TOTAL 2-RAINBOW DOMINATION NUMBERS OF TREES

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Abstract

A 2-rainbow dominating function (2RDF) of a graph $G = (V(G), E(G))$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of $v$. A total 2-rainbow dominating function $f$ of a graph with no isolated vertices is a 2RDF with the additional condition that the subgraph of $G$ induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total 2-rainbow domination number, $\gamma_{tr2}(G)$, is the minimum weight of a total 2-rainbow dominating function of $G$. In this paper, we establish some sharp upper and lower bounds on the total 2-rainbow domination number of a tree. Moreover, we show that the decision problem associated with $\gamma_{tr2}(G)$ is NP-complete for bipartite and chordal graphs.

Keywords: 2-rainbow dominating function, 2-rainbow domination number, total 2-rainbow dominating function, total 2-rainbow domination number.

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Throughout this paper, \( G \) is a simple graph with vertex set \( V(G) \) and edge set \( E(G) \) (briefly \( V, E \)) such that \( G \) has no isolated vertices. The order of a graph \( G \) is the number of vertices in \( G \), denoted by \( n = n(G) \). For every vertex \( v \in V(G) \), the open neighborhood of \( v \) is the set \( N_G(v) = N(v) = \{ u \in V(G) \mid uv \in E(G) \} \) and its closed neighborhood is the set \( N_G[v] = N[v] = N(v) \cup \{ v \} \). The degree of a vertex \( v \in V \) is \( \deg_G(v) = |N(v)| \). The maximum degree of a graph \( G \) is denoted by \( \Delta = \Delta(G) \). The open neighborhood of a set \( S \subseteq V \) is the set \( N_G(S) = N(S) = \bigcup_{v \in S} N(v) \), and the closed neighborhood of \( S \) is the set \( N_G[S] = N[S] = N(S) \cup S \). The diameter of \( G \), denoted by \( \text{diam}(G) \), is the maximum value among minimum distances between all pairs of vertices of \( G \). A leaf of a tree \( T \) is a vertex of degree one, a support vertex is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. If \( v \) is a support vertex, then \( L(v) \) will denote the set of the leaves attached to \( v \). For a vertex \( v \) in a rooted tree \( T \), let \( C(v) \) denote the set of children of \( v \), \( D(v) \) denote the set of descendants of \( v \) and \( D[v] = D(v) \cup \{ v \} \). Also, the depth of \( v \), \( \text{depth}(v) \), is the maximum distance from \( v \) to a vertex in \( D(v) \). We denote by \( T_v \) the induced subgraph of \( T \) with vertex set \( D[v] \). The independence number of a graph \( G \), denoted \( \alpha(G) \), is the order of a largest subset of vertices in which no two are adjacent. A vertex cover of \( G \) is a set of vertices \( S \) that covers all the edges, i.e., every edge is incident with a vertex of \( S \). The vertex cover number \( \beta(G) \) is the minimum cardinality of a vertex cover of \( G \). It is well-known that for every graph \( G \) of order \( n \), \( \beta(G) + \alpha(G) = n \).

A total Roman dominating function of a graph \( G \) is a function \( f : V(G) \to \{0, 1, 2\} \) satisfying the following conditions: (i) every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \), and (ii) the subgraph of \( G \) induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function \( f \) is the value \( w(f) = \sum_{u \in V(G)} f(u) \), and the total Roman domination number \( \gamma_{\text{TR}}(G) \) is the minimum weight of a total Roman dominating function of \( G \). The concept of total Roman domination in graphs was introduced by Liu and Chang [11] and studied for example in [2].

A 2-rainbow dominating function (2RDF) of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of all subsets of the set \( \{1, 2\} \) such that for any vertex \( v \in V(G) \) with \( f(v) = \emptyset \) the condition \( \bigcup_{u \in N(v)} f(u) = \{1, 2\} \) is fulfilled. The weight of a 2RDF \( f \) is defined as \( w(f) = \sum_{v \in V(G)} |f(v)| \), and the minimum weight of a 2RDF is called the 2-rainbow domination number of \( G \), denoted by \( \gamma_{\text{tr2}}(G) \). The concept of 2-rainbow domination was introduced by Bresar et al. [6], and has been studied by several authors, for example [4, 5, 7, 8, 10, 12, 13].

A 2RDF \( f \) is called a total 2-rainbow dominating function, or just \( \text{T2RDF} \), if the subgraph of \( G \) induced by \( \{ v \in V(G) \mid f(v) \neq \emptyset \} \) has no isolated vertices. The total 2-rainbow domination number, \( \gamma_{\text{ttr2}}(G) \), is the minimum weight of a total
2-rainbow dominating function of \( G \), and a T2RDF of \( G \) with weight \( \gamma_{tr2}(G) \) is called a \( \gamma_{tr2}(G) \)-function. We note that if \( f \) is a T2RDF of a graph \( G \) and \( H \) is a subgraph of \( G \), then we denote the restriction of \( f \) to \( H \) by \( f|_{V(H)} \). Total 2-rainbow domination was recently introduced by Abdollahzadeh Ahangar et al. in [1] and has been studied in [3].

Before presenting our main results, we present some straightforward observations.

Observation 1. If \( v \) is a strong support vertex in a graph \( G \), then there exists a \( \gamma_{tr2}(G) \)-function \( f \) such that \( f(v) = \{1, 2\} \).

Observation 2. If \( u_1 \) and \( u_2 \) are two adjacent support vertices in a graph \( G \), then there exists a \( \gamma_{tr2}(G) \)-function \( f \) such that \( f(u_1) = f(u_2) = \{1, 2\} \).

Observation 3. If \( v \) is a leaf neighbor of a support vertex of degree 2 in a graph \( G \), then there exists a \( \gamma_{tr2}(G) \)-function \( f \) such that \( |f(v)| = 1 \).

2. Lower Bounds

In this section, we establish some sharp lower bounds on the total 2-rainbow domination number of a tree. We begin by recalling the following result given in [1] for paths.

Proposition 4. For \( n \geq 2 \), \( \gamma_{tr2}(P_n) = \left\lceil \frac{2n+2}{3} \right\rceil \).

Our first lower bound on \( \gamma_{tr2}(T) \) is in terms of the order and the number of leaves of a tree \( T \).

Theorem 5. Let \( T \) be a non-trivial tree of order \( n \) with \( \ell(T) \) leaves. Then

\[
\gamma_{tr2}(T) \geq \left\lceil \frac{2(n + 3 - \ell(T))}{3} \right\rceil.
\]

This bound is sharp for paths, stars and double stars.

Proof. We use an induction on \( n \). It is easy to check that the statement holds for all trees of order \( n \leq 4 \). Let \( n \geq 5 \) and assume that for every non-trivial tree \( T \) of order at most \( n - 1 \) the result is true. Let \( T \) be a tree of order \( n \geq 5 \). If \( T \) is a star, then \( \gamma_{tr2}(T) = 3 = \left\lceil \frac{2(n+3-(n-1))}{3} \right\rceil \). If \( T \) is a double star, then \( \gamma_{tr2}(T) = 4 = \left\lfloor \frac{2(n+3-(n-2))}{3} \right\rceil \). Henceforth we can assume that \( T \) has diameter at least 4.

Suppose that \( T \) has a strong support vertex \( u \). Let \( T' = T - u' \), where \( u' \) is a leaf neighbor of \( u \). By Observation 1, there exists a \( \gamma_{tr2}(T) \)-function \( g \) such
that \( g(u) = \{1, 2\} \). We may assume, without loss of generality, that \( g(u') = \emptyset \). Then the function \( g \), restricted to \( T' \) is a T2RDF. We can apply the inductive hypothesis to the tree \( T' \) and deduce that
\[
\gamma_{tr2}(T) = \omega(g) \geq \gamma_{tr2}(T') \geq \left\lceil \frac{2(n - 1) + 3 - (\ell(T) - 1)}{3} \right\rceil = \left\lceil \frac{2(n + 3 - \ell(T))}{3} \right\rceil.
\]
Therefore, from now on we suppose that \( T \) has no strong support vertex.

Let \( v_1, v_2, \ldots, v_k \) be a diametral path of rooted tree \( T \) with root vertex \( v_k \). Since \( T \) has no strong support vertex, each child of \( v_3 \) is either a leaf or a support vertex of degree 2. Let \( f \) be a \( \gamma_{tr2}(T) \)-function, and consider the following cases.

Case 1. \( \deg_T(v_3) \geq 3 \). Assume first that \( v_3 \) is a support vertex. By Observation 2, we may assume that \( f(v_2) = f(v_3) = \{1, 2\} \). Let \( T' = T - v_1 \) and define \( h : V(T') \to \mathcal{P}(\{1, 2\}) \) by \( h(v_2) = \{1\} \) and \( h(x) = f(x) \) for \( x \in V(T') - \{v_2\} \).

Clearly, \( h \) is a T2RDF of \( T' \). Using the fact that \( n' = n - 1 \) and \( \ell(T') = \ell(T) \), it follows from the induction hypothesis that
\[
\gamma_{tr2}(T) = \omega(f) = \omega(h) + 1 \geq \gamma_{tr2}(T') + 1
\]
\[
\geq \left\lceil \frac{2(n - 1) + 3 - (\ell(T))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n + 3 - \ell(T)) + 1}{3} \right\rceil,
\]
as desired. Hence we assume that \( v_3 \) is not a support vertex, and thus every child of \( v_3 \) is a support vertex of degree 2. Let \( u_2 \neq v_2 \) be a child of \( v_3 \) and \( u_1 \) the leaf neighbor of \( u_2 \). Clearly, \( |f(u_1)| + |f(u_2)| \geq 2 \) and \( |f(v_1)| + |f(v_2)| \geq 2 \). Let \( T' = T - \{u_1, v_2, u_2\} \) and define \( h : V(T') \to \mathcal{P}(\{1, 2\}) \) by \( h(v_3) = \{1\} \cup f(v_3) \) and \( h(x) = f(x) \) for \( x \in V(T') - \{v_3\} \). Clearly, \( h \) is a T2RDF of \( T' \), \( n' = n - 2 \) and \( \ell(T') = \ell(T) - 1 \). It follows from the induction hypothesis that
\[
\gamma_{tr2}(T) = \omega(f) \geq \omega(h) + 1 \geq \gamma_{tr2}(T') + 1
\]
\[
\geq \left\lceil \frac{2(n - 2) + 3 - (\ell(T))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n + 3 - \ell(T)) + 1}{3} \right\rceil,
\]
as desired.

Case 2. \( \deg_T(v_3) = 2 \). As above we have \( |f(v_1)| + |f(v_2)| \geq 2 \). Suppose first that \( |f(v_1)| + |f(v_2)| \geq 3 \), and let \( T' = T - v_1 \). Then the function \( h : V(T') \to \mathcal{P}(\{1, 2\}) \) defined by \( h(v_3) = \{1\} \) and \( h(x) = f(x) \) for \( x \in V(T') - \{v_3\} \) is a T2RDF of \( T' \). By induction on \( T' \) and using the fact that \( n' = n - 1 \), \( \ell(T') = \ell(T) \), we obtain
\[
\gamma_{tr2}(T) \geq \left\lceil \frac{2(n + 3 - \ell(T)) + 1}{3} \right\rceil,
\]
as desired. Therefore, we assume for the next that \( |f(v_1)| + |f(v_2)| = 2 \). Now, if \( f(v_3) \neq \emptyset \), then the function \( f \), restricted to \( T - v_1 \) is a T2RDF of \( T - v_1 \) of weight \( \gamma_{tr2}(T) - 1 \), and by the induction hypothesis on \( T - v_1 \) we obtain
\[
\gamma_{tr2}(T) \geq \left\lceil \frac{2(n + 3 - \ell(T)) + 1}{3} \right\rceil.
\]
Hence let \( f(v_3) = \emptyset \). Let \( T' = T - \{v_1, v_2, v_3\} \) and recall that \( T \) has diameter at least four. If \( T' \) has order \( n' = 2 \), then \( T = P_5 \), and by Proposition 4 the result is valid. Hence let \( n' \geq 3 \). Then \( f|_{V(T')} \) is a T2RDF of \( T' \) of weight \( \omega(f) - 2 \). Using the fact that \( n' = n - 3 \) and \( \ell(T') \leq \ell(T) \), and by applying the induction on \( T' \), we obtain

\[
\gamma_{tr2}(T) = \omega(f) = \omega(f|_{V(T')}) + 2 \geq \gamma_{tr2}(T') + 2 \\
\geq \left\lceil \frac{2(n-3)+3-\ell(T)}{3} \right\rceil + 2 = \left\lceil \frac{2(n+3-\ell(T))}{3} \right\rceil.
\]

This completes the proof.

**Theorem 6.** If \( T \) is a tree of order \( n \geq 3 \) with \( \ell(T) \) leaves and \( s(T) \) support vertices, then

\[
\gamma_{tr2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta} \right\rceil,
\]

and this bound is sharp.

**Proof.** The proof is by induction on \( n \). One can easily check that the statement holds for all trees of order \( n \leq 4 \). Let \( n \geq 5 \) and assume that the result is true for every non-trivial tree \( T' \) of order \( n' \), with \( 3 \leq n' < n \). Let \( T \) be a tree of order \( n \) with \( \ell(T) \) leaves and \( s(T) \) support vertices. If \( \text{diam}(T) = 2 \), then \( T \) is a star, where \( \gamma_{tr2}(T) = 3 = \left\lceil \frac{n-4}{\Delta} \right\rceil \). If \( \text{diam}(T) = 3 \), then \( T \) is a double star, where \( 4 = \gamma_{tr2}(T) \geq 2 + \left\lceil \frac{n-4}{\Delta} \right\rceil \), and clearly the result is valid since \( \left\lceil \frac{n-4}{\Delta} \right\rceil \leq 2 \). Henceforth we may assume that \( \text{diam}(T) \geq 4 \).

Let \( v_1v_2 \cdots v_k \) be a diametral path of \( T \) and \( f \) be a \( \gamma_{tr2}(T) \)-function. Without loss of generality, we assume \( \text{deg}_T(v_2) \leq \text{deg}_T(v_{k-1}) \). Consider the following situations.

Suppose first that \( v_3 \) is a support vertex adjacent to another support vertex different from \( v_2, v_3 \) and \( v_4 \) is adjacent to a strong support vertex different from \( v_2, v_4 \). Let \( T' = T - T_{v_2} \). Clearly, \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - |L(v_2)| \) and \( s(T') = s(T) - 1 \). Moreover, it is easy to see that \( \gamma_t(T) \leq \gamma_t(T') + 1 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2 \). By the induction hypothesis on \( T' \) we obtain that

\[
\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 2 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2 \\
\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil.
\]

Next, suppose that \( v_3 \) is not a support vertex and it is adjacent to a support vertex of degree two different from \( v_2 \). Let \( T' = T - T_{v_2} \). Clearly, \( \Delta(T) \geq \Delta(T') \), \( \ell(T') = \ell(T) - |L(v_2)| \) and \( s(T') = s(T) - 1 \). On the other hand, if \( \text{deg}_T(v_2) \geq 3 \), then \( \gamma_t(T) \leq \gamma_t(T') + 1 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2 \), and if \( \text{deg}_T(v_2) = 2 \), then
\(\gamma_t(T) \leq \gamma_t(T') + 1\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 1\). Using the induction on \(T'\) and according to each situation, the result follows.

Suppose now that \(v_3\) is a support vertex having no neighbor as support vertex besides \(v_2\) and (possibly) \(v_4\). If \(|f(x)| \geq 1\) for some \(x \in N(v_3) - \{v_2\}\), then let \(T' = T - T_{v_3}\). Clearly, \(\Delta(T) \geq \Delta(T')\), \(\ell(T') = \ell(T) - |L(v_3)|\) and \(s(T') = s(T) - 1\). Moreover, one can see that \(\gamma_t(T) \leq \gamma_t(T') + 1\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2\). By induction on \(T'\), we obtain as above \(\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 2\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2\). By the induction hypothesis on \(T'\) we obtain that

\[
\gamma_{tr2}(T') \geq \gamma_{tr2}(T') + 2 \geq \gamma_t(T') + \left[\frac{\ell(T') - s(T')}{\Delta(T')}\right] + 2
\]

\[
\geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right].
\]

Therefore, we can assume that all children of \(v_4\) have depth 2. According the diametral path and the situations already considered, we conclude that each child of \(v_4\) is a support vertex or has degree 2. If \(z\) is a child of \(v_4\) with degree 2 with \(z_1 \in N(z) - v_4\), then let \(T' = T - T_z\). Clearly, \(\Delta(T) \geq \Delta(T')\), \(\ell(T') = \ell(T) - |L(z_1)|\) and \(s(T') = s(T) - 1\). On the other hand, \(\gamma_t(T) \leq \gamma_t(T') + 2\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3\). Using the induction on \(T'\), we obtain desired result. Hence, each child of \(v_4\) is a support vertex assigned \(\{1, 2\}\) under \(f\). Let \(T' = T - T_{v_3}\). Then \(\Delta(T) \geq \Delta(T')\), \(\ell(T') = \ell(T) - (|L(v_2)| + |L(v_3)|)\) and \(s(T') = s(T) - 2\). On the other hand, \(\gamma_t(T) \leq \gamma_t(T') + 2\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4\). By the induction hypothesis on \(T'\) we obtain that

\[
\gamma_{tr2}(T') \geq \gamma_{tr2}(T') + 4 \geq \gamma_t(T') + \left[\frac{\ell(T') - s(T')}{\Delta(T')}\right] + 4
\]

\[
\geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right].
\]

Now, let \(\deg_{T}(v_{4}) = 2\) and \(T' = T - T_{v_4}\). Note \(T'\) has order \(n' \geq 1\) since \(\text{diam}(T) \geq 4\). It is a routine matter to check that the result holds if \(n' \in \{1, 2\}\). Hence let \(n' \geq 3\). Then \(\Delta(T) \geq \Delta(T')\), \(\ell(T') \geq \ell(T) - (|L(v_2)| + |L(v_3)|)\) and \(s(T') \leq s(T) - 1\). On the other hand, \(\gamma_t(T) \leq \gamma_t(T') + 2\) and \(\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4\). Using the induction on \(T'\), the result follows.

Finally, assume that \(\deg_{T}(v_3) = 2\). First, assume that \(f|_{T'}\) is a T2RDF of \(T' = T - T_{v_3}\). Recall that \(T\) has diameter at least four. If \(T'\) has order 2, then \(T\)
is obtained from a star of order at least three and a path $P_2$ by adding an edge joining their leaves, and clearly the result holds. So assume that $T'$ has order at least three. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - |L(v_2)|$ and $s(T') \leq s(T)$. Moreover, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 3 \geq \gamma_t(T) + \left[\frac{\ell(T') - s(T')}{\Delta(T')}\right] + 3$$

$$\geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right] + 1 \geq \gamma_t(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right].$$

Suppose now that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$. Hence, we have the following cases.

Case 1. $f(v_4) = \emptyset$. Then $v_4$ is not a support vertex and has no child of depth 1 which is a strong support vertex. Seeing the previous cases, it follows that any child of $v_4$ other than $v_3$ is either a support vertex of degree two or a vertex with depth 2 and degree 2. Moreover, since every child of $v_4$ is assigned a non-empty set, we conclude from our assumption that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$ and that $\deg_T(v_4) \in \{2, 3\}$. We consider the following.

Subcase 1.1. $\deg_T(v_4) = 3$. Observe that $T_{v_4}$ has exactly two support vertices, $v_2$ and say $z$. We note that $z$ is a either at distance one or two from $v_4$. Let $T'' = T - T_{v_4}$. Clearly, $T''$ has order at least three, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(z)|)$, $s(T'') \leq s(T) - 1$ and $\gamma_t(T) \leq \gamma_t(T'') + 4$. Now, if $z$ is at distance one from $v_4$, then $|L(z)| = 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Also, if $z$ is at distance two from $v_4$, then $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 6$. Whatever the case, using the induction on $T''$, the result follows.

Subcase 1.2. $\deg_T(v_4) = 2$. Let $T'' = T - T_{v_4}$. It is easy to check the result if $n(T'') \in \{1, 2\}$. Hence let $n(T'') \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - |L(v_2)|$ and $s(T'') \leq s(T)$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on $T''$, the result follows.

Case 2. $|f(v_4)| \geq 1$ and thus $f(x) = \emptyset$ for each vertex $x \in N(v_4) - \{v_3\}$. Then every child of $v_4$ besides $v_3$ (if any) is leaf. To avoid the previous case when $f(v_4) = \emptyset$ we can assume that $v_4$ is a support vertex (else substitute the assignments of $v_4$ and $v_5$). Now if $f|_{T''}$ is a T2RDF of $T'' = T - T_{v_4}$, then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$, the result follows by using the induction on $T''$. Hence suppose that $f|_{T''}$ is not a T2RDF of $T'' = T - T_{v_4}$ and so $v_3$ has no child of depth 3 other than $v_4$. Since $f(v_5) = \emptyset$, we conclude that $v_5$ is not a support vertex and has no child of depth 1 which is a strong support vertex. Consider the following situations.
Subcase 2.1. $v_5$ has a child of depth 1. Let $u_2$ be such a child of depth 1 and $u_1$ its the leaf neighbor. Note that $\deg_T(u_2) = 2$. Let $T'' = T - \{u_1, u_2\}$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Since $\gamma_T(T) \leq \gamma_{\rho_1}(T'') + 2$ and $\gamma_{\rho_2}(T'') \leq \gamma_{\rho_2}(T) - 2$, the result follows by using the induction on $T''$.

Subcase 2.2. All children of $v_5$ different to $v_4$ have depth 2. Since $|f(x)| \leq 1$ for $x \in N(v_5) \setminus \{v_4\}$, we deduce that every child of $v_5$ other than $u_2$ is not a support vertex. Let $z \neq v_4$ be a child of $v_5$. If $\deg(z) = 2$ and $z' \in N(z) \setminus \{v_5\}$, then let $T'' = T - T_z$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - |L(z')|$ and $s(T'') = s(T) - 1$. Also, $\gamma_T(T) \leq \gamma_{\rho_1}(T'') + 2$ and $\gamma_{\rho_2}(T'') \leq \gamma_{\rho_2}(T) - 3$. Using the induction on $T''$, the result follows. Hence suppose that $\deg_T(z) \geq 3$. If $z$ has a child of depth 1 say, $u_2$, of degree two, with $u_1$ as the leaf neighbor of $u_2$, then let $T'' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Also, $\gamma_T(T) \leq \gamma_{\rho_1}(T'') + 2$ and $\gamma_{\rho_2}(T'') \leq \gamma_{\rho_2}(T) - 2$. Using the induction on $T''$, the result follows. Hence, all children of $z$ are strong support vertex. Let $|C(z)| = k$ and $x_1, \ldots, x_k$ be the children of $z$, and let $T'' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - (\sum_{i=1}^{k} |L(x_i)|)$ and $s(T'') = s(T) - k$. On the other hand, $\gamma_T(T) \leq \gamma_{\rho_1}(T'') + k + 1$ and $\gamma_{\rho_2}(T'') \leq \gamma_{\rho_2}(T) - 2k - 1$. It follows from the induction hypothesis that

$$\gamma_{\rho_2}(T) \geq \gamma_{\rho_2}(T'') + 2k + 1 \geq \gamma_T(T'') + \left[\frac{\ell(T'') - s(T'')}{\Delta(T'')}\right] + 2k + 1$$

$$\geq \gamma_T(T) + \left[\frac{\ell(T'') - s(T'')}{\Delta(T''')}\right] + k \geq \gamma_T(T) + \left[\frac{\ell(T) - s(T)}{\Delta(T)}\right].$$

Subcase 2.3. $\deg_T(v_5) = 2$. Let $T'' = T - T_{v_5}$. Note that $T''$ may have order $n'' = 0$. However, it is easy to check that the result is valid for $n'' \leq 2$. Hence, let $n'' \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_1)|)$ and $s(T'') \leq s(T) - 1$. Also, $\gamma_T(T) \leq \gamma_{\rho_1}(T'') + 3$ and $\gamma_{\rho_2}(T'') \leq \gamma_{\rho_2}(T) - 5$. Using the induction on $T''$, the result follows. This completes the proof.

Obviously, $\gamma_{\rho_2}(G) \leq \gamma_{\rho_1}(G)$ for every graph $G$ without isolated vertices. In the following, we provide an upper bound on the ratio $\gamma_{\rho_1}(G)/\gamma_{\rho_2}(G)$ for arbitrary graphs $G$. Moreover, this ratio will be slightly improved for the class of trees.

**Theorem 7.** If $G$ is a graph without isolated vertices, then $\gamma_{\rho_1}(G) \leq \frac{3}{2}\gamma_{\rho_2}(G)$. This bound is sharp for the graph in Figure 1.

**Proof.** Let $f$ be a $\gamma_{\rho_2}(G)$-function. For every $i \in \{1, 2\}$, let $X_i$ be the set of all vertices $u$ for which $i \in f(u)$. Clearly, if a vertex of $G$ is assigned $\{1, 2\}$ under $f$, then $X_1 \cap X_2 \neq \emptyset$. Also, it is obvious that $|X_1| + |X_2| = \gamma_{\rho_2}(G)$. Now assume, without loss of generality, that $|X_1| \leq |X_2|$. Then $|X_1| \leq \frac{|X_1| + |X_2|}{2} = \frac{\gamma_{\rho_2}(G)}{2}$, and
the function \( g : V(G) \rightarrow \{0, 1, 2\} \) defined by \( g(x) = 0 \) if \( f(x) = \emptyset \), \( g(x) = 1 \) if \( f(x) = \{2\} \), and \( g(x) = 2 \) if \( 1 \in f(x) \), is a total Roman dominating function on \( G \), implying that

\[
\gamma_{tR}(G) \leq 2|X_1| + |X_2| \leq \frac{|X_1| + |X_2|}{2} + |X_1| + |X_2| \leq \frac{3}{2} \gamma_{tr2}(G).
\]

**Theorem 8.** For every non-trivial tree \( T \),

\[
\gamma_{tR}(T) \leq \frac{3}{2} \gamma_{tr2}(T) - 1,
\]

and this bound is sharp for \( P_n \) such that \( n \equiv 2 \pmod{3} \).

**Proof.** The proof is by induction on \( n \). The statement is valid for all trees of order \( n \in \{2, 3, 4\} \). Let \( n \geq 5 \) and assume that for every tree \( T' \) of order at most \( n - 1 \), \( \gamma_{tR}(T') \leq \frac{3}{2} \gamma_{tr2}(T') - 1 \). Let \( T \) be a tree of order \( n \). Since stars and double stars \( T \) satisfy \( \gamma_{tr2}(T) = 3 = \gamma_{tR}(T) \), the result holds. Therefore, we can assume that \( \text{diam}(T) \geq 4 \).

If \( T \) has a support vertex, say \( u \), with \( |L(u)| \geq 3 \), then let \( T' = T - u' \), where \( u' \) is a leaf neighbor of \( u \). Clearly \( \gamma_{tR}(T) \leq \gamma_{tR}(T') \). On the other hand, by Observation 1, there exists a \( \gamma_{tr2}(T) \)-function \( g \) such that \( g(u) = \{1, 2\} \). Also, we can assume that \( g(u') = \emptyset \). It follows that \( g|_{V(T')} \) is a \( T2RDF \) of \( T' \), and thus \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) \). By the inductive hypothesis on \( T' \), we obtain

\[
2 \gamma_{tR}(T) \leq 2 \gamma_{tR}(T') \leq 3 \gamma_{tr2}(T') - 2 \leq 3 \gamma_{tr2}(T) - 2.
\]

Hence we assume that every support vertex in \( T \) is adjacent to at most two leaves. Let \( v_1 v_2 \cdots v_k \) be a diametral path in \( T \) with root vertex \( v_k \). We consider the following cases.

**Case 1.** \( \deg_T(v_3) = 2 \). Let \( T' = T - v_3 \). Then \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2 \). It follows from the induction hypothesis that

\[
2 \gamma_{tR}(T) \leq 2 \gamma_{tR}(T') + 6 \leq 3 \gamma_{tr2}(T') + 4 \leq 3 \gamma_{tr2}(T) - 2.
\]

**Case 2.** \( \deg_T(v_3) \geq 3 \). Consider the following subcases.
Subcase 2.1. Suppose that \( v_3 \) is a support vertex adjacent to another support vertex different from \( v_2 \) and \( v_4 \), or \( v_3 \) is adjacent to a strong support vertex different from \( v_2 \) and \( v_4 \). Let \( T' = T - T_{v_3} \). It is easy to see that \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 2 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2 \). It follows from the induction hypothesis that
\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 4 \leq 3\gamma_{tr2}(T') + 2 \leq 3\gamma_{tr2}(T) - 4 < 3\gamma_{tr2}(T) - 2.
\]

Subcase 2.2. \( v_3 \) is not a support vertex. Since \( \text{deg}_T(v_3) \geq 3 \), every child of \( v_3 \) is a support vertex. Moreover, according to Subcase 2.1, all support vertices of \( T_{v_3}, \) but possibly \( v_2 \), have degree two. Let \( t = \text{deg}_T(v_3) - 1 \geq 2 \). Let \( T' = T - T_{v_3} \).

It is easy to see that \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 2t + 1 \). Among all \( \gamma_{tr2}(T) \)-functions, let \( g \) be one for which \( |g(v_3)| \) is as small as possible. Clearly, for every child \( x \) of \( v_3 \) we have \( |g(N[x])| \geq 2 \). Now, if \( g(v_3) = \emptyset \), then \( g|_{V(T')} \) is a T2RDF of \( T' \) of weight \( \omega(g) - 2t \), and thus \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t \). Hence assume that \( g(v_3) \neq \emptyset \). The choice of \( g \) implies that \( g(v_3) = 1 \), and thus the weight of \( T_{v_3} \) under \( g \) is \( 2t + 1 \). The choice of \( g \) also implies that \( g(v_4) = \emptyset \). In that case, the function \( g' \) defined on \( V(T') \) defined by \( g'(v_4) = g(v_3) \) and \( g'(x) = g(x) \) for all \( x \in V(T') - \{v_4\} \) is a T2RDF of \( T' \) of weight \( \gamma_{tr2}(T) - 2t \), and thus \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t \). In all cases, it follows from the induction hypothesis on \( T' \) that
\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 2 + 4t \leq 3\gamma_{tr2}(T') + 4t \leq 3\gamma_{tr2}(T) - 6t + 4t < 3\gamma_{tr2}(T) - 2.
\]

Subcase 2.3. \( v_3 \) is a support vertex adjacent to no support vertex besides \( v_2 \) and (possibly) \( v_4 \). Let \( f \) be a \( \gamma_{tr2}(T) \)-function. If \( |f(v_4)| \geq 1 \) or there exists a vertex \( x \in N_T(v_4) - \{v_3\} \) with \( |f(x)| \geq 1 \), then let \( T' = T - T_{v_3} \). Obviously, \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 4 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3 \). It follows from the induction hypothesis that
\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 9 + 6 < 3\gamma_{tr2}(T) - 2.
\]

Hence we can assume that \( f(x) = \emptyset \) for each \( x \in N_T(v_4) - \{v_3\} \). Therefore, all children of \( v_4 \) have depth 2. According to Case 1 and the diametral path, we conclude that each child of \( v_4 \) is a support vertex. Since we assumed that \( f(x) = \emptyset \) for each \( x \in N_T[v_4] - \{v_3\} \), we deduce that \( d_T(v_4) = 2 \). In this case, let \( T' = T - T_{v_4} \). Recall that \( T \) has diameter at least four. Suppose that \( T' \) has order one. Clearly, \( T \) is a tree with three support vertices \( v_2, v_3, v_4 \) and the remaining vertices are leaves. Hence \( \gamma_{tR}(T) = \gamma_{tR}(T') = 6 \), and thus the result holds. So suppose that \( T' \) is nontrivial. Then \( \gamma_{tR}(T) \leq \gamma_{tR}(T') + 4 \) and \( \gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4 \). By induction on \( T' \) we deduce that
\[
2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 12 + 6 < 3\gamma_{tr2}(T) - 2.
\]

This completes the proof. 

\( \blacksquare \)
3. Upper Bounds

In this section, we provide two upper bounds on the total 2-rainbow domination number of a tree. The first one we present is in terms of the order and the number of support vertices of a tree.

**Theorem 9.** If $T$ is a tree of order $n \geq 4$ with $s$ support vertices, then

$$
\gamma_{tr2}(T) \leq \frac{2(n + s)}{3},
$$

and this bound is sharp for $P_n$ such that $n \equiv 1 \pmod{3}$.

**Proof.** The proof is by induction on $n$. It is a routine matter to check that the statement holds if $n \in \{4, 5\}$. Hence, let $n \geq 6$ and assume that for every $T'$ or order $n'$ or with $s'$ support vertices satisfies $\gamma_{tr2}(T') \leq \frac{2(n' + s')}{3}$. Let $T$ be a tree of order $n$. If $T$ is a star, then $\gamma_{tr2}(T) = 3 < \frac{2(n+1)}{3}$. Likewise, if $T$ is a double star, then $\gamma_{tr2}(T) = 4 < \frac{2(n+2)}{3}$. Henceforth we can assume $T$ has diameter at least four.

If $T$ has a strong support vertex $u$ adjacent to at least three leaves, then let $T' = T - u'$, where $u'$ is a leaf neighbor of $u$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of $T$ by assigning $\emptyset$ to vertex $u'$, and thus $\gamma_{tr2}(T) \leq \gamma_{tr2}(T')$. The result follows by using the induction on $T'$, with $n' = n - 1$ and $s' = s$.

Therefore, we will assume that every support vertex of $T$ is adjacent to at most two leaves.

Let $v_1 v_2 \cdots v_k$ be a diametral path in $T$ and root $T$ in $v_k$. We consider the following cases.

**Case 1.** $\deg_T(v_2) = 3$. Thus $v_2$ has two leaf neighbors. We distinguish between the following situations.

**Subcase 1.1.** $\deg_T(v_3) \geq 3$. Suppose first that $v_3$ is a support vertex. Let $T' = T - T_{v_2}$. Then $n' = n - 3$ and $s' = s - 1$. Let $f$ be a $\gamma_{tr2}(T')$-function.

Since $v_3$ is a support vertex of $T'$, we must have $|f(v_3)| \geq 1$. Then the function $g : V(T) \to \mathcal{P}\{1, 2\}$ defined by $g(v_2) = \{1, 2\}$, $g(x) = \emptyset$ for $x \in L(v_2)$ and $g(x) = f(x)$ otherwise, is a T2RDF of $T$ of weight $\gamma_{tr2}(T') + 2$. By induction on $T'$, we have

$$
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.
$$

Suppose now that $v_3$ is not a support vertex. Thus every child of $v_3$ is a support vertex with degree either 2 or 3. Let $u_2$ be a child of $v_3$ different from $v_2$. If $\deg_T(u_2) = 3$, then let $T' = T - T_{u_2}$. By using a similar argument to that used above, we obtain $\gamma_{tr2}(T) < \frac{2(n+s)}{3}$. Thus let $\deg_T(u_2) = 2$ with $u_1$ as the unique
leaf of \(v_2\). Let \(T' = T - \{u_1, u_2\}\). Clearly, any \(\gamma_{tr2}(T')\)-function can be extended to a T2RDF of \(T\) by assigning the set \(\{1\}\) to both \(u_1\) and \(u_2\). Since \(n' = n - 2\) and \(s' = s - 1\), using the induction on \(T'\) we obtain
\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.
\]

Subcase 1.2. \(\deg_T(v_3) = 2\). Recall that since \(T\) has diameter at least four, \(\deg_T(v_4) \geq 2\). Assume that \(\deg_T(v_4) \geq 3\), and let \(T' = T - T_{v_3}\). Observe that \(T'\) has order \(n' \geq 3\). If \(n' = 3\), then \(T\) is a tree of order 7 with 2 support vertices, where \(\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = \frac{16}{3}\). Hence we assume that \(n' \geq 4\). Clearly, any \(\gamma_{tr2}(T')\)-function can be extended to a T2RDF of \(T\) by assigning \(\{1, 2\}\) to \(v_2\), \(\{1\}\) to \(v_3\) and \(\emptyset\) to the leaves of \(L(v_2)\). By induction on \(T'\) and using the fact that \(n = n - 4\) and \(s' = s - 1\) we obtain
\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.
\]
So, suppose for the sequel that \(\deg_T(v_4) = 2\). Let \(T' = T - T_{v_3}\). Note that \(n' \geq 3\). If \(n' = 3\), then \(T'\) has order 6 with 2 support vertices, where \(\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = \frac{16}{3}\). Hence let \(n' \geq 4\). By Observation 3, there exists a \(\gamma_{tr2}(T')\)-function \(f\) such that \(|f| = 1\) and clearly such a function can be extended to a T2RDF of \(T\) by assigning \(\{1, 2\}\) to \(v_2\) and \(\emptyset\) to the leaves of \(L(v_2)\). Hence \(\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2\). By induction on \(T'\) and using the fact that \(n = n - 3\) and \(s' = s\), we obtain
\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.
\]

Case 2. \(\deg_T(v_2) = 2\). Seeing the previous case, we may assume that every child of \(v_3\) which is a support vertex has degree two. Consider the following subcases.

Subcase 2.1. \(\deg_T(v_3) \geq 3\). Let \(T' = T - \{v_1, v_2\}\). Since any \(\gamma_{tr2}(T')\)-function can be extended to a T2RDF of \(T\) by assigning the set \(\{1\}\) to \(v_1\) and \(v_2\), \(\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2\). Using the induction on \(T'\), where \(n = n - 2\) and \(s' = s - 1\), we obtain
\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.
\]

Subcase 2.2. \(\deg_T(v_3) = 2\). We consider some additional subcases.

Subcase 2.2.1. \(\deg_T(v_4) \geq 3\). Let \(T' = T - \{v_1, v_2, v_3\}\). Note that \(n' \geq 3\). If \(n' = 3\), then \(T\) is a tree of order 6 with two support vertices, where \(\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3} = \frac{16}{3}\), and thus the result is valid. Hence let \(n' \geq 4\). Among all \(\gamma_{tr2}(T')\)-functions, let \(f\) be one such that \(|f| = 1\) as large as possible. If \(|f| \geq 1\),
then define the function $g$ on $V(T)$ as follows: $g(x) = f(x)$ for all $x \in V(T')$, $g(v_3) = \emptyset$ and $g(v_1) = g(v_2) = \{1\}$ or $\{2\}$ so that $g(N[v_3]) = \{1, 2\}$. Clearly, $g$ is a $T2RDF$ of $T'$ of weight $\gamma_{tr2}(T') + 2$. By induction on $T'$ and using the fact that $n' = n - 3$ and $s' = s - 1$ we deduce that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$

For the sequel we can assume that $f(v_4) = \emptyset$. Clearly in that case, $v_4$ is not a support vertex. By the choice of the diametral path and taking into account the previous cases, we can assume that every child of $v_4$ with depth two and different from $v_3$ has degree 2. We consider the following.

(i) $v_4$ has a child $u_2$ which is a support vertex. Since $f(v_4) = \emptyset$, we conclude that $\deg_T(u_2) = 2$. Let $v_1$ be the leaf neighbor of $u_2$ and let $T'' = T - \{u_1, u_2\}$. Clearly, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 2$ and $s'' = s - 1$. By induction on $T''$, it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

(ii) There is a pendant path $v_4v_3u_2v_1$ in $T$, where $u_3 \neq v_3$. Since $|f(v_4)| = 0$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| = 3$. Define the function $g$ on $T'$ by $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(v_4) = \{2\}$, and $g(x) = f(x)$ otherwise. Clearly $g$ is a $\gamma_{tr2}(T')$-function $|g(v_4)| > |f(v_4)| = 0$, contradicting our choice of $f$.

Subcase 2.2.2. $\deg_T(v_4) = 2$. If $\deg_T(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Note that $T'$ has order $n' \geq 3$. If $n = 3$, then $T$ is a path $P_6$, where $\gamma_{tr2}(P_6) = 5$ (by Proposition 4) and the result is valid. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$-function $f$ such that $|f(v_4)| = 1$, and such a function can be extended to a $T2RDF$ of $T$ by assigning $\emptyset$ to $v_3$, $\{1\}$ to $v_1$ and $\{1, 2\} - f(v_4)$ to $v_2$. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

Assume now that $\deg_T(v_5) \geq 3$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that $T'$ has order $n' \geq 3$. If $n' = 3$, then $T$ is a tree of order 7 obtained from a path $P_6$ by adding a new vertex attached to one of the two support vertices of the path $P_6$. It is easy to check that $\gamma_{tr2}(T) = 5 < \frac{2(n + s)}{3}$. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$-functions, let $f$ be one such that $|f(v_5)|$ is as large as possible. If $|f(v_5)| \geq 1$, then $f$ can be extended to a $T2RDF$ of $T$ by assigning $\emptyset$ to $v_3$, $\{1\}$ to $v_1$ and $v_2$, and either $\{1\}$ or $\{2\}$ to $v_3$ so that $f(N[v_4]) = \{1, 2\}$. By induction on $T'$, it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$
For the sequel, we can assume that \( f(v_5) = \emptyset \). Trivially, \( v_5 \) is not a support vertex. Also, every child of \( v_5 \) with depth one has degree two. We consider the following.

(i) \( v_5 \) has a child with depth 3. Let \( u_1 \neq v_1 \) be a leaf at distance four from \( v_5 \) and let \( v_5 u_4 u_3 u_2 u_1 \) be the unique path between \( u_1 \) and \( v_5 \). According to Cases 1 and 2 and Subcases 2.1 and 2.2, we must assume that each of \( u_4 \), \( u_3 \) and \( u_2 \) has degree two. Moreover, since \( f(v_5) = \emptyset \) as assumed and according to the choice of \( f \) maximizing \( |f(v_5)| \), we conclude that \( |f(u_1)| + |f(u_2)| + |f(u_3)| + |f(u_4)| = 4 \).

Define the function \( g \) on \( V(T') \) as follows: \( g(u_1) = g(u_2) = \{1\} \), \( g(u_3) = \emptyset \), \( g(u_4) = g(v_5) = \{2\} \) and \( g(x) = f(x) \) otherwise. Clearly, \( g \) is a \( \gamma_{tr2}(T') \)-function with \( |g(v_5)| > |f(v_5)| = 0 \), a contradiction.

(ii) \( v_5 \) has a child \( u_2 \) with depth one. Let \( u_1 \) be the leaf neighbor of \( u_2 \). Let \( T'' = T - \{u_1, u_2\} \). Obviously, \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \). It follows by induction on \( T'' \) that

\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq 2\left( \frac{n'' + s''}{3} \right) + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.
\]

(iii) \( v_5 \) has a child, say \( w \), with depth two having degree at least 3. Suppose first that \( w \) has at least two children as support vertices and let \( z \) be one of them having minimum degree. Note that \( \deg_T(z) \in \{2, 3\} \) since every support vertex of \( T \) has at most two leaves. Let \( T'' = T - \{z \cup L(z)\} \). Then \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \), \( n'' = n - 1 - |L(z)| \) and \( s'' = s - 1 \). Using the induction on \( T' \) we obtain the desired result. Now, let \( w \) has exactly one child, say \( t \), as a support neighbor. Since \( \deg_T(w) \geq 3 \), we deduce that \( w \) is a support vertex. Let \( T''' = T - T_w \). Note that \( T_w \) has order \( n_w \in \{4, 5, 6\} \). Moreover, it is clear that \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T''') + 4 \). It follows from the induction hypothesis that

\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T''') + 4 \leq \frac{2(n'' + s'')}{3} + 4 + \frac{2(n - n_w + s - 2)}{3} + 4 \leq \frac{2(n + s)}{3}.
\]

(iv) \( v_5 \) has a child, say \( w \), with depth two and having degree 2. Suppose first that the child \( z \) of \( w \) is a strong support. Let \( L(z) = \{z_1, z_2\} \) and let \( T'' = T - \{w, z_1, z_2\} \). Then \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3 \), \( n'' = n - 4 \) and \( s'' = s - 1 \). It follows from the induction on \( T'' \) that

\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3 \leq \frac{2(n'' + s'')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.
\]

Now, suppose that the child \( z \) of \( w \) is a support vertex of degree two. Let \( \deg_T(v_5) = k \geq 3 \) and \( H_t \) for \( t \geq 2 \) be the tree obtained from a star \( K_{1,k} \) by subdividing one edge three times and each of the remaining edges exactly twice. Seeing the previous situations, clearly \( T_{v_5} \) is isomorphic to \( H_{k-1} \). Now let \( T'' = T - T_{v_5} \). We note that \( T'' \) has order \( n' \geq 3 \). If \( n' = 3 \), then \( T'' = H_k \), where \( n = 3k + 2 \),
s(T) = k and \( \gamma_{tr2}(T) = 2k + 2 < \frac{2(n+s)}{3} \). Hence we can assume that \( n' \geq 4 \). Then \( \gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k, n' = n - 3k + 1 \) and \( s(T') \leq s(T) - (k - 1) + 1 \). It follows from the induction on \( T' \) that

\[
\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k \leq \frac{2(n' + s')}{3} + 2k
\]

\[
= \frac{2(n - 3k + 1 + s - k + 2)}{3} + 2k \leq \frac{2(n + s)}{3}.
\]

This completes the proof.

Next we establish an upper bound on the total 2-rainbow domination number of a tree in terms of the vertex cover number. We first give an upper bound for arbitrary graphs.

**Lemma 10.** Let \( G \) be a graph of order \( n \geq 2 \) with no isolated vertex and \( V_c \) a minimum vertex cover of \( G \). Then

\[
\gamma_{tr2}(G) \leq 2\beta(G) + r,
\]

where \( r \) is the number of isolated vertices in the subgraph induced by \( V_c \). This bound is sharp for the graphs in Figure 2.

![Figure 2](image-url)

**Figure 2.** Two graphs \( G \) with \( \gamma_{tr2}(G) = 2\beta(G) + r \).

**Proof.** Let \( V_c \) be a minimum vertex cover of \( G \) and \( I \) the set of isolated vertices in \( G[V_c] \). Let \( K = V(G) - V_c \). Since \( K \) is a maximum independent set, every vertex of \( V_c \) has a neighbor in \( K \). Let \( D \) be a smallest subset of vertices of \( K \) that dominates all vertices of \( I \). Obviously, \( |D| \leq |I| = r \). Now define a function \( f : V(G) \to \mathcal{P}\{1,2\} \) by \( f(x) = \{1,2\} \) if \( x \in V_c \), \( f(x) = \{1\} \) if \( x \in D \) and \( f(x) = \emptyset \) otherwise. Clearly, \( f \) is a T2RDF of \( G \) of weight \( 2|V_c| + |D| \leq 2|V_c| + r \).

The proof of the next the result is inspired by the proof of Theorem 2 in [9].
Theorem 11. Let $T$ be a tree of order $n \geq 3$ and let $S'$ be the set of isolated vertices in the subgraph induced by the set of support vertices of $T$. Then

$$\gamma_{tr2}(T) \leq 2\beta(T) + |S'|.$$ 

This bound is sharp for the graph in Figure 3.

Figure 3. A tree $T$ with $\gamma_{tr2}(T) = 2\beta(T) + |S'|$.

Proof. Let $L$ and $S$ denote the set of leaves and support vertices of a tree $T$, respectively. Let $V_f$ be a maximum independent set that contains all leaves of $T$. Then $V_c = V - V_f$ is a vertex cover set of $T$. Note that $S \subseteq V_c$. If no support vertex of $T$ is isolated in $T[V_c]$, then the result holds by Lemma 10. Hence, assume that $u$ is a support vertex which is isolated in $T[V_c]$. Root $T$ at $u$ and let $A_1 = \{u\}$ and $A_2 = N(u)$. Clearly, $A_1 \subseteq V_c$ and $A_2 \subseteq V_f$. Assume that $A_3 = (N(A_2) - A_1) \cup B_{N(A_2) - A_1}$, where $B_{N(A_2) - A_1} = \{v \in V_c \mid v \text{ is in a component of } T[V_c] \text{ with a vertex of } N(A_2) - A_1\}$. Set $A_4 = N(A_3) - A_2$. Then we have $A_3 \subseteq V_c$ and $A_4 \subseteq V_f$.

We repeat this process so that at some odd number step $2k + 1$, we put

$$A_{2k+1} = (N(A_{2k}) - A_{2k-1}) \cup B_{N(A_{2k}) - A_{2k-1}},$$

where $B_{N(A_{2k}) - A_{2k-1}} = \{v \in V_c \mid v \text{ is in a component of } T[V_c] \text{ with a vertex of } N(A_{2k}) - A_{2k-1}\}$ and we set $A_{2k+2} = N(A_{2k+1}) - A_{2k}$. This process will terminate at some $m$th step where $m$ is even and $A_m$ composed only of leaves. Note that $A_1 \cup \cdots \cup A_m$ is a partition of $V(T)$. Obviously, $V_f = A_2 \cup \cdots \cup A_{m-2} \cup A_m$ and $V_c = A_1 \cup A_3 \cup \cdots \cup A_{m-3} \cup A_{m-1}$. Note that if $v \in A_i$, for $i > 1$, has a neighbor in $A_{i-1}$, then it has only one neighbor in $A_{i-1}$.

Let $D_1 = V_c$. If $T[V_c]$ has isolated vertices that are support vertices in $T$, then let $K$ be a smallest subset of vertices of $V_f - L$ that dominates these isolated support vertices. Clearly, $|K| \leq |S'|$. Now we consider the isolated vertices of $T[V_c]$ that are not support vertex in $T$. In decreasing order, we visit each $A_i$ with odd index $i$, where $3 \leq i \leq m - 1$. We start with $A_{m-1}$ and observe that if there is an isolate of $T[V_c]$ in $A_{m-1}$, then it is a support vertex and some vertex of $K$ is adjacent to it. Now for each non-support isolated vertex $v$ of $T[V_c]$ which is in $A_{m-3}$, if $N(v) \cap A_{m-2}$ is dominated by $A_{m-1} \cap V_c$, then remove $v$ from $D_1$ and add to $D_1$ its unique neighbor in $A_{m-4}$, otherwise we leave $v$ in $D_1$. Continue this way for each odd $i$ in decreasing order. That is, in general for $A_i$ where $i$ is odd,
if a non-support isolated vertex \( v \) of \( T[V_c] \) is in \( A_i \) and \( N(u) \cap A_{i+1} \) are dominated by \( A_{i+2} \cap V_c \), then remove \( v \) from \( D_1 \) and add its unique neighbor in \( A_{i-1} \) to \( D_1 \), otherwise we leave \( v \) in \( D_1 \). This process terminates after \( i = 3 \). Now, if some vertex of \( A_2 \) is in \( K \), then we are done. Otherwise remove \( u \) from \( D_1 \) and add to \( D_1 \) one of its neighbors. Note that \( |D_1| \) has not increased. Now let \( D_2 = D_1 \cup K \).

Using an argument similar to that described in the proof of Theorem 2 in [9], we see that the induced subgraph \( T[D_2] \) has no isolated vertex. Define the function \( f : V(T) \to \mathcal{P}\{1, 2\} \) by \( f(x) = \{1, 2\} \) for \( x \in D_1 \), \( f(x) = \{1\} \) for \( x \in K \) and \( f(x) = \emptyset \) otherwise. Clearly, \( f \) is a T2RDF of \( T \) and thus

\[
\gamma_{tr2}(T) \leq 2|V_c| + |K| \leq 2\beta(T) + |S'|.
\]

This achieves that proof.

\[\square\]

4. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL 2-RAINBOW DOMINATION:

**TOTAL 2-RAINBOW DOMINATION**

**Instance.** Graph \( G = (V, E) \), positive integer \( k \leq |V| \).

**Question.** Does \( G \) have a total 2-rainbow dominating function of weight at most \( k \)?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, EXACT-3-COVER (X3C), to TOTAL 2-RAINBOW DOMINATION.

**EXACT 3-COVER (X3C)**

**Instance.** A finite set \( X \) with \( |X| = 3q \) and a collection \( C \) of 3-element subsets of \( X \).

**Question.** Is there a subset \( C' \) of \( C \) such that every element of \( X \) appears in exactly one element of \( C' \)?

**Theorem 12.** TOTAL 2-RAINBOW DOMINATION is NP-complete for bipartite graphs.

**Proof.** TOTAL 2-RAINBOW DOMINATION is a member of NP, since we can check in polynomial time that a function \( f : V \to \{0, 1, 2\} \) has weight at most \( k \) and is a T2RDF. Now let us show how to transform any instance of X3C into an instance of TOTAL 2-RAINBOW DOMINATION so that one of them has a
solution if and only if the other one has a solution. Let \( X = \{x_1, x_2, \ldots, x_{3q}\} \) and \( C = \{C_1, C_2, \ldots, C_t\} \) be an arbitrary instance of X3C.

For each \( x_i \in X \), we build a graph \( H_i \) obtained from a path \( P_2 : x_i-y_i \) and two stars \( K_{1,3} \) with centers \( a_i \) and \( b_i \), by adding edges \( y_i a_i \) and \( y_i b_i \). Hence, each \( H_i \) has order 10. For each \( C_j \in C \), we build a double star \( S_{3,3} \) with support vertices \( u_j \) and \( v_j \). Let \( c_j \) be a leaf of the double star \( S_{3,3} \). Let \( Y = \{c_1, c_2, \ldots, c_t\} \). Now to obtain a graph \( G \), we add edges \( c_j x_i \) if \( x_i \in C_j \). Clearly, \( G \) is a bipartite graph (for example, see Figure 4). Set \( k = 4t + 16q \). Observe that for every T2RDF \( f \) on \( G \), each \( H_i \) has weight at least 5 and each double star \( S_{3,3} \) has weight at least 4.

![Figure 4. NP-completeness for bipartite graphs.](image)

Suppose that the instance \( X, C \) of X3C has a solution \( C' \). We construct a T2RDF \( f \) on \( G \) of weight \( k \). For each \( i \), assign the set \( \{1, 2\} \) to \( a_i, b_i \), the set \( \{1\} \) to \( y_i \), and \( \emptyset \) to the remaining vertices of \( H_i \). For every \( j \), assign \( \{1, 2\} \) to \( u_j \) and \( v_j \), and \( \emptyset \) to each leaf. In addition, if for every \( C_j \), assign to \( c_j \) the set \( \{2\} \) if \( C_j \subseteq C' \) and \( \emptyset \) if \( C_j \not\subseteq C' \). Note that since \( C' \) exists, its cardinality is precisely \( q \), and so the number of \( c_j \)'s assigned \( \{2\} \) is \( q \), having disjoint neighborhoods in \( \{x_1, x_2, \ldots, x_{3q}\} \). Since \( C' \) is a solution for X3C, every vertex \( x_i \) in \( X \) satisfies \( f(N[x_i]) = \{1, 2\} \). Hence, it is straightforward to see that \( f \) is a T2RDF with weight \( f(V) = 4t + q + 15q = k \).

Conversely, suppose that \( G \) has a T2RDF with weight at most \( k \). Among all such functions, let \( g = (V_0, V_1, V_2, V_{12}) \) be one such that the number of vertices of \( \{y_1, y_2, \ldots, y_{3q}\} \) assigned \( \{1, 2\} \) is as small as possible. As observed above, since each \( H_i \) has weight at least 5, we may assume that \( g(a_i) = g(b_i) = \{1, 2\} \) and \( |g(y_i)| > 0 \) so that vertices \( a_i, b_i \) are not isolated in the subgraph induced by \( V_1 \cup V_2 \cup V_{12} \). Hence each leaf neighbor of \( a_i \) or \( b_i \) is assigned \( \emptyset \) under \( g \). Assume...
that $g(y_i) = \{1, 2\}$ for some $i$. Observe that if $|g(x_i)| > 0$, then reassigning $\{1\}$ to $y_i$ provides a T2RDF $g'$ with less vertices $y_i$ assigned $\{1, 2\}$ than under $g$, contradicting our choice of $g$. Hence $g(x_i) = \emptyset$. But then reassigning $\{1\}$ to each of $y_i$ and $x_i$ instead of $\{1, 2\}$ and $\emptyset$, respectively, provides a T2RDF $g'$ with less vertices $y_i$ assigned $\{1, 2\}$ than under $g$, a contradiction too. Therefore $|g(y_i)| = 1$ for every $i \in \{1, 2, \ldots, 3q\}$. On the other hand, the total weight of all double stars corresponding to elements of $C$ is $4t$. In this case, we can assume that $g(u_j) = g(v_j) = \{1, 2\}$ and so each leaf neighbor of $u_j$ or $v_j$ is assigned $\emptyset$ under $g$. Note that each $c_j$ can be assigned $\emptyset$ since $g(u_j) = \{1, 2\}$. Since $w(g) \leq 4t + 16q$ and the total weight assigned to vertices of $V(G) - (X \cup Y)$ is $4t + 15q$, we have to assign to vertices of $(X \cup Y)$ sets whose total cardinalities not exceeding $q$ so that each vertex $x_i \in X$ has either $|g(x_i)| > 0$ or has two neighbors in $V_1 \cup V_2$ so that $f(N[x_i]) = \{1, 2\}$. Since $|X| = 3q$, it is clear that this is only possible if there are $q$ vertices of $\{c_1, c_2, \ldots, c_t\}$ belonging to $V_1 \cup V_2$. Since each $c_j$ has a exactly three neighbors in $\{x_1, x_2, \ldots, x_{3q}\}$, we deduce that $C' = \{c_j : |g(c_j)| = 1\}$ is an exact cover for $C$.

The next result is obtained by using the same proof as for Theorem 12 on the (same) graph $G$ built for the transformation by adding all edges between the $c_j$’s so that the resulting graph is chordal.

**Theorem 13.** TOTAL 2-RAINFLOW DOMINATION is NP-complete for chordal graphs.

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