2$. So, we obtain $\gamma_{oiR}(T) = w(g) = g(V(T')) + g(u_1) + g(u_2) \geq \gamma_{oiR}(T') + 2$, and consequently, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$.

By using again Observation 9, Proposition 3 and Theorem 4 we see that $\alpha(T) = |S(T)| = |S(T')| + 1 = \alpha(T') + 1$. By hypothesis we know $\gamma_{oiR}(T') = 2\alpha(T')$ (because T' is a VC-Roman tree). Therefore, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2 = 2\alpha(T') + 2 = 2(\alpha(T) - 1) + 2 = 2\alpha(T)$ and T is a VC-Roman tree.

Case 3. T is obtained from T' by operation F_3. Assume T is obtained from T' by adding the path $u_1u_2u_3$ and the edge u_2v where $v \in SS(T')$. Notice that $u_2 \in S_s(T)$ and $u_1, u_3 \in L_s(T)$. By Observation 9 we get that the function $f = (L(T') \cup SS(T') \cup \{u_1, u_3\}, \emptyset, S(T') \cup \{u_2\}) = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF on T with weight $w(f) = \gamma_{oiR}(T') + 2$ and so, $\gamma_{oiR}(T) \leq w(f) = \gamma_{oiR}(T') + 2$. On the other hand, based on Proposition 7 and Corollary 8, we consider a $\gamma_{oiR}(T)$-function g satisfying that $g(u_2) = 2$ and $g(u_1) = g(u_3) = 0$, and such that number of vertices labeled with one is minimum. Again, we consider the function g restricted to $V(T')$, say g'. If g' is not an OIRDF on T', then this can only happen when $g(v) = 0$ and all its neighbors in T' have labels different from two. Let u be a support adjacent to v and let u' be a leaf adjacent to u. It must clearly happen that $g(u) = 1$ (it cannot be $g(u) = 0$ because $g(v) = 0$) and $g(u') = 1$. Thus, by a similar reasoning as in some cases above, we redefine g by making $g(u) = 2$ and $g(u') = 0$, which is a contradiction with the choice of g, since we obtain a function with a smaller number of vertices labeled with one. Thus, g' is an OIRDF on T' and so, $g(V(T')) = g'(V(T')) = w(g') \geq \gamma_{oiR}(T')$. Moreover, we see that $g(u_1) + g(u_2) + g(u_3) \geq 2$. Therefore, $\gamma_{oiR}(T) = w(g) = g(V(T')) + g(u_1) + g(u_2) + g(u_3) \geq \gamma_{oiR}(T') + 2$ and, as a consequence, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$.

Again, by Observation 9, Proposition 3 and Theorem 4 we get $\alpha(T) = |S(T)| = |S(T')| + 1 = \alpha(T') + 1$. It is known by hypothesis that $\gamma_{oiR}(T') = 2\alpha(T')$ (because T' is a VC-Roman tree). Therefore, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2 = 2\alpha(T') + 2 = 2(\alpha(T) - 1) + 2 = 2\alpha(T)$ and T is a VC-Roman tree.

Case 4. T is obtained from T' by operation F_4. Assume T is obtained from T' by adding the path $u_1u_2u_3$ and the edge u_2v where $v \in S_s(T') \cup S_w^{ns}(T')$. Also, let w be a leaf-neighbor to v. Notice that $u_2 \in S_s(T)$ and $u_1, u_3 \in L_s(T)$. By using Observation 9, it is readily seen that the function $f = (L(T') \cup SS(T') \cup \{u_1, u_3\}, \emptyset, S(T') \cup \{u_2\}) = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF on T with weight $w(f) = \gamma_{oiR}(T') + 2$ and so, $\gamma_{oiR}(T) \leq w(f) = \gamma_{oiR}(T') + 2$. Now, we consider the next two cases.

Case 4.1. $v \in S_s(T')$. In concordance with Proposition 7 and Corollary 8, we consider a $\gamma_{oiR}(T)$-function g satisfying that $g(v) = g(u_2) = 2$ and $g(u_1) = g(u_3) = 0$. Now, we notice that the function g restricted to $V(T')$, say g', is an OIRDF on T'. Thus, it is satisfied that $\gamma_{oiR}(T') \leq g'(V(T')) = g(V(T'))$. Moreover, since $g(u_1) + g(u_2) + g(u_3) = 2$ we get $\gamma_{oiR}(T) = w(g) = g(V(T')) + g(u_1) + g(u_2) + g(u_3) \geq \gamma_{oiR}(T') + 2$ and, as a consequence, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$.

Case 4.2. $v \in S_w^{ns}(T')$. In concordance with Proposition 7 and Corollary 8, we consider a $\gamma_{oiR}(T)$-function h such that the number of vertices labeled with one under h is minimum and satisfying $h(u_2) = 2$, $h(u_1) = h(u_3) = 0$. If $h(v) = 2$, then, by using some similar procedure as in Case 4.1 we obtain $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$. Thus, we notice that $h(v) = 0$ (otherwise, if $h(v) = 1$, then the function f' defined by $f'(v) = 2$, $f'(w) = 0$ and $f'(x) = h(x)$ for every $x \in V(T) \setminus \{v, w\}$, is a $\gamma_{oiR}(T)$-function with a smaller number of vertices labeled with one under h, which is a contradiction). Now consider the function h restricted to $V(T')$, say h'. It is easy to see that h' is a near-OIRDF on T', and as $v \in S_w^{ns}(T')$, then, $\gamma_{oiR}(T') \leq \gamma_{oiR}^n(T'; v) \leq h'(V(T')) = h(V(T'))$. Moreover, since $h(u_1) + h(u_2) + h(u_3) = 2$, we get $\gamma_{oiR}(T) = w(h) = h(V(T')) + h(u_1) + h(u_2) + h(u_3) \geq \gamma_{oiR}(T') + 2$ and, as a consequence, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$.

In addition, for both subcases, by Observation 9, Proposition 3 and Theorem 4 we observe that $\alpha(T) = |S(T)| = |S(T')| + 1 = \alpha(T') + 1$. By hypothesis, we know that $\gamma_{oiR}(T') = 2\alpha(T')$ (because T' is a VC-Roman tree). Therefore, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2 = 2\alpha(T') + 2 = 2(\alpha(T) - 1) + 2 = 2\alpha(T)$ and T is a VC-Roman tree.

Case 5. T is obtained from T' by operation F_5. Assume T is obtained from T' by adding the path $u_1u_2u_3u_4u_5$ and the edge u_3v where $v \in S_s(T') \cup S_w^{ns}(T')$. Notice that $u_3 \in SS(T)$, $u_2, u_4 \in S(T)$ and $u_1, u_5 \in L(T)$. By Observation 9 we deduce that the function $f = (L(T') \cup SS(T') \cup \{u_1, u_3, u_5\}, \emptyset, S(T') \cup \{u_2, u_4\}) = (L(T) \cup SS(T), \emptyset, S(T))$ is an OIRDF on T with weight $w(f) = \gamma_{oiR}(T') + 4$ and so, $\gamma_{oiR}(T) \le w(f) = \gamma_{oiR}(T') + 4$. Now, we consider the following two cases.

Case 5.1. $v \in S_s(T')$. We consider a $\gamma_{oiR}(T)$-function g for which $g(v) = 2$ (this can be asserted based on the fact that $v \in S_s(T')$ together with Proposition 7 and Corollary 8). In concordance with this, we readily seen that g restricted to $V(T')$, say g', is an OIRDF on T', which means $g((V(T')) = g'(V(T')) \ge \gamma_{oiR}(T')$. Also, we see that $g(u_1) + g(u_2) + g(u_3) + g(u_4) + g(u_5) \ge 4$. Thus, $\gamma_{oiR}(T) = w(g) = g(V(T')) + g(u_1) + g(u_2) + g(u_3) + g(u_4) + g(u_5) \ge \gamma_{oiR}(T') + 4$, which allows to claim $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 4$.

Case 5.2. $v \in S_w^{ns}(T')$. Let z be a leaf neighbor of v. We consider a $\gamma_{oiR}(T)$-function h such that the number of vertices labeled with one under h is minimum. Again, we note that if $h(v) = 2$, then by using some similar procedure, as in Case 5.1, we obtain that $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 4$. On the other hand, we notice that $h(v) = 0$ (otherwise, if $h(v) = 1$, then the function h_1 defined by $h_1(v) = 2$, $h_1(z) = 0$ and $h_1(x) = h(x)$ for every $x \in V(T) \setminus \{v, z\}$, is a $\gamma_{oiR}(T)$-function with a smaller number of vertices labeled with one under h_1, which is a contradiction). Now consider the function h restricted to $V(T')$, say h'. It is easy to see that h' is a near-OIRDF on T'. As $v \in S_w^{ns}(T')$, we get $\gamma_{oiR}(T') \le \gamma_{oiR}^n(T'; v) \le h'(V(T')) = h(V(T'))$. Moreover, since $h(u_1) + h(u_2) + h(u_3) + h(u_4) + h(u_5) \ge 4$, we deduce $\gamma_{oiR}(T) = w(h) = h(V(T')) + h(u_1) + h(u_2) + h(u_3) + h(u_4) + h(u_5) \ge \gamma_{oiR}(T') + 4$ and, as a consequence, $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 4$.

Again, for both subcases, by Observation 9, Proposition 3 and Theorem 4 it follows $\alpha(T) = |S(T)| = |S(T')| + 2 = \alpha(T') + 2$. By hypothesis we know that $\gamma_{oiR}(T') = 2\alpha(T')$ (because T' is a VC-Roman tree), which leads to $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 4 = 2\alpha(T') + 4 = 2(\alpha(T) - 2) + 4 = 2\alpha(T)$ and therefore, T is a VC-Roman tree. ■

We now turn our attention to the opposite direction concerning the lemma above. In this sense, from now on we shall need the following terminology and notation in our results. Given a tree T and a set $S \subset V(T)$, by $T - S$ we denote a tree obtained from T by removing from T all the vertices in S and all the edges incident with a vertex in S (if $S = \{v\}$ for some vertex v, then we simply write $T - v$). For a vertex x of a tree T, a subtree T_x at x of a rooted tree T is the subtree induced by the descendants of x together with x (rooted tree and descendants are understood as it is common in the literature). Moreover, we denote by $P(x, y)$ the set of vertices of one shortest path between x and y, including x and y. We next show that every VC-Roman tree belongs to the family $\mathcal{F}$.

Lemma 11. *If T is a VC-Roman tree, then $T \in \mathcal{F}$.*

Proof. We proceed by induction on the order $n \geq 3$ of the VC-Roman tree T. If $n = 3$, then $T = P_3$ that belongs to $\mathcal{F}$. If $n = 4$, then T is either a path P_4 or a star S_3 with three leaves. Notice that P_4 is not a VC-Roman tree and that the star S_3 can be obtained from P_3 by applying operation F_1. More in general, if T is any star S_n, then T can be obtained from P_3 by repeatedly applying operation F_1. These facts establish the base case of the induction procedure. We assume next that $k > 4$ is an integer and that each VC-Roman tree T' with $|V(T')| < k$ satisfies $T' \in \mathcal{F}$.

Let T be a VC-Roman tree and $|V(T)| = k$. Then, by Proposition 1 and Theorem 4, there exists a $\gamma_{oiR}(T)$-function $f = (V_0, \emptyset, V_2)$, where $V_0 = L(T) \cup SS(T)$ (note that this implies that $SS(T)$ induces a subgraph without edges), $V_2 = S(T)$ and $V(T) = L(T) \cup S(T) \cup SS(T)$. We consider now several situations.

Case 1. $|S_s(T)| = 0$. Clearly, any support vertex is adjacent to exactly one leaf. Also, since $k > 4$, and by Proposition 6, $|SS(T)| > 0$. Let h, h' be two leaves at the maximum possible distance in T such that there is $v \in SS(T) \cap P(h, h')$ with $d(v, h) = 2$ or $d(v, h') = 2$. Without loss of generality assume that $d(v, h) = 2$. Let s be the support vertex adjacent to h, $P(h, h') \cap (N(v) \setminus \{s\}) = \{w\}$ (w is also a support vertex since v cannot have other kind of neighbor) and assume T is rooted at h'. We have now some possible scenarios.

Case 1.1. $|N(v)| = 2$. We first observe that $|N(s) \cap S(T)| = 0$. That is, if there exists $r \in N(s) \cap S(T)$ such that $N(r) \cap L(T) = \{h_r\}$, then the function $g = (((L(T) \cup SS(T)) \setminus \{h_r\}) \cup \{r\}, \{h_r\}, S(T) \setminus \{r\})$ is an OIRDF on T satisfying that $w(g) < w(f) = \gamma_{oiR}(T)$, which is a contradiction. Hence $|N(s)| = 2$, where $N(s) = \{v, h\}$ and $N(v) = \{s, w\}$. We consider the tree $T' = T - \{s, h\}$. In T', the vertex v is a strong leaf and the vertex w is a strong support. By using Proposition 7 and Corollary 8, we can deduce that the function f restricted to $V(T')$, say $f' = (V_0', V_1', V_2')$ is a $\gamma_{oiR}(T')$-function which has $V_1' = \emptyset$. Thus, by Proposition 1, T' is a VC-Roman tree and, by inductive hypothesis, $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_2, we get $T \in \mathcal{F}$.

Case 1.2. $|N(v)| = 3$. In this case, we first note that $N(v) \subset S(T)$. Let $N(v) = \{s, w, s_1\}$. As $|S_s(T)| = 0$, let $N(s_1) \cap L(T) = \{h_1\}$ and $N(w) \cap L(T) = \{h_w\}$. By the maximality of $P(h, h')$, if $|N(s_1)| > 2$, then every neighbor of s_1

other than v is a support. In such case, if there exists a neighbor of s_1 other than v, then we proceed with s_1 instead of s, to construct a function g as in Case 1.1, and obtain a contradiction. Thus, it follows that $|N(s_1)| = 2$.

Suppose now that $N(w) \cap (SS(T) \setminus \{v\}) = \emptyset$ and consider the function g for which $g(v) = 2$, $g(s) = g(s_1) = g(w) = 0$ and $g(h) = g(h_1) = g(h_w) = 1$ and $g(u) = f(u)$ for $u \in V(T) \setminus \{v, s, h, s_1, h_1, w, h_w\}$. It can be easily checked that g is an OIRDF on T and that $w(g) < w(f) = \gamma_{oiR}(T)$, a contradiction.

In this sense, we may assume $N(w) \cap (SS(T) \setminus \{v\}) \neq \emptyset$ and we consider the tree $T' = T - \{h, s, v, s_1, h_1\}$. In T', the vertex w is also a weak support vertex. Moreover, we claim that the $\gamma_{oiR}(T)$-function f restricted to $V(T')$, say f', is a $\gamma_{oiR}(T')$-function. It is clear that f' is an OIRDF on T'. We consider a $\gamma_{oiR}(T')$-function g' and suppose that $w(g') < w(f') = \gamma_{oiR}(T) - 4$. Now, we consider the function g on T, defined by $g(v) = 2$, $g(s) = g(s_1) = 0$, $g(h) = g(h_1) = 1$ and $g(x) = g'(x)$ for every $x \in V(T')$. We observe that g is an OIRDF on T satisfying that $w(g) = w(g') + 4 < w(f') + 4 = \gamma_{oiR}(T)$, a contradiction. Thus, $f' = (V_0', V_1', V_2')$ is a $\gamma_{oiR}(T')$-function with $V_1' = \emptyset$. So, T' is a VC-Roman tree, and by inductive hypothesis, $T' \in \mathcal{F}$.

On the other hand, let f'' be a $\gamma^n_{oiR}(T', w)$-function and we consider the function g'' on T, defined by $g''(v) = 2$, $g''(s) = g''(s_1) = 0$, $g''(h) = g''(h_1) = 1$ and $g''(x) = f''(x)$ for every $x \in V(T')$. We observe that g'' is an OIRDF on T, and so, we obtain $\gamma_{oiR}(T') + 4 = \gamma_{oiR}(T) \leq w(g'') = \gamma^n_{oiR}(T', w) + 4$. Thus, $\gamma_{oiR}(T') \leq \gamma^n_{oiR}(T', w)$, which means $w \in S^{ns}_w(T')$. Since T can be obtained from T' by operation F_5, we deduce that $T \in \mathcal{F}$.

Case 1.3. $|N(v)| > 3$. Let $N(v) = \{s, w, s_1, \ldots, s_r\}$ with $r \geq 2$. As $|S_s(T)| = 0$ and the neighbors of v are only support vertices, we assume $N(s_i) \cap L(T) = \{h_i\}$ for $1 \leq i \leq r$. By the maximality of $P(h, h')$, every neighbor of s_i other than v is a support with $1 \leq i \leq r$. If there exists a support neighbor of s_i for some i, other than v, then we proceed with s_i instead of s, to construct a function g as in Case 1.1, and obtain a contradiction. Thus, it follows that $|N(s_i)| = 2$ for $1 \leq i \leq r$. Now, we consider the function g for which $g(v) = 2$, $g(s) = g(s_i) = 0$ and $g(h) = g(h_i) = 1$ for $1 \leq i \leq r$ and $g(u) = f(u)$ for $u \in V(T) \setminus \{v, s, h, s_1, h_1, \ldots, s_r, h_r\}$. It can be easily checked that g is an OIRDF on T and that $w(g) < w(f) = \gamma_{oiR}(T)$, a contradiction again.

Case 2. $|S_s(T)| > 0$ and $2|S_s(T)| < |L_s(T)|$. Let v be a strong support vertex satisfying $|N(v) \cap L_s(T)| \geq 3$. Let $h \in N(v) \cap L_s(T)$ and $T' = T - h$. Since $f(v) = 2$ and $f(h) = 0$, we note that the $\gamma_{oiR}(T)$-function f restricted to $V(T')$ is an OIRDF on T' and so, $\gamma_{oiR}(T') \leq f(V(T')) = w(f) - f(h) = \gamma_{oiR}(T)$. Now, let $g = (V_0', V_1', V_2')$ be a $\gamma_{oiR}(T')$-function satisfying $g(v) = 2$, which can be claimed by Proposition 7, since v is a strong support vertex of T'. It can be checked that the function g' on T defined as $g'(h) = 0$ and $g'(x) = g(x)$ otherwise, is an OIRDF on T. Thus, $\gamma_{oiR}(T) \leq w(g') = \gamma_{oiR}(T')$ and, so $\gamma_{oiR}(T) = \gamma_{oiR}(T')$. It is then possible to deduce that g can be understood as the restriction of f to $V(T')$ (for which $V_1' = \emptyset$). Thus, by Proposition 1, it follows that T' is a VC-Roman tree. By inductive hypothesis, $T' \in \mathcal{F}$, and since T can be obtained from T' by operation F_1, we obtain that also $T \in \mathcal{F}$.

Case 3. $|S_s(T)| > 0$, $2|S_s(T)| = |L_s(T)|$ and $|SS(T)| = 0$. In this case, by Proposition 6, it follows that every support vertex is a strong support vertex. Let s, s' be two strong support vertices at the maximum possible distance in T. It is easy to see that $|N(s) \cap S_s(T)| = 1$ (since $|SS(T)| = 0$ by assumption) and $|N(s) \cap L(T)| = 2$. Let $N(s) \cap L(T) = \{h_1, h_2\}$ and consider the tree $T' = T - \{s, h_1, h_2\}$. By Proposition 7 and Corollary 8, we can assume $f(s) = 2$ and, in this sense, we can deduce that the function f restricted to $V(T')$, say $f' = (V_0', V_1', V_2')$ is a $\gamma_{oiR}(T')$-function which has $V_1' = \emptyset$. Thus, by Proposition 1, T' is a VC-Roman tree and, by inductive hypothesis, $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_4, we get $T \in \mathcal{F}$.

Case 4. $|S_s(T)| > 0$, $2|S_s(T)| = |L_s(T)|$ and $|SS(T)| > 0$. Let h, h' be two leaves at the maximum possible distance in T such that there is $v \in SS(T) \cap P(h, h')$ with $d(v, h) = 2$ or $d(v, h') = 2$. Without loss of generality assume that $d(v, h) = 2$. Let s be the support vertex adjacent to h and assume T is rooted at h'. Note that $N(v) \subset S(T)$ (since $SS(T) \subset V_0$) and $|N(v)| \geq 2$. We have now some possible scenarios.

Case 4.1. $|N(v)| = 2$ and $|N(s) \cap S(T)| \geq 1$. If there exists a vertex $r \in N(s) \cap S(T)$ such that $N(r) \cap L(T) = \{h_r\}$ (r is not a strong support), then the function $g = ((L(T) \backslash \{h_r\}) \cup \{r\}, \{h_r\}, S(T) \backslash \{r\})$ is an OIRDF on T satisfying that $w(g) < w(f) = \gamma_{oiR}(T)$, a contradiction. Thus, $N(s) \cap S(T) \subset S_s(T)$ (every support neighbor of s is a strong support). Now, this fact together with the maximality of $P(h, h')$ allows to claim that there is a subtree T_q, with $q \in N(s)$, which is a tree whose vertices are only strong support vertices: the vertex q itself together with other k ones, say $r_1, r_2, \ldots, r_k$, (notice that such vertices belong to $S_s(T)$) where $|N(r_k) \cap S(T_q)| \geq 1$, and leaves such that $N(r_k) \cap L(T_q) = \{h_{k_1}, h_{k_2}\}$ (since $2|S_s(T)| = |L_s(T)|$). Moreover, note that there is at least one of such strong supports, say r_j, such that $|N(r_j) \cap S(T_q)| = 1$.

If $k \geq 1$, then the strong support r_k is adjacent to another strong support (which could be the vertex q). Thus, by using a similar procedure as in Case 3 and, without loss of generality, assuming that r_k satisfies $|N(r_k) \cap S(T_q)| = 1$, we obtain that $T' = T - \{r_k, h_{k_1}, h_{k_2}\}$ is a VC-Roman tree. Thus, by the inductive hypothesis, $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_4, we get $T \in \mathcal{F}$.

On the other hand, assume that $k = 0$. Let $N(s) \cap S(T) = \{q\}$ and let $h_{q_1}, h_{q_2} \in N(q) \cap L(T)$. Let $T' = T - \{q, h_{q_1}, h_{q_2}\}$. We note that f restricted to $V(T')$, say $f' = (V_0', V_1', V_2')$, is an OIRDF on T', and so $\gamma_{oiR}(T') \leq w(f) -$

$(f(q) + f(h_{q_1}) + f(h_{q_2})) = \gamma_{oiR}(T) - 2$. Now, suppose that f' restricted to $V(T')$ is not a $\gamma_{oiR}(T')$-function. Let g be a $\gamma_{oiR}(T')$-function. It follows that $w(g) = \gamma_{oiR}(T') < f(V(T')) = \gamma_{oiR}(T) - 2$. Moreover, we consider the function g' such that $g'(x) = g(x)$ for every $x \in V(T')$ and $g'(q) = 2$, $g'(h_{q_1}) = g'(h_{q_2}) = 0$. It is easy to see that g' is an OIRDF on T satisfying that $w(g') = w(g) + 2 = \gamma_{oiR}(T')+2 < \gamma_{oiR}(T)$, a contradiction. Thus, as $V_1' \subset V_1 = \emptyset$, and by Proposition 1, it follows that T' is a VC-Roman tree. By inductive hypothesis $T' \in \mathcal{F}$.

Now, we may assume $s \in S_w(T')$ (otherwise, if $s \in S_s(T')$, then T can be obtained from T' by operation F_4, and we obtain $T \in \mathcal{F}$), and we consider a $\gamma^n_{oiR}(T'; s)$-function g. Notice that $g(s) = 0$ and $g(h) = g(v) = 1$. Now, we consider the function g' on T' defined by $g'(s) = 2$, $g'(h) = g'(v) = 0$ and $g'(x) = g(x)$ for every $x \in V(T') \setminus \{h, s, v\}$. We notice that g' is an OIRDF on T' and so, $\gamma_{oiR}(T') \leq \gamma^n_{oiR}(T'; s)$. Therefore, $s \in S^{ns}_w(T')$, and since T can be obtained from T' by operation F_4, we obtain that $T \in \mathcal{F}$.

Case 4.2. $|N(v)| = 2$, $|N(s) \cap S(T)| = 0$ and $s \in S_s(T)$. Hence, by the maximality of $P(h, h')$ and since $2|S_s(T)| = |L_s(T)|$, it must happen that $N(s) = \{h, h_1, v\}$ where h_1 is a strong leaf. Let $T' = T - \{h, h_1, s\}$. We note that f restricted to $V(T')$ is an OIRDF on T', and so $\gamma_{oiR}(T') \leq f(V(T')) = w(f) - (f(h) + f(h_1) + f(s)) = \gamma_{oiR}(T) - 2$ (according to the choice of f). Let g' be a $\gamma_{oiR}(T')$-function. Since v has degree two in T and is adjacent to a support vertex w other than s, the vertex v is a strong leaf and w is a strong support in T'. Hence, by Proposition 7 and Corollary 8, we may consider that $g'(w) = 2$ and $g'(v) = 0$. Let the function g on T be such that $g(x) = g'(x)$ for every $x \in V(T')$, $g(s) = 2$ and $g(h) = g(h_1) = 0$. It is easy to see that g is an OIRDF on T satisfying that $\gamma_{oiR}(T) \leq w(g) = w(g') + 2 = \gamma_{oiR}(T') + 2$, and so $\gamma_{oiR}(T') = \gamma_{oiR}(T) - 2$. Thus, the function $g' = (V_0', V_1', V_2')$, which is a $\gamma_{oiR}(T')$-function, can be understood as f restricted to $V(T')$. Consequently, as $V_1' \subset V_1 = \emptyset$, by Proposition 1, it follows that T' is a VC-Roman tree and, by inductive hypothesis $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_2 and F_1, we obtain $T \in \mathcal{F}$.

Case 4.3. $|N(v)| = 2$, $|N(s) \cap S(T)| = 0$ and $s \in S_w(T)$. Let $T' = T - \{s, h\}$ and let $f' = (V_0', V_1', V_2')$ be the restriction of f to $V(T')$. Notice that f' is an OIRDF, and so $\gamma_{oiR}(T') \leq w(f') = f(V(T')) = w(f) - (f(s)+f(h)) = \gamma_{oiR}(T) - 2$. Let s' be the other support vertex adjacent to v. It is not difficult to see that s' is a strong support vertex in T', since s' is a support in T, and also v becomes a leaf in T', which is also adjacent to s'. Also $v \in L_s(T')$. Suppose that f restricted to $V(T')$ is not an OIRDF of minimum weight on T'. By Proposition 7, there exists a $\gamma_{oiR}(T')$-function g' satisfying $g'(s') = 2$ and $g'(v) = 0$. Also, it is satisfied $w(g') < w(f') = f(V(T')) = \gamma_{oiR}(T) - 2$. Consider now the function g such that $g(x) = g'(x)$ for every $x \in V(T')$, $g(s) = 2$ and $g(h) = 0$. Thus, it is easy to see that g is an OIRDF on T satisfying that $w(g) = w(g') + 2 < f(V(T')) + 2 = \gamma_{oiR}(T)$, a contradiction. Therefore, f restricted to $V(T')$ is a $\gamma_{oiR}(T')$-function.

Since $V_1' \subset V_1 = \emptyset$, by Proposition, 1, it follows that T' is a VC-Roman tree. By inductive hypothesis, it is known that $T' \in \mathcal{F}$ and, since T can be obtained from T' by operation F_2, we deduce that $T \in \mathcal{F}$.

Case 4.4. $|N(v)| > 2$ and $|N(s) \cap S(T)| \geq 1$. We proceed analogously to Case 4.1 to obtain that $T \in \mathcal{F}$.

Case 4.5. $|N(v)| > 2$, $|N(s) \cap S(T)| = 0$ and $s \in S_s(T)$. Assume $N(s) \cap L(T) = \{h, h_1\}$ (since s is a strong support). Let $T' = T - \{s, h, h_1\}$. Notice that $v \in SS(T')$ since $v \in SS(T)$. Also, note that f restricted to $V(T')$ is an OIRDF on T', and so $\gamma_{oiR}(T') \leq f(V(T')) = w(f) - (f(s) + f(h) + f(h_1)) = \gamma_{oiR}(T) - 2$. Let g' be a $\gamma_{oiR}(T')$-function and consider the function g such that $g(x) = g'(x)$ for every $x \in V(T')$, $g(s) = 2$ and $g(h) = g(h_1) = 0$. We observe that g is an OIRDF on T satisfying that $\gamma_{oiR}(T) \leq w(g) = w(g') + 2 = \gamma_{oiR}(T') + 2$, which leads to $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 2$. Also, by Observation 9, we see that $\alpha(T') = \alpha(T) - 1$. From these above equalities and the fact that T is a VC-Roman tree, we get $\gamma_{oiR}(T') = 2\alpha(T')$, which implies that T' is a VC-Roman tree. By the inductive hypothesis $T' \in \mathcal{F}$. In addition, since T can be obtained from T' by operation F_3, we obtain that $T \in \mathcal{F}$.

Case 4.6. $|N(v)| > 2$, $|N(s) \cap S(T)| = 0$ and $s \in S_w(T)$. Note that $N(v) \subset S(T)$. We can consider that $N(v) = \{u, s, s_1, s_2, \ldots, s_r\}$ with $r \geq 1$ where $u, s \in P(h, h') \cap S(T)$ and for all $i \in \{1, 2, \ldots, r\}$, it follows that s_i is a support vertex with $N(s_i) = \{v, h_i\}$ and h_i is a leaf adjacent to s_i. Notice that, if there is one of such support vertices, say s_i, with $|N(s_i)| > 2$, then by the maximality of $P(h, h')$ it follows that $N(s_i) \subset S(T) \cup L(T)$, and by using some similar procedures as above we get the desired results. These are Case 4.4 if $|N(s_i) \cap S(T)| \geq 1$, or Case 4.5 if $|N(s_i) \cap S(T)| = 0$ and $s \in S_s(T)$. Thus, without loss of generality we can make the previous assumption concerning the degrees of the supports $s_1, \ldots, s_r$.

Since T is a VC-Roman tree, it must happen $r = 1$. Otherwise, if $r > 1$, then we consider a function g on T, satisfying $g(v) = 2$, $g(s) = g(s_1) = g(s_2) = \cdots = g(s_r) = 0$, $g(h) = g(h_1) = g(h_2) = \cdots = g(h_r) = 1$, and $g(x) = f(x)$ for every vertex $x \in V(T) \setminus \{v, s, s_1, h_1, \ldots, s_r, h_r\}$. Clearly, g is an OIRDF on T for which $w(g) < w(f) = \gamma_{oiR}(T)$, and this is a contradiction. Thus, $|N(v)| = \{u, s, s_1\}$ and let $T' = T - \{v, s, h, s_1, h_1\}$. We also notice that $u \in S(T')$ (since $u \in S(T)$), and that f restricted to $V(T')$ is an OIRDF on T'. So, $\gamma_{oiR}(T') \leq f(T') = w(f) - (f(v) + f(s) + f(h) + f(s_1) + f(h_1)) = \gamma_{oiR}(T) - 4$. Now, suppose $u \in S_s(T')$.

Let g' be a $\gamma_{oiR}(T')$-function. By Proposition 7, we know that $g'(u) = 2$. Consider the function g such that $g(x) = g'(x)$ for every $x \in V(T')$ and $g(v) = 0, g(s) = g(s_1) = 2$ and $g(h) = g(h_1) = 0$. It is easy to see that g is an OIRDF on T satisfying that $\gamma_{oiR}(T) \leq w(g) = w(g') + 4 = \gamma_{oiR}(T') + 4$. Thus $\gamma_{oiR}(T) = \gamma_{oiR}(T') + 4$. By Observation 9, we also get that $\alpha(T') = \alpha(T) - 2$.

Since $\gamma_{oiR}(T) = 2\alpha(T)$ (T is a VC-Roman tree), we deduce $\gamma_{oiR}(T') = 2\alpha(T')$, and this implies T' is a VC-Roman tree and by inductive hypothesis $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_5, we obtain that $T \in \mathcal{F}$.

Finally, suppose $u \in S_w(T')$, and consider a $\gamma^n_{oiR}(T'; u)$-function g'. Notice that in such case $g'(u) = 0$. Let g be a function on T defined by $g(v) = 2$, $g(s) = g(s_1) = 0$, $g(h) = g(h_1) = 1$ and $g(x) = g'(x)$ for every $x \in V(T) \setminus \{h, s, h_1, s_1, v\}$. We notice that g is an OIRDF on T, and so, $\gamma_{oiR}(T')+4 = \gamma_{oiR}(T) \le \gamma^n_{oiR}(T'; u)+ 4$. Thus, $\gamma_{oiR}(T') \le \gamma^n_{oiR}(T'; u)$, which implies that $u \in S^{ns}_w(T')$. Since T can be obtained from T' by operation F_5, we obtain that $T \in \mathcal{F}$, which completes all the cases of the proof. ■

As an immediate consequence of Lemmas 10 and 11, we have the desired characterization, which is the goal of this article.

Theorem 12. *Let T be a tree. Then T is a VC-Roman tree if and only if $T \in \mathcal{F}$.*

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