TOTAL 2-RAINBOW DOMINATION NUMBERS OF TREES

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Abstract

A 2-rainbow dominating function (2RDF) of a graph $G = (V(G), E(G))$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of $v$. A total 2-rainbow dominating function $f$ of a graph with no isolated vertices is a 2RDF with the additional condition that the subgraph of $G$ induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total 2-rainbow domination number, $\gamma_{tr_2}(G)$, is the minimum weight of a total 2-rainbow dominating function of $G$. In this paper, we establish some sharp upper and lower bounds on the total 2-rainbow domination number of a tree. Moreover, we show that the decision problem associated with $\gamma_{tr_2}(G)$ is NP-complete for bipartite and chordal graphs.

Keywords: 2-rainbow dominating function, 2-rainbow domination number, total 2-rainbow dominating function, total 2-rainbow domination number.

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1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V, E$) such that $G$ has no isolated vertices. The order of a graph $G$ is the number of vertices in $G$, denoted by $n = n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The maximum degree of a graph $G$ is denoted by $\Delta = \Delta(G)$. The open neighborhood of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N_G[S] = N[S] = N(S) \cup S$. The diameter of $G$, denoted by $\text{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of $G$. A leaf of a tree $T$ is a vertex of degree one, a support vertex is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. If $v$ is a support vertex, then $L(v)$ will denote the set of the leaves attached to $v$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$, $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. Also, the depth of $v$, $\text{depth}(v)$, is the maximum distance from $v$ to a vertex in $D(v)$. We denote by $T_v$ the induced subgraph of $T$ with vertex set $D[v]$. The independence number of a graph $G$, denoted $\alpha(G)$, is the order of a largest subset of vertices in which no two are adjacent. A vertex cover of $G$ is a set of vertices $S$ that covers all the edges, i.e., every edge is incident with a vertex of $S$. The vertex cover number $\beta(G)$ is the minimum cardinality of a vertex cover of $G$. It is well-known that for every graph $G$ of order $n$, $\beta(G) + \alpha(G) = n$.

A total Roman dominating function of a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$, and (ii) the subgraph of $G$ induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function $f$ is the value $w(f) = \sum_{u \in V(G)} f(u)$, and the total Roman domination number $\gamma_{tr}(G)$ is the minimum weight of a total Roman dominating function of $G$. The concept of total Roman domination in graphs was introduced by Liu and Chang [11] and studied for example in [2].

A $2$-rainbow dominating function (2RDF) of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled. The weight of a 2RDF $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$, and the minimum weight of a 2RDF is called the 2-rainbow domination number of $G$, denoted by $\gamma_{tr2}(G)$. The concept of 2-rainbow domination was introduced by Bresar et al. [6], and has been studied by several authors, for example [4, 5, 7, 8, 10, 12, 13].

A 2RDF $f$ is called a total 2-rainbow dominating function, or just T2RDF, if the subgraph of $G$ induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertices. The total 2-rainbow domination number, $\gamma_{tr2}(G)$, is the minimum weight of a total
2-rainbow dominating function of $G$, and a T2RDF of $G$ with weight $\gamma_{tr2}(G)$ is called a $\gamma_{tr2}(G)$-function. We note that if $f$ is a T2RDF of a graph $G$ and $H$ is a subgraph of $G$, then we denote the restriction of $f$ to $H$ by $f|_{V(H)}$. Total 2-rainbow domination was recently introduced by Abdollahzadeh Ahangar et al. in [1] and has been studied in [3].

Before presenting our main results, we present some straightforward observations.

**Observation 1.** If $v$ is a strong support vertex in a graph $G$, then there exists a $\gamma_{tr2}(G)$-function $f$ such that $f(v) = \{1, 2\}$.

**Observation 2.** If $u_1$ and $u_2$ are two adjacent support vertices in a graph $G$, then there exists a $\gamma_{tr2}(G)$-function $f$ such that $f(u_1) = f(u_2) = \{1, 2\}$.

**Observation 3.** If $v$ is a leaf neighbor of a support vertex of degree 2 in a graph $G$, then there exists a $\gamma_{tr2}(G)$-function $f$ such that $|f(v)| = 1$.

2. Lower Bounds

In this section, we establish some sharp lower bounds on the total 2-rainbow domination number of a tree. We begin by recalling the following result given in [1] for paths.

**Proposition 4.** For $n \geq 2$, $\gamma_{tr2}(P_n) = \left\lceil \frac{2n+2}{3} \right\rceil$.

Our first lower bound on $\gamma_{tr2}(T)$ is in terms of the order and the number of leaves of a tree $T$.

**Theorem 5.** Let $T$ be a non-trivial tree of order $n$ with $\ell(T)$ leaves. Then

$$\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))}{3} \right\rceil.$$ 

This bound is sharp for paths, stars and double stars.

**Proof.** We use an induction on $n$. It is easy to check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that for every non-trivial tree $T$ of order at most $n - 1$ the result is true. Let $T$ be a tree of order $n \geq 5$. If $T$ is a star, then $\gamma_{tr2}(T) = 3 = \left\lceil \frac{2(n+3-(n-1))}{3} \right\rceil$. If $T$ is a double star, then $\gamma_{tr2}(T) = 4 = \left\lceil \frac{2(n+3-(n-2))}{3} \right\rceil$. Henceforth we can assume that $T$ has diameter at least 4.

Suppose that $T$ has a strong support vertex $u$. Let $T' = T - u'$, where $u'$ is a leaf neighbor of $u$. By Observation 1, there exists a $\gamma_{tr2}(T)$-function $g$ such
that $g(u) = \{1, 2\}$. We may assume, without loss of generality, that $g(u') = \emptyset$. Then the function $g$, restricted to $T'$ is a T2RDF. We can apply the inductive hypothesis to the tree $T'$ and deduce that

$$
\gamma_{tr2}(T) = \omega(g) \geq \gamma_{tr2}(T') \geq \left\lceil \frac{2((n-1)+3-(\ell(T)-1))}{3} \right\rceil = \left\lceil \frac{2(n+3-\ell(T))}{3} \right\rceil.
$$

Therefore, from now on we suppose that $T$ has no strong support vertex.

Let $v_1v_2 \cdots v_k$ be a diametral path of rooted tree $T$ with root vertex $v_1$. Since $T$ has no strong support vertex, each child of $v_3$ is either a leaf or a support vertex of degree 2. Let $f$ be a $\gamma_{tr2}(T)$-function, and consider the following cases.

**Case 1.** $\text{deg}_T(v_3) = 3$. Assume first that $v_3$ is a support vertex. By Observation 2, we may assume that $f(v_2) = f(v_3) = \{1, 2\}$. Let $T' = T - v_1$ and define $h : V(T') \to P(\{1, 2\})$ by $h(v_2) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_2\}$. Clearly, $h$ is a T2RDF of $T'$. Using the fact that $n' = n - 1$ and $\ell(T') = \ell(T)$, it follows from the induction hypothesis that

$$
\gamma_{tr2}(T) = \omega(f) = \omega(h) + 1 \geq \gamma_{tr2}(T') + 1
$$

$$
\geq \left\lceil \frac{2((n-2)+3-(\ell(T)-1))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n+3-\ell(T))}{3} + 1 \right\rceil,
$$

as desired. Hence we assume that $v_3$ is not a support vertex, and thus every child of $v_3$ is a support vertex of degree 2. Let $u_2 \neq v_2$ be a child of $v_3$ and $u_1$ the leaf neighbor of $u_2$. Clearly, $|f(u_1)| + |f(u_2)| \geq 2$ and $|f(u_1)| + |f(v_2)| \geq 2$. Let $T' = T - \{u_1, u_2\}$ and define $h : V(T') \to P(\{1, 2\})$ by $h(v_3) = \{1\} \cup f(v_3)$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$. Clearly, $h$ is a T2RDF of $T'$, $n' = n - 2$ and $\ell(T') = \ell(T) - 1$. It follows from the induction hypothesis that

$$
\gamma_{tr2}(T) = \omega(f) \geq \omega(h) + 1 \geq \gamma_{tr2}(T') + 1
$$

$$
\geq \left\lceil \frac{2((n-2)+3-(\ell(T)-1))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n+3-\ell(T))}{3} + 1 \right\rceil,
$$

as desired.

**Case 2.** $\text{deg}_T(v_3) = 2$. As above we have $|f(u_1)| + |f(v_2)| \geq 2$. Suppose first that $|f(u_1)| + |f(v_2)| \geq 3$, and let $T' = T - v_1$. Then the function $h : V(T') \to P(\{1, 2\})$ defined by $h(v_3) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$ is a T2RDF of $T'$. By induction on $T'$ and using the fact that $n' = n - 1$, $\ell(T') = \ell(T)$, we obtain

$$
\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))}{3} + 1 \right\rceil,
$$

as desired. Therefore, we assume for the next that $|f(u_1)| + |f(v_2)| = 2$. Now, if $f(v_3) \neq \emptyset$, then the function $f$, restricted to $T - v_1$ is a T2RDF of $T - v_1$ of weight $\gamma_{tr2}(T) - 1$, and by the induction hypothesis on $T - v_1$ we obtain

$$
\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))}{3} + 1 \right\rceil.
$$
Hence let $f(v_3) = \emptyset$. Let $T' = T - \{v_1, v_2, v_3\}$ and recall that $T$ has diameter at least four. If $T'$ has order $n' = 2$, then $T = P_3$, and by Proposition 4 the result is valid. Hence let $n' \geq 3$. Then $f|_{V(T')} \in \mathcal{F}$ is a T2RDF of $T'$ of weight $\omega(f) - 2$. Using the fact that $n' = n - 3$ and $\ell(T') \leq \ell(T)$, and by applying the induction on $T'$, we obtain

$$\gamma_{tr2}(T) = \omega(f) = \omega(f|_{V(T')}) + 2 \geq \gamma_{tr2}(T') + 2 \geq \left\lceil \frac{2(n - 3) + 3 - \ell(T)}{3} \right\rceil + 2 = \left\lceil \frac{2(n + 3 - \ell(T))}{3} \right\rceil .$$

This completes the proof. $

**Theorem 6.** If $T$ is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then

$$\gamma_{tr2}(T) \geq \gamma_l(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta} \right\rceil ,$$

and this bound is sharp.

**Proof.** The proof is by induction on $n$. One can easily check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that the result is true for every non-trivial tree $T'$ of order $n'$, with $3 \leq n' < n$. Let $T$ be a tree of order $n$ with $\ell(T)$ leaves and $s(T)$ support vertices. If $\text{diam}(T) = 2$, then $T$ is a star, where $\gamma_{tr2}(T) = 3 = 2 + \left\lceil \frac{n - 4}{\Delta} \right\rceil$. If $\text{diam}(T) = 3$, then $T$ is a double star, where $4 = \gamma_{tr2}(T) \geq 2 + \left\lceil \frac{n - 4}{\Delta} \right\rceil$, and clearly the result is valid since $\left\lceil \frac{n - 4}{\Delta} \right\rceil \leq 2$. Henceforth we may assume that $\text{diam}(T) \geq 4$.

Let $v_1v_2 \cdots v_k$ be a diametral path of $T$ and $f$ be a $\gamma_{tr2}(T)$-function. Without loss of generality, we assume $\deg_T(v_2) \leq \deg_T(v_{k-1})$. Consider the following situations.

Suppose first that $v_3$ is a support vertex adjacent to another support vertex different from $v_2, v_4$ or $v_3$ is adjacent to a strong support vertex different from $v_2, v_4$. Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. Moreover, it is easy to see that $\gamma_l(T) \leq \gamma_l(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By the induction hypothesis on $T'$ we obtain that

$$\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 2 \geq \gamma_l(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2 \geq \gamma_l(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil .$$

Next, suppose that $v_3$ is not a support vertex and it is adjacent to a support vertex of degree two different from $v_2$. Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. On the other hand, if $\deg_T(v_2) \geq 3$, then $\gamma_l(T) \leq \gamma_l(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$, and if $\deg_T(v_2) = 2$, then
\( \gamma_t(T) \leq \gamma_t(T') + 1 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 1 \). Using the induction on \( T' \) and according to each situation, the result follows.

Suppose now that \( v_3 \) is a support vertex having no neighbor as support vertex besides \( v_2 \) and (possibly) \( v_4 \). If \( |f(x)| \geq 1 \) for some \( x \in N(v_3) - \{v_2\} \), then let \( T' = T - T_{v_3} \). Clearly, \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - |L(v_3)| \) and \( s(T') = s(T) - 1 \). Moreover, one can see that \( \gamma_t(T) \leq \gamma_t(T') + 1 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 2 \). By induction on \( T' \), we obtain as above \( \gamma_{tr^2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil - 1 \). Hence, we assume that \( f(x) = \emptyset \) for all \( x \in N(v_3) - \{v_2\} \). Thus \( f(v_3) = \{1, 2\} \). Since, \( f(v_4) = \emptyset \), we conclude that \( v_4 \) is not a support vertex and has no child of depth 1 which is a strong support vertex. Assume that \( \deg_T(v_4) \geq 3 \). If \( v_4 \) has a child of depth 1 say, \( u_2 \), with \( u_1 \) as a leaf neighbor of \( u_2 \), then let \( T' = T - \{u_1, u_2\} \).

Clearly, \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - 1 \) and \( s(T') = s(T) - 1 \). On the other hand, \( \gamma_t(T) \leq \gamma_t(T') + 2 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 2 \). By the induction hypothesis on \( T' \) we obtain that

\[
\gamma_{tr^2}(T) \geq \gamma_{tr^2}(T') + 2 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2
\]

\[
\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil + 2
\]

Therefore, we can assume that all children of \( v_4 \) have depth 2. According the diametral path and the situations already considered, we conclude that each child of \( v_4 \) is a support vertex or has degree 2. If \( z \) is a child of \( v_4 \) with degree 2 with \( z_1 \in N(z) - v_4 \), then let \( T' = T - T_z \). Clearly, \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - |L(z_1)| \) and \( s(T') = s(T) - 1 \). On the other hand, \( \gamma_t(T) \leq \gamma_t(T') + 2 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 3 \). Using the induction on \( T' \), we obtain desired result. Hence, each child of \( v_3 \) is a support vertex assigned \( \{1, 2\} \) under \( f \). Let \( T' = T - T_{v_3} \). Then \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - (|L(v_2)| + |L(v_3)|) \) and \( s(T') = s(T) - 2 \). On the other hand, \( \gamma_t(T) \leq \gamma_t(T') + 2 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 4 \). By the induction hypothesis on \( T' \) we obtain that

\[
\gamma_{tr^2}(T) \geq \gamma_{tr^2}(T') + 4 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 4
\]

\[
\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil + 4
\]

Now, let \( \deg_T(v_4) = 2 \) and \( T' = T - T_{v_4} \). Note \( T' \) has order \( n' \geq 1 \) since \( \text{diam}(T) \geq 4 \). It is a routine matter to check that the result holds if \( n' \in \{1, 2\} \). Hence let \( n' \geq 3 \). Then \( \Delta(T) \geq \Delta(T') \), \( \ell(T') \geq \ell(T) - (|L(v_2)| + |L(v_3)|) \) and \( s(T') \leq s(T) - 1 \). On the other hand, \( \gamma_t(T) \leq \gamma_t(T') + 2 \) and \( \gamma_{tr^2}(T') \leq \gamma_{tr^2}(T) - 4 \). Using the induction on \( T' \), the result follows.

Finally, assume that \( \deg_T(v_3) = 2 \). First, assume that \( f|_{T'} \) is a \( T2RDF \) of \( T' = T - T_{v_3} \). Recall that \( T \) has diameter at least four. If \( T' \) has order 2, then \( T \)
is obtained from a star of order at least three and a path $P_2$ by adding an edge joining their leaves, and clearly the result holds. So assume that $T'$ has order at least three. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - |L(v_2)|$ and $s(T') \leq s(T)$. Moreover, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$
\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 3 \geq \gamma_t(T') + \left\lceil \frac{(\ell(T') - s(T'))}{\Delta(T')} \right\rceil + 3
\geq \gamma_t(T) + \left\lceil \frac{(\ell(T) - s(T))}{\Delta(T)} \right\rceil + 1 \geq \gamma_t(T) + \left\lceil \frac{(\ell(T) - s(T))}{\Delta(T)} \right\rceil .
$$

Suppose now that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$. Hence, we have the following cases.

**Case 1.** $f(v_4) = \emptyset$. Then $v_4$ is not a support vertex and has no child of depth 1 which is a strong support vertex. Seeing the previous cases, it follows that any child of $v_4$ other than $v_3$ is either a support vertex of degree two or a vertex with depth 2 and degree 2. Moreover, since every child of $v_4$ is assigned a non-empty set, we conclude from our assumption that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$ and that $\deg_T(v_4) \in \{2, 3\}$. We consider the following.

**Subcase 1.1.** $\deg_T(v_4) = 3$. Observe that $T_{v_4}$ has exactly two support vertices, $v_2$ and $z$. We note that $z$ is a either at distance one or two from $v_4$. Let $T'' = T - T_{v_4}$. Clearly, $T''$ has order at least three, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(z)|)$, $s(T'') \leq s(T) - 1$ and $\gamma_t(T') \leq \gamma_t(T'') + 4$. Now, if $z$ is at distance one from $v_4$, then $|L(z)| = 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Also, if $z$ is at distance two from $v_4$, then $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 6$. Whatever the case, using the induction on $T''$, the result follows.

**Subcase 1.2.** $\deg_T(v_4) = 2$. Let $T'' = T - T_{v_4}$. It is easy to check the result if $n(T'') \in \{1, 2\}$. Hence let $n(T'') \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - |L(v_2)|$ and $s(T'') \leq s(T)$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on $T''$, the result follows.

**Case 2.** $|f(v_4)| \geq 1$ and thus $f(x) = \emptyset$ for each vertex $x \in N(v_4) - \{v_3\}$. Then every child of $v_4$ besides $v_3$ (if any) is leaf. To avoid the previous case when $f(v_4) = \emptyset$ we can assume that $v_4$ is a support vertex (else substitute the assignments of $v_4$ and $v_5$). Now if $f|_{T''}$ is a T2RDF of $T'' = T - T_{v_4}$, then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$, the result follows by using the induction on $T''$. Hence suppose that $f|_{T''}$ is not a T2RDF of $T'' = T - T_{v_4}$ and so $v_3$ has no child of depth 3 other than $v_4$. Since $f(v_5) = \emptyset$, we conclude that $v_5$ is not a support vertex and has no child of depth 1 which is a strong support vertex. Consider the following situations.
Subcase 2.1. $v_5$ has a child of depth 1. Let $u_2$ be such a child of depth 1 and $u_1$ its the leaf neighbor. Note that $\text{deg}_T(u_2) = 2$. Let $T'' = T - \{u_1, u_2\}$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$, the result follows by using the induction on $T''$.

Subcase 2.2. All children of $v_5$ different to $v_4$ have depth 2. Since $|f(x)| \leq 1$ for $x \in N(v_5) - \{v_4\}$, we deduce that every child of $v_5$ other than $v_4$ is not a support vertex. Let $z \neq v_4$ be a child of $v_5$. If $\text{deg}(z) = 2$ and $z' \in N(z) - \{v_5\}$, then let $T'' = T - T_z$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - |L(z')|$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on $T''$, the result follows. Hence suppose that $\text{deg}_T(z) \geq 3$. If $z$ has a child of depth 1 say, $u_2$, of degree two, with $u_1$ as the leaf neighbor of $u_2$, then let $T'' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$. Using the induction on $T''$, the result follows. Hence, all children of $z$ are strong support vertex. Let $|C(z)| = k$ and $x_1, \ldots, x_k$ be the children of $z$, and let $T'' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - \left(\sum_{i=1}^{k} |L(x_i)|\right)$ and $s(T'') = s(T) - k$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + k + 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2k - 1$. It follows from the induction hypothesis that

$$
\gamma_{tr2}(T) \geq \gamma_{tr2}(T'') + 2k + 1 \geq \gamma_t(T'') + \left[\frac{\ell(T'') - s(T'')}{\Delta(T'')}\right] + 2k + 1
$$

$$
\geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right] + k \geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right].
$$

Subcase 2.3. $\text{deg}_T(v_5) = 2$. Let $T'' = T - T_{v_5}$. Note that $T''$ may have order $n'' = 0$. However, it is easy to check that the result is valid for $n'' \leq 2$. Hence, let $n'' \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_1)|)$ and $s(T'') \leq s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Using the induction on $T''$, the result follows. This completes the proof.

Obviously, $\gamma_{tr2}(G) \leq \gamma_{tr}(G)$ for every graph $G$ without isolated vertices. In the following, we provide an upper bound on the ratio $\gamma_{tr}(G)/\gamma_{tr2}(G)$ for arbitrary graphs $G$. Moreover, this ratio will be slightly improved for the class of trees.

**Theorem 7.** If $G$ is a graph without isolated vertices, then $\gamma_{tr}(G) \leq \frac{3}{2}\gamma_{tr2}(G)$. This bound is sharp for the graph in Figure 1.

**Proof.** Let $f$ be a $\gamma_{tr2}(G)$-function. For every $i \in \{1, 2\}$, let $X_i$ be the set of all vertices $u$ for which $i \in f(u)$. Clearly, if a vertex of $G$ is assigned $\{1, 2\}$ under $f$, then $X_1 \cap X_2 \neq \emptyset$. Also, it is obvious that $|X_1| + |X_2| = \gamma_{tr2}(G)$. Now assume, without loss of generality, that $|X_1| \leq |X_2|$. Then $|X_1| \leq \frac{|X_1| + |X_2|}{2} = \frac{\gamma_{tr2}(G)}{2}$, and
the function \( g : V(G) \rightarrow \{0, 1, 2\} \) defined by \( g(x) = 0 \) if \( f(x) = 0 \), \( g(x) = 1 \) if \( f(x) = \{2\} \), and \( g(x) = 2 \) if \( 1 \in f(x) \), is a total Roman dominating function on \( G \), implying that

\[
\gamma_{tR}(G) \leq \omega(g) = 2|X_1| + |X_2| \leq \frac{|X_1| + |X_2|}{2} + |X_1| + |X_2| \leq \frac{3}{2} \gamma_{tr2}(G).
\]

**Theorem 8.** For every non-trivial tree \( T \),

\[
\gamma_{tR}(T) \leq \frac{3}{2} \gamma_{tr2}(T) - 1,
\]

and this bound is sharp for \( P_n \) such that \( n \equiv 2 \pmod{3} \).

**Proof.** The proof is by induction on \( n \). The statement is valid for all trees of order \( n \in \{2, 3, 4\} \). Let \( n \geq 5 \) and assume that for every tree \( T' \) of order at most \( n-1 \), \( \gamma_{tR}(T') \leq \frac{3}{2} \gamma_{tr2}(T') - 1 \). Let \( T \) be a tree of order \( n \). Since stars and double stars \( T \) satisfy \( \gamma_{tr2}(T) = 3 = \gamma_{tR}(T) \), the result holds. Therefore, we can assume that \( \text{diam}(T) \geq 4 \).

If \( T \) has a support vertex, say \( u \), with \( |L(u)| \geq 3 \), then let \( T' = T - u' \), where \( u' \) is a leaf neighbor of \( u \). Clearly \( \gamma_{tR}(T) \leq \gamma_{tR}(T') \). On the other hand, by Observation 1, there exists a \( \gamma_{tr2}(T) \)-function \( g \) such that \( g(u) = \{1, 2\} \). Also, we can assume that \( g(u') = \emptyset \). It follows that \( g|_{V(T')} \) is a T2RDF of \( T' \), and thus \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) \). By the inductive hypothesis on \( T' \), we obtain

\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') \leq 3\gamma_{tr2}(T') - 2 \leq 3\gamma_{tr2}(T) - 2.
\]

Hence we assume that every support vertex in \( T \) is adjacent to at most two leaves.

Let \( v_1v_2 \cdots v_k \) be a diametral path in \( T \) with root vertex \( v_k \). We consider the following cases.

**Case 1.** \( \deg_T(v_3) = 2 \). Let \( T' = T - T_{v_3} \). Then \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2 \). It follows from the induction hypothesis that

\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 6 \leq 3\gamma_{tr2}(T') + 4 \leq 3\gamma_{tr2}(T) - 2.
\]

**Case 2.** \( \deg_T(v_3) \geq 3 \). Consider the following subcases.
Subcase 2.1. Suppose that \( v_3 \) is a support vertex adjacent to another support vertex different from \( v_2 \) and \( v_4 \), or \( v_3 \) is adjacent to a strong support vertex different from \( v_2 \) and \( v_4 \). Let \( T' = T - T_{v_3} \). It is easy to see that \( \gamma_{tr}(T) \leq \gamma_{tr}(T') + 2 \) and \( \gamma_{tr_2}(T') \leq \gamma_{tr_2}(T) - 2 \). It follows from the induction hypothesis that

\[
2\gamma_{tr}(T) \leq 2\gamma_{tr}(T') + 4 \leq 3\gamma_{tr_2}(T') + 2 \leq 3\gamma_{tr_2}(T) - 12 + 6 < 3\gamma_{tr_2}(T) - 2.
\]

Subcase 2.2. \( v_3 \) is not a support vertex. Since \( \deg_T(v_3) \geq 3 \), every child of \( v_3 \) is a support vertex. Moreover, according to Subcase 2.1, all support vertices of \( T_{v_3} \), but possibly \( v_2 \), have degree two. Let \( t = \deg_T(v_3) - 1 \geq 2 \). Let \( T' = T - T_{v_3} \). It is easy to see that \( \gamma_{tr}(T) \leq \gamma_{tr}(T') + 2t + 1 \). Among all \( \gamma_{tr_2}(T) \)-functions, let \( g \) be one for which \( |g(v_3)| \) is as small as possible. Clearly, for every child \( x \) of \( v_3 \) we have \( |g(N[x])| \geq 2 \). Now, if \( g(v_3) = 0 \), then \( g|_{V(T')} \) is a \( T_2 \)-RDF of \( T' \) of weight \( \omega(g) - 2t \), and thus \( \gamma_{tr_2}(T') \leq \gamma_{tr_2}(T) - 2t \). Hence assume that \( g(v_3) \neq 0 \). The choice of \( g \) implies that \( |g(v_3)| = 1 \), and thus the weight of \( T_{v_3} \) under \( g \) is \( 2t + 1 \). The choice of \( g \) also implies that \( g(v_4) = 0 \). In that case, the function \( g' \) defined on \( V(T') \) defined by \( g'(v_4) = g(v_3) \) and \( g'(x) = g(x) \) for all \( x \in V(T') - \{v_4\} \) is a \( T_2 \)-RDF of \( T' \) of weight \( \gamma_{tr_2}(T) - 2t \), and thus \( \gamma_{tr_2}(T') \leq \gamma_{tr_2}(T) - 2t \). In all cases, it follows from the induction hypothesis on \( T' \) that

\[
2\gamma_{tr}(T) \leq 2\gamma_{tr}(T') + 2 + 4t \leq 3\gamma_{tr_2}(T') + 4t \leq 3\gamma_{tr_2}(T) - 6t + 4t < 3\gamma_{tr_2}(T) - 2.
\]

Subcase 2.3. \( v_3 \) is a support vertex adjacent to no support vertex besides \( v_2 \) and (possibly) \( v_4 \). Let \( f \) be a \( \gamma_{tr_2}(T) \)-function. If \( |f(v_4)| \geq 1 \) or there exists a vertex \( x \in N_T(v_4) - \{v_3\} \) with \( |f(x)| \geq 1 \), then let \( T' = T - T_{v_3} \). Obviously, \( \gamma_{tr}(T) \leq \gamma_{tr}(T') + 4 \) and \( \gamma_{tr_2}(T') \leq \gamma_{tr_2}(T) - 3 \). It follows from the induction hypothesis that

\[
2\gamma_{tr}(T) \leq 2\gamma_{tr}(T') + 8 \leq 3\gamma_{tr_2}(T') + 6 \leq 3\gamma_{tr_2}(T) - 9 + 6 < 3\gamma_{tr_2}(T) - 2.
\]

Hence we can assume that \( f(x) = 0 \) for each \( x \in N_T[v_4] - \{v_3\} \). Therefore, all children of \( v_4 \) have depth 2. According to Case 1 and the diametral path, we conclude that each child of \( v_4 \) is a support vertex. Since we assumed that \( f(x) = 0 \) for each \( x \in N_T[v_4] - \{v_3\} \), we deduce that \( d_T(v_4) = 2 \). In this case, let \( T' = T - T_{v_4} \). Recall that \( T \) has diameter at least four. Suppose that \( T' \) has order one. Clearly, \( T \) is a tree with three support vertices \( v_2, v_3, v_4 \) and the remaining vertices are leaves. Hence \( \gamma_{tr}(T) = \gamma_{tr}(T') = 6 \), and thus the result holds. So suppose that \( T' \) is nontrivial. Then \( \gamma_{tr}(T) \leq \gamma_{tr}(T') + 4 \) and \( \gamma_{tr_2}(T') \leq \gamma_{tr_2}(T) - 4 \). By induction on \( T' \) we deduce that

\[
2\gamma_{tr}(T) \leq 2\gamma_{tr}(T') + 8 \leq 3\gamma_{tr_2}(T') + 6 \leq 3\gamma_{tr_2}(T) - 12 + 6 < 3\gamma_{tr_2}(T) - 2.
\]

This completes the proof.

\[\blacksquare\]
3. Upper Bounds

In this section, we provide two upper bounds on the total 2-rainbow domination number of a tree. The first one we present is in terms of the order and the number of support vertices of a tree.

**Theorem 9.** If $T$ is a tree of order $n \geq 4$ with $s$ support vertices, then

$$\gamma_{tr2}(T) \leq \frac{2(n+s)}{3},$$

and this bound is sharp for $P_n$ such that $n \equiv 1 \pmod{3}$.

**Proof.** The proof is by induction on $n$. It is a routine matter to check that the statement holds if $n \in \{4, 5\}$. Hence, let $n \geq 6$ and assume that for every $T'$ or order $n' < n$ with $s'$ support vertices satisfies $\gamma_{tr2}(T') \leq \frac{2(n'+s')}{3}$. Let $T$ be a tree of order $n$. If $T$ is a star, then $\gamma_{tr2}(T) = 3 < \frac{2(n+1)}{3}$. Likewise, if $T$ is a double star, then $\gamma_{tr2}(T) = 4 < \frac{2(n+2)}{3}$. Henceforth we can assume $T$ has diameter at least four.

If $T$ has a strong support vertex $u$ adjacent to at least three leaves, then let $T' = T - u'$, where $u'$ is a leaf neighbor of $u$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to T2RDF of $T$ by assigning $\emptyset$ to vertex $u'$, and thus $\gamma_{tr2}(T) \leq \gamma_{tr2}(T')$. The result follows by using the induction on $T'$, with $n' = n - 1$ and $s' = s$. Therefore, we will assume that every support vertex of $T$ is adjacent to at most two leaves.

Let $v_1v_2 \cdots v_k$ be a diametral path in $T$ and root $T$ in $v_k$. We consider the following cases.

**Case 1.** $\deg_T(v_2) = 3$. Thus $v_2$ has two leaf neighbors. We distinguish between the following situations.

**Subcase 1.1.** $\deg_T(v_3) \geq 3$. Suppose first that $v_3$ is a support vertex. Let $T' = T - v_3$. Then $n' = n - 3$ and $s' = s - 1$. Let $f$ be a $\gamma_{tr2}(T')$-function. Since $v_3$ is a support vertex of $T'$, we must have $|f(v_3)| \geq 1$. Then the function $g : V(T) \rightarrow \mathcal{P}\{1, 2\}$ defined by $g(v_2) = \{1, 2\}$, $g(x) = \emptyset$ for $x \in L(v_2)$ and $g(x) = f(x)$ otherwise, is a T2RDF of $T$ of weight $\gamma_{tr2}(T') + 2$. By induction on $T'$, we have

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$

Suppose now that $v_3$ is not a support vertex. Thus every child of $v_3$ is a support vertex with degree either 2 or 3. Let $u_2$ be a child of $v_3$ different from $v_2$. If $\deg_T(u_2) = 3$, then let $T' = T - T_{v_2}$. By using a similar argument to that used above, we obtain $\gamma_{tr2}(T) < \frac{2(n+s)}{3}$. Thus let $\deg_T(u_2) = 2$ with $u_1$ as the unique
leaf of $u_2$. Let $T' = T - \{u_1, u_2\}$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of $T$ by assigning the set $\{1\}$ to both $u_1$ and $u_2$. Since $n' = n - 2$ and $s' = s - 1$, using the induction on $T'$ we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

\textbf{Subcase 1.2.} $\deg_T(v_3) = 2$. Recall that since $T$ has diameter at least four, $\deg_T(v_4) \geq 2$. Assume that $\deg_T(v_4) \geq 3$, and let $T' = T - T_{v_3}$. Observe that $T'$ has order $n' \geq 3$. If $n' = 3$, then $T$ is a tree of order 7 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = 6$. Hence we assume that $n' \geq 4$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of $T$ by assigning $\{1, 2\}$ to $v_2$, $\{1\}$ to $v_3$ and $\emptyset$ to the leaves of $L(v_2)$. By induction on $T'$ and using the fact that $n = n - 4$ and $s' = s - 1$ we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 4 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

So, suppose for the sequel that $\deg_T(v_4) = 2$. Let $T' = T - T_{v_3}$. Note that $n' \geq 3$. If $n' = 3$, then $T'$ has order 6 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = \frac{16}{3}$. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$-function $f$ such that $|f(v_4)| = 1$ and clearly such a function can be extended to a T2RDF of $T$ by assigning $\{1, 2\}$ to $v_2$ and $\emptyset$ to the leaves of $L(v_2)$. Hence $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. By induction on $T'$ and using the fact that $n = n - 3$ and $s' = s$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

\textbf{Case 2.} $\deg_T(v_2) = 2$. Seeing the previous case, we may assume that every child of $v_3$ which is a support vertex has degree two. Consider the following subcases.

\textbf{Subcase 2.1.} $\deg_T(v_3) \geq 3$. Let $T' = T - \{v_1, v_2\}$. Since any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of $T$ by assigning the set $\{1\}$ to $v_1$ and $v_2$, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. Using the induction on $T'$, where $n = n - 2$ and $s' = s - 1$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

\textbf{Subcase 2.2.} $\deg_T(v_3) = 2$. We consider some additional subcases.

\textbf{Subcase 2.2.1.} $\deg_T(v_4) \geq 3$. Let $T' = T - \{v_1, v_2, v_3\}$. Note that $n' \geq 3$. If $n' = 3$, then $T$ is a tree of order 6 with two support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = \frac{16}{3}$, and thus the result is valid. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$-functions, let $f$ be one such that $|f(v_4)|$ is as large as possible. If $|f(v_4)| \geq 1,$
then define the function $g$ on $V(T)$ as follows: $g(x) = f(x)$ for all $x \in V(T')$, $g(v_3) = \emptyset$ and $g(v_1) = g(v_2) = \{1\}$ or $\{2\}$ so that $g(N[v_3]) = \{1, 2\}$. Clearly, $g$ is a T2RDF of $T$ of weight $\gamma_{tr2}(T') + 2$. By induction on $T'$ and using the fact that $n' = n - 3$ and $s' = s - 1$ we deduce that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$  

For the sequel we can assume that $f(v_4) = \emptyset$. Clearly in that case, $v_4$ is not a support vertex. By the choice of the diametral path and taking into account the previous cases, we can assume that every child of $v_4$ with depth two and different from $v_3$ has degree 2. We consider the following.

(i) $v_4$ has a child $u_2$ which is a support vertex. Since $f(v_4) = \emptyset$, we conclude that $\deg_T(u_2) = 2$. Let $u_1$ be the leaf neighbor of $u_2$ and let $T'' = T - \{u_1, u_2\}$. Clearly, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 2$ and $s'' = s - 1$. By induction on $T''$, it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$  

(ii) There is a pendant path $v_4u_3v_2v_1$ in $T$, where $u_3 \neq v_3$. Since $|f(v_4)| = 0$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| = 3$. Define the function $g$ on $T'$ by $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(v_4) = \{2\}$, and $g(x) = f(x)$ otherwise. Clearly $g$ is a $\gamma_{tr2}(T')$-function $|g(v_4)| > |f(v_4)| = 0$, contradicting our choice of $f$.

Subcase 2.2.2. $\deg_T(v_4) = 2$. If $\deg_T(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Note that $T'$ has order $n' \geq 3$. If $n = 3$, then $T$ is a path $P_6$, where $\gamma_{tr2}(P_6) = 5$ (by Proposition 4) and the result is valid. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$-function $f$ such that $|f(v_4)| = 1$, and such a function can be extended to a T2RDF of $T$ by assigning $\emptyset$ to $v_3$, $\{1\}$ to $v_1$ and $\{1, 2\} - f(v_4)$ to $v_2$. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$  

Assume now that $\deg_T(v_3) \geq 3$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that $T'$ has order $n' \geq 3$. If $n' = 3$, then $T$ is a tree of order 7 obtained from a path $P_6$ by adding a new vertex attached to one of the two support vertices of the path $P_6$. It is easy to check that $\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3}$. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$-functions, let $f$ be one such that $|f(v_3)|$ is as large as possible. If $|f(v_3)| \geq 1$, then $f$ can be extended to a T2RDF of $T$ by assigning $\emptyset$ to $v_4$, $\{1\}$ to $v_1$ and $v_2$, and either $\{1\}$ or $\{2\}$ to $v_3$ so that $f(N[v_4]) = \{1, 2\}$. By induction on $T'$, it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$  


For the sequel, we can assume that $f(v_5) = \emptyset$. Trivially, $v_5$ is not a support vertex. Also, every child of $v_5$ with depth one has degree two. We consider the following.

(i) $v_5$ has a child with depth 3. Let $u_1 \neq v_1$ be a leaf at distance four from $v_5$ and let $v_5 u_4 u_3 u_2 u_1$ be the unique path between $u_1$ and $v_5$. According to Cases 1 and 2 and Subcases 2.1 and 2.2, we must assume that each of $u_4$, $u_3$ and $u_2$ has degree two. Moreover, since $f(v_5) = \emptyset$ as assumed and according to the choice of $f$ maximizing $|f(v_5)|$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| + |f(u_4)| = 4$.

Define the function $g$ on $V(T')$ as follows: $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(u_4) = g(v_5) = \{2\}$ and $g(x) = f(x)$ otherwise. Clearly, $g$ is a $\gamma_{tr_2}(T')$-function with $|g(v_5)| > |f(v_5)| = 0$, a contradiction.

(ii) $v_5$ has a child $v_2$ with depth one. Let $u_1$ be the leaf neighbor of $u_2$. Let $T'' = T - \{u_1, u_2\}$. Obviously, $\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 2$. It follows by induction on $T''$ that

$$\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$ 

(iii) $v_5$ has a child, say $w$, with depth two having degree at least 3. Suppose first that $w$ has at least two children as support vertices and let $z$ be one of them having minimum degree. Note that deg$_T(z) \in \{2, 3\}$ since every support vertex of $T$ has at most two leaves. Let $T'' = T - \{\{z\} \cup L(z)\}$. Then $\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 2$, $n'' = n - 1 - |L(z)|$ and $s'' = s - 1$. Using the induction on $T'$ we obtain the desired result. Now, let $w$ has exactly one child, say $t$, as a support neighbor. Since deg$_T(w) \geq 3$, we deduce that $w$ is a support vertex. Let $T'' = T - T_w$. Note that $T_w$ has order $n_w \in \{4, 5, 6\}$. Moreover, it is clear that $\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 4$. It follows from the induction hypothesis that

$$\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 4 \leq \frac{2(n'' + s'')}{3} + 4 \leq \frac{2(n - n_w + s - 2)}{3} + 4 \leq \frac{2(n + s)}{3}.$$ 

(iv) $v_5$ has a child, say $w$, with depth two and having degree 2. Suppose first that the child $z$ of $w$ is a strong support. Let $L(z) = \{z_1, z_2\}$ and let $T'' = T - \{w, z, z_1, z_2\}$. Then $\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 3$, $n'' = n - 4$ and $s'' = s - 1$. It follows from the induction on $T'$ that

$$\gamma_{tr_2}(T) \leq \gamma_{tr_2}(T'') + 3 \leq \frac{2(n'' + s'')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$ 

Now, suppose that the child $z$ of $w$ is a support vertex of degree two. Let deg$_T(v_5) = k \geq 3$ and $H_t$ for $t \geq 2$ be the tree obtained from a star $K_{1,t}$ by subdividing one edge three times and each of the remaining edges exactly twice. Seeing the previous situations, clearly $T_{v_5}$ is isomorphic to $H_{k-1}$. Now let $T' = T - T_{v_5}$. We note that $T'$ has order $n' \geq 3$. If $n' = 3$, then $T = H_k$, where $n = 3k + 2$,
s(T) = k and \( \gamma_{tr2}(T) = 2k + 2 < \frac{2(n+s)}{3} \). Hence we can assume that \( n' \geq 4 \). Then \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k, n' = n - 3k + 1 \) and \( s(T') \leq s(T) - (k - 1) + 1 \). It follows from the induction on \( T' \) that

\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k \leq \frac{2(n' + s')}{3} + 2k
\]

\[= \frac{2(n - 3k + 1 + s - k + 2)}{3} + 2k \leq \frac{2(n + s)}{3}.
\]

This completes the proof.

Next we establish an upper bound on the total 2-rainbow domination number of a tree in terms of the vertex cover number. We first give an upper bound for arbitrary graphs.

**Lemma 10.** Let \( G \) be a graph of order \( n \geq 2 \) with no isolated vertex and \( V_c \) a minimum vertex cover of \( G \). Then

\[\gamma_{tr2}(G) \leq 2\beta(G) + r,\]

where \( r \) is the number of isolated vertices in the subgraph induced by \( V_c \). This bound is sharp for the graphs in Figure 2.

![Figure 2](image-url)

**Proof.** Let \( V_c \) be a minimum vertex cover of \( G \) and \( I \) the set of isolated vertices in \( G[V_c] \). Let \( K = V(G) - V_c \). Since \( K \) is a maximum independent set, every vertex of \( V_c \) has a neighbor in \( K \). Let \( D \) be a smallest subset of vertices of \( K \) that dominates all vertices of \( I \). Obviously, \( |D| \leq |I| = r \). Now define a function \( f : V(G) \rightarrow \mathcal{P}({1,2}) \) by \( f(x) = \{1,2\} \) if \( x \in V_c \), \( f(x) = \{1\} \) if \( x \in D \) and \( f(x) = \emptyset \) otherwise. Clearly, \( f \) is a T2RDF of \( G \) of weight \( 2|V_c| + |D| \leq 2|V_c| + r \).

The proof of the next theorem is inspired by the proof of Theorem 2 in [9].
Theorem 11. Let $T$ be a tree of order $n \geq 3$ and let $S'$ be the set of isolated vertices in the subgraph induced by the set of support vertices of $T$. Then

$$\gamma_{tr\,2}(T) \leq 2\beta(T) + |S'|.$$  

This bound is sharp for the graph in Figure 3.

![Figure 3. A tree $T$ with $\gamma_{tr\,2}(T) = 2\beta(T) + |S'|$.](image_url)

**Proof.** Let $L$ and $S$ denote the set of leaves and support vertices of a tree $T$, respectively. Let $V_I$ be a maximum independent set that contains all leaves of $T$. Then $V_c = V - V_I$ is a vertex cover set of $T$. Note that $S \subseteq V_c$. If no support vertex of $T$ is isolated in $T[V_c]$, then the result holds by Lemma 10.

Hence, assume that $u$ is a support vertex which is isolated in $T[V_c]$. Root $T$ at $u$ and let $A_1 = \{u\}$ and $A_2 = N(u)$. Clearly, $A_1 \subseteq V_c$ and $A_2 \subseteq V_I$. Assume that $A_3 = (N(A_2) - A_1) \cup B_{N(A_2)} - A_1$, where $B_{N(A_2)} - A_1 = \{v \in V_c | v \text{ is in a component of } T[V_c]\}$ with a vertex of $N(A_2) - A_1$. Set $A_4 = N(A_4) - A_2$. Then we have $A_3 \subseteq V_c$ and $A_4 \subseteq V_I$.

We repeat this process so that at some odd number step $2k + 1$, we put

$$A_{2k+1} = (N(A_{2k}) - A_{2k-1}) \cup B_{N(A_{2k}) - A_{2k-1}},$$

where $B_{N(A_{2k}) - A_{2k-1}} = \{v \in V_c | v \text{ is in a component of } T[V_c]\}$ with a vertex of $N(A_{2k}) - A_{2k-1}$ and we set $A_{2k+2} = N(A_{2k+1}) - A_{2k}$. This process will terminate at some $m$th step where $m$ is even and $A_m$ composed only of leaves. Note that $A_1 \cup \cdots \cup A_m$ is a partition of $V(T)$. Obviously, $V_I = A_2 \cup A_4 \cup \cdots \cup A_{m-2} \cup A_m$ and $V_c = A_1 \cup A_3 \cup \cdots \cup A_{m-3} \cup A_{m-1}$. Note that if $v \in A_i$, for $i > 1$, has a neighbor in $A_{i-1}$, then it has only one neighbor in $A_{i-1}$.

Let $D_1 = V_c$. If $T[V_c]$ has isolated vertices that are support vertices in $T$, then let $K$ be a smallest subset of vertices of $V_I - L$ that dominates these isolated support vertices. Clearly, $|K| \leq |S'|$. Now we consider the isolated vertices of $T[V_c]$ that are not support vertex in $T$. In decreasing order, we visit each $A_i$ with odd index $i$, where $3 \leq i \leq m - 1$. We start with $A_{m-1}$ and observe that if there is an isolate of $T[V_c]$ in $A_{m-1}$, then it is a support vertex and some vertex of $K$ is adjacent to it. Now for each non-support isolated vertex $v$ of $T[V_c]$ which is in $A_{m-3}$, if $N(v) \cap A_{m-2}$ is dominated by $A_{m-1} \cap V_c$, then remove $v$ from $D_1$ and add to $D_1$ its unique neighbor in $A_{m-4}$, otherwise we leave $v$ in $D_1$. Continue this way for each odd $i$ in decreasing order. That is, in general for $A_i$ where $i$ is odd,
if a non-support isolated vertex \( v \) of \( T[V_c] \) is in \( A_i \) and \( N(u) \cap A_{i+1} \) are dominated by \( A_{i+2} \cap V_c \), then remove \( v \) from \( D_1 \) and add its unique neighbor in \( A_{i-1} \) to \( D_1 \), otherwise we leave \( v \) in \( D_1 \). This process terminates after \( i = 3 \). Now, if some vertex of \( A_2 \) is in \( K \), then we are done. Otherwise remove \( u \) from \( D_1 \) and add to \( D_1 \) one of its neighbors. Note that \( |D_1| \) has not increased. Now let \( D_2 = D_1 \cup K \). Using an argument similar to that described in the proof of Theorem 2 in [9], we see that the induced subgraph \( T[D_2] \) has no isolated vertex. Define the function \( f : V(T) \rightarrow P(\{1, 2\}) \) by \( f(x) = \{1, 2\} \) for \( x \in D_1 \), \( f(x) = \{1\} \) for \( x \in K \) and \( f(x) = \emptyset \) otherwise. Clearly, \( f \) is a T2RDF of \( T \) and thus

\[
\gamma_{tr2}(T) \leq 2|V_c| + |K| \leq 2\beta(T) + |S'|.
\]

This achieves that proof.

### 4. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL 2-RAINBOW DOMINATION:

**TOTAL 2-RAINBOW DOMINATION**

**Instance.** Graph \( G = (V, E) \), positive integer \( k \leq |V| \).

**Question.** Does \( G \) have a total 2-rainbow dominating function of weight at most \( k \)?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, EXACT-3-COVER (X3C), to TOTAL 2-RAINBOW DOMINATION.

**EXACT 3-COVER (X3C)**

**Instance.** A finite set \( X \) with \( |X| = 3q \) and a collection \( C \) of 3-element subsets of \( X \).

**Question.** Is there a subset \( C' \) of \( C \) such that every element of \( X \) appears in exactly one element of \( C' \)?

**Theorem 12.** TOTAL 2-RAINBOW DOMINATION is NP-complete for bipartite graphs.

**Proof.** TOTAL 2-RAINBOW DOMINATION is a member of NP, since we can check in polynomial time that a function \( f : V \rightarrow \{0, 1, 2\} \) has weight at most \( k \) and is a T2RDF. Now let us show how to transform any instance of X3C into an instance of TOTAL 2-RAINBOW DOMINATION so that one of them has a
solution if and only if the other one has a solution. Let \( X = \{x_1, x_2, \ldots, x_{3q}\} \) and \( C = \{C_1, C_2, \ldots, C_t\} \) be an arbitrary instance of X3C.

For each \( x_i \in X \), we build a graph \( H_i \) obtained from a path \( P_2 : x_i-y_i \) and two stars \( K_{1,3} \) with centers \( a_i \) and \( b_i \), by adding edges \( y_i a_i \) and \( y_i b_i \). Hence, each \( H_i \) has order 10. For each \( C_j \in C \), we build a double star \( S_{3,3} \) with support vertices \( u_j \) and \( v_j \). Let \( c_j \) be a leaf of the double star \( S_{3,3} \). Let \( Y = \{c_1, c_2, \ldots, c_t\} \). Now to obtain a graph \( G \), we add edges \( c_j x_i \) if \( x_i \in C_j \). Clearly, \( G \) is a bipartite graph (for example, see Figure 4). Set \( k = 4t + 16q \). Observe that for every T2RDF \( f \) on \( G \), each \( H_i \) has weight at least 5 and each double star \( S_{3,3} \) has weight at least 4.

![Figure 4. NP-completeness for bipartite graphs.](image_url)

Suppose that the instance \( X, C \) of X3C has a solution \( C' \). We construct a T2RDF \( f \) on \( G \) of weight \( k \). For each \( i \), assign the set \( \{1, 2\} \) to \( a_i, b_i \), the set \( \{1\} \) to \( y_i \), and \( \emptyset \) to the remaining vertices of \( H_i \). For every \( j \), assign \( \{1, 2\} \) to \( u_j \) and \( v_j \), and \( \emptyset \) to each leaf. In addition, if for every \( C_j \), assign to \( c_j \) the set \( \{2\} \) if \( C_j \in C' \) and \( \emptyset \) if \( C_j \notin C' \). Note that since \( C' \) exists, its cardinality is precisely \( q \), and so the number of \( c_j \)'s assigned \( \{2\} \) is \( q \), having disjoint neighborhoods in \( \{x_1, x_2, \ldots, x_{3q}\} \). Since \( C' \) is a solution for X3C, every vertex \( x_i \) in \( X \) satisfies \( f(N[x_i]) = \{1, 2\} \). Hence, it is straightforward to see that \( f \) is a T2RDF with weight \( f(V) = 4t + q + 15q = k \).

Conversely, suppose that \( G \) has a T2RDF with weight at most \( k \). Among all such functions, let \( g = (V_0, V_1, V_2, V_{12}) \) be one such that the number of vertices of \( \{y_1, y_2, \ldots, y_{3q}\} \) assigned \( \{1, 2\} \) is as small as possible. As observed above, since each \( H_i \) has weight at least 5, we may assume that \( g(a_i) = g(b_i) = \{1, 2\} \) and \( |g(y_i)| > 0 \) so that vertices \( a_i, b_i \) are not isolated in the subgraph induced by \( V_1 \cup V_2 \cup V_{12} \). Hence each leaf neighbor of \( a_i \) or \( b_i \) is assigned \( \emptyset \) under \( g \). Assume
that \( g(y_i) = \{1, 2\} \) for some \( i \). Observe that if \( |g(x_i)| > 0 \), then reassigning \( \{1\} \) to \( y_i \) provides a T2RDF \( g' \) with less vertices \( y_i \) assigned \( \{1, 2\} \) than under \( g \), contradicting our choice of \( g \). Hence \( g(x_i) = \emptyset \). But then reassigning \( \{1\} \) to each of \( y_i \) and \( x_i \) instead of \( \{1, 2\} \) and \( \emptyset \), respectively, provides a T2RDF \( g' \) with less vertices \( y_i \) assigned \( \{1, 2\} \) than under \( g \), a contradiction too. Therefore \( |g(y_i)| = 1 \) for every \( i \in \{1, 2, \ldots, 3q\} \). On the other hand, the total weight of all double stars corresponding to elements of \( C \) is \( 4t \). In this case, we can assume that \( g(u_j) = g(v_j) = \{1, 2\} \) and so each leaf neighbor of \( u_j \) or \( v_j \) is assigned \( \emptyset \) under \( g \). Note that each \( c_j \) can be assigned \( \emptyset \) since \( g(u_j) = \{1, 2\} \). Since \( w(g) \leq 4t + 16q \) and the total weight assigned to vertices of \( V(G) \setminus (X \cup Y) \) is \( 4t + 15q \), we have to assign to vertices of \( (X \cup Y) \) sets whose total cardinalities not exceeding \( q \) so that each vertex \( x_i \in X \) has either \( |g(x_i)| > 0 \) or has two neighbors in \( V_1 \cup V_2 \) so that \( f(N[x_i]) = \{1, 2\} \). Since \( |X| = 3q \), it is clear that this is only possible if there are \( q \) vertices of \( \{c_1, c_2, \ldots, c_t\} \) belonging to \( V_1 \cup V_2 \). Since each \( c_j \) has exactly three neighbors in \( \{x_1, x_2, \ldots, x_{3q}\} \), we deduce that \( C'' = \{c_j : |g(c_j)| = 1\} \) is an exact cover for \( C \).

The next result is obtained by using the same proof as for Theorem 12 on the (same) graph \( G \) built for the transformation by adding all edges between the \( c_j \)’s so that the resulting graph is chordal.

**Theorem 13.** TOTAL 2-RAINBOW DOMINATION is NP-complete for chordal graphs.

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