HAMILTONIAN CYCLE PROBLEM IN STRONG
k-QUASI-TRANSITIVE DIGRAPHS
WITH LARGE DIAMETER

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Abstract

Let $k$ be an integer with $k \geq 2$. A digraph is $k$-quasi-transitive, if for any path $x_0x_1\ldots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. Let $D$ be a strong $k$-quasi-transitive digraph with even $k \geq 4$ and diameter at least $k + 2$. It has been shown that $D$ has a Hamiltonian path. However, the Hamiltonian cycle problem in $D$ is still open. In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for $D$ to be Hamiltonian.

Keywords: quasi-transitive digraph, $k$-quasi-transitive digraph, Hamiltonian cycle.

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1. Terminology and Introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \rightarrow y$ if $xy \in A(D)$, and also, we will write $\overrightarrow{xy}$ if $x \rightarrow y$ or $y \rightarrow x$. For disjoint subsets $X$ and $Y$ of $V(D)$, $X \rightarrow Y$ means that every vertex of $X$ dominates every vertex of $Y$, $X \Rightarrow Y$ means that there is no arc from $Y$ to $X$ and $X \Rightarrow Y$ means that both of $X \rightarrow Y$ and $X \Rightarrow Y$ hold. For subsets $X, Y$ of $V(D)$, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $X = \{x\}$, then we write $(x, Y)$ instead of $(\{x\}, Y)$. Likewise, if $Y = \{y\}$, then we write $(X, y)$ instead of $(X, \{y\})$. Let
Let $x$ and $y$ be two vertices of $V(D)$. The distance from $x$ to $y$ in $D$, denoted $d(x, y)$, is the minimum length of an $(x, y)$-path, if $y$ is reachable from $x$, and otherwise $d(x, y) = \infty$. The distance from a set $X$ to a set $Y$ of vertices in $D$ is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The diameter of $D$ is $\text{diam}(D) = \text{diam}(V(D), V(D))$. Clearly, $D$ has finite diameter if and only if it is strong.

Let $P = v_1v_2\cdots v_p$ be a path or a cycle of $D$. If $i \neq j$, $v_i, v_j \in V(P)$ we denote by $P[v_i, v_j]$ the subpath of $P$ from $v_i$ to $v_j$. Let $Q = u_1u_2\cdots u_q$ be a vertex-disjoint path or cycle with $P$ in $D$. If there exist $v_i \in V(P)$ and $u_j \in V(Q)$ such that $v_iu_j \in A(D)$, then we will use $P[v_i, v_j]Q[u_j, u_q]$ to denote the path $v_1v_2\cdots v_iu_ju_{j+1}\cdots u_q$. A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, $x_0$ and $x_2$ are adjacent. The concept of $k$-quasi-transitive digraphs was introduced in [2] as a generalization of quasi-transitive digraphs. A digraph is $k$-quasi-transitive, if for any path $x_0x_1\cdots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. The $k$-quasi-transitive digraphs have been studied in [2–7].

In [7], Wang and Zhang showed that a strong $k$-quasi-transitive digraph $D$ with even $k \geq 4$ and $\text{diam}(D) \geq k + 2$ has a Hamiltonian path and proposed the following problem. Let $k$ be an even integer with $k \geq 4$. Is it true that every strong $k$-quasi-transitive digraph with diameter at least $k + 2$ is Hamiltonian?

In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for it to be Hamiltonian.

2. Main Results

For the rest of this paper, let $k$ be an even integer with $k \geq 4$ and $D$ denote a strong $k$-quasi-transitive digraph with $\text{diam}(D) \geq k + 2$. There exist two vertices $u, v$ such that $d(u, v) = k + 2$ in $D$. Let $P = x_0x_1\cdots x_{k+2}$ denote a shortest $(u, v)$-path in $D$, where $u = x_0$ and $v = x_{k+2}$.

**Theorem 1** [7]. The subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq k + 2$.

**Lemma 2** [5]. Let $k$ be an integer with $k \geq 2$ and $D$ be a strong $k$-quasi-transitive digraph. Suppose that $C = x_0x_1\cdots x_{n-1}x_0$ is a cycle of length $n$ with $n \geq k$ in $D$. Then for any $x \in V(D) \setminus V(C)$, $x$ and $C$ are adjacent.

By Theorem 1, $x_{k+2} \to x_0$. So $x_0x_1\cdots x_{k+2}x_0$ is a cycle of length $k + 3$. By Lemma 2, every vertex of $V(D) \setminus V(P)$ is adjacent to $P$. Hence we can divide
Hamiltonian Cycle Problem in Strong $k$-Quasi-Transitive ... 3

$V(D) \setminus V(P)$ into three subsets:

$$O = \{x \in V(D) \setminus V(P) : V(P) \Rightarrow x\},$$

$$I = \{x \in V(D) \setminus V(P) : x \Rightarrow V(P)\},$$

and

$$B = V(D) \setminus (V(P) \cup O \cup I).$$

One of $I, O$ and $B$ may be empty.

**Theorem 3** [7]. The subdigraph induced by $V(D) \setminus V(P)$ is a semicomplete digraph.

**Lemma 4** [7]. For any $x \in B$, either $x$ is adjacent to every vertex of $V(P)$ or $\{x_{k+2}, x_{k+1}, x_k, x_{k-1}\} \rightarrow x \Rightarrow \{x_0, x_1, x_2, x_3\}$. In particular, if $k = 4$, then $x$ is adjacent to every vertex of $V(P)$.

From the proof of Lemma 2.11 in [7], we have the following result.

**Lemma 5** [7]. $V(P) \rightarrow O$ and $I \rightarrow V(P)$.

By Theorems 1 and 3 and Lemmas 4 and 5, a strong 4-quasi-transitive digraph $D$ with $\text{diam}(D) \geq 6$ is in fact a semicomplete digraph. It is well known that a strong semicomplete digraph is Hamiltonian. Hence, for the rest of this paper, we consider the case $k \geq 6$.

**Lemma 6.** Let $H$ be a digraph and $u, v \in V(H)$ such that $d(u, v) = n$ with $n \geq 4$. Let $Q = x_0, x_1 \cdots x_n$ be a shortest $(u, v)$-path in $H$. If $H[V(Q)]$ is a semicomplete digraph, then, for any $x_i, x_j \in V(Q)$ with $0 \leq i < j \leq n$, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-1\}$ in $H[V(Q)]$.

**Proof.** We prove the result by induction on $n$. For $n = 4$, it is not difficult to check that the result is true. Suppose $n \geq 5$. Assume $j - i = n$. It must be $j = n$ and $i = 0$. Then the length of the path $x_n P[x_2, x_p] x_0$ is $p$ with $p \in \{2, 3, \ldots, n-1\}$. Now assume $1 \leq j - i \leq n - 1$. Then $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$ or $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, assume that $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$. By induction, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-2\}$. Now we only need to show that there exists a path of length $n - 1$ from $x_j$ to $x_i$. If $j - i = 1$, then $P[x_j, x_{n-1}] P[x_0, x_1]$ is the desired path. If $j - i = 2$, then $P[x_j, x_n] P[x_0, x_1]$ is the desired path. If $3 \leq j - i \leq n - 1$, then $P[x_j, x_n] P[x_{i+2}, x_{j-1}] P[x_0, x_1]$ is the desired path.

By Lemma 6, we can obtain the following lemma.
Lemma 7. For any \( x \in V(D) \setminus V(P) \) and \( x_i \in V(P) \), if \( x \to x_i \), then \( x \) and every vertex of \( \{x_0, x_1, \ldots, x_{i-1}\} \) are adjacent; if \( x_i \to x \), then \( x \) and every vertex of \( \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \) are adjacent.

Proof. If \( x \to x_i \), then for any \( x_j \in \{x_0, x_1, \ldots, x_{i-1}\} \), by Lemma 6, there exists a path \( Q \) of length \( k - 1 \) from \( x_i \) to \( x_j \). Then the path \( xQ \) implies \( x \to x_j \). If \( x_i \to x \), then for any \( x_j \in \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \), by Lemma 6, there exists a path \( R \) of length \( k - 1 \) from \( x_j \) to \( x_i \). Then the path \( Rx \) implies \( x \to x_i \).

Using Lemma 7, Lemma 4 can be improved to the following result.

Lemma 8. For any \( x \in B \), either \( x \) and every vertex of \( V(P) \) are adjacent or there exist two vertices \( x_1, x_s \in V(P) \) with \( 4 \leq t + 1 < s \leq k - 1 \) such that \( \{x_s, \ldots, x_{k+2}\} \not\to x \not\to \{x_0, \ldots, x_t\} \).

Proof. If \( x \) and every vertex of \( V(P) \) are adjacent, then we are done. Suppose not. By the definition of \( B \), \((x, V(P)) = \emptyset \) and \((V(P), x) \neq \emptyset \). Take \( t = \max\{i : x \to x_i\} \) and \( s = \min\{j : x_j \to x\} \). By Lemma 7, \( x \) and every vertex \( \{x_0, \ldots, x_t\} \cup \{x_s, \ldots, x_{k+2}\} \) are adjacent. Moreover, since \( x \) and some vertex of \( V(P) \) are not adjacent, we can conclude \( s > t + 1 \) and \( \{x_s, \ldots, x_{k+2}\} \not\to x \not\to \{x_0, \ldots, x_t\} \).

Lemma 9. Let \( Q = z_0z_1 \cdots z_n \) be a path of length \( n \) with \( 1 \leq n \leq k - 1 \) in \( D - V(P) \). For some \( x_i \in V(P) \), if \( z_n \to x_i \), then \( z_0 \) and \( x_i \) are adjacent; if \( x_i \to z_0 \), then \( z_n \) and \( x_i \) are adjacent, where the subscripts are taken modulo \( k + 3 \).

Proof. Using the definition of \( k \)-quasi-transitive digraphs, the proof is easy and so we omit it.
exist \( x_j \in V(P) \) and \( y' \in O \) such that \( y' \to v \to x_j \). Then the path \( x_0y'vx_j \) implies that \( d(x_0, x_j) \leq 3 \). Hence \( j \leq 3 \), which means \( \{x_4, x_5, \ldots, x_{k+2}\} \to v \).

Note that \( k - 2 \geq 4 \). Thus \( u \neq v \), which implies \( B_2' \cap B_2'' = \emptyset \). Hence \( B_2' \to O \) and \( I \to B_2'' \). If \( z \to u \), then considering the path \( zux' \), by Lemma 9, \( z \) and every vertex of \( V(P) \) are adjacent, a contradiction. Hence \( B_2' \to B_1 \). If \( v \to z \), then considering the path \( y'vx \), by Lemma 9, \( z \) and every vertex of \( V(P) \) are adjacent, a contradiction. Hence \( B_1 \to B_2'' \). If \( k = 6 \), denote the path \( R = x_{k+2}x_n \). If \( k \geq 7 \), by Lemma 6, there exists a path of length \( k - 5 \) from \( x_{k+2} \) to \( x_n \), denote it by \( R \). If \( v \to u \), then \( zy'vx'R \) implies \( xy_n \), a contradiction. Hence \( B_2' \to B_2'' \).

**Theorem 10.** If \( D - V(P) \) is strong, then \( D \) is Hamiltonian.

**Proof.** By Lemma 3, \( D - V(P) \) is a semicomplete digraph. Hence \( D - V(P) \) contains a Hamiltonian cycle, denote it by \( H = y_0y_1 \cdots y_my_0 \). Clearly, if there exists a pair of arcs \( x_i, x_{i+1} \in A(P) \) and \( y_jy_{j+1} \in A(H) \) such that \( x_i \to y_j \to x_{j+1} \), then \( D \) contains a Hamiltonian cycle \( x_iH[y_{j+1}, y_j]P[x_{j+1}, x_i] \). Next we shall find out such a pair of arcs. Suppose \( O \neq \emptyset \). Since \( D \) is strong, \( B \cup I \neq \emptyset \) and there exists a Hamiltonian cycle \( x_iH[y_j, y_{j+1}]P[x_{j+1}, x_i] \). There exists \( x_i \in V(P) \) such that \( y_j \to x_i \). Then \( y_jy_{j+1} \) and \( x_{j+1} \) are the desired arcs. Now assume \( O = \emptyset \). Analogously, assume \( I = \emptyset \) and so \( V(D) \setminus V(P) = B \). If \( B_1 = \emptyset \), then \( D \) is semicomplete and so \( D \) is Hamiltonian. Now assume \( B_1 \neq \emptyset \). If \( |V(H)| = 1 \), then \( y_0 \in B_1 \) and \( x_{k+2}y_0x_0 \) is a Hamiltonian cycle of \( D \). Assume \( |V(H)| \geq 2 \). If there exist two consecutive vertices \( y_j, y_{j+1} \in B_1 \), then \( y_jy_{j+1} \) and \( x_{k+2}x_0 \) are the desired arcs. Assume there is no such a pair of arcs. So there exists a pair of vertices \( y_j, y_{j+1} \in V(H) \) such that \( y_j \in B_2 \) and \( y_{j+1} \in B_1 \). If \( y_j \to x_0 \), then \( y_j \to x_0 \) and \( y_{j+1} \to x_{k+2} \) are the desired arcs. Assume \( x_0 \to y_j \). If \( |V(H)| = 2 \), then \( x_0y_jy_{j+1}x_1x_2 \cdots x_{k+2}x_0 \) is a Hamiltonian cycle of \( D \). Assume \( |V(H)| \geq 3 \). According to the above argument, \( y_{j+2} \in B_2 \). If \( y_{j+2} \to x_{k+2} \), then \( x_0y_jy_{j+1}y_{j+2}x_{k+2} \) is a path of length \( 4 \) from \( x_0 \) to \( x_{k+2} \), a contradiction to \( d(x_0, x_{k+2}) \geq 8 \). Thus \( x_{2k+2} \to y_{j+2} \). Then \( y_{j+1}y_{j+2} \) and \( x_{k+2}x_0 \) are the desired arcs.

**Theorem 11.** If \( B_2 = \emptyset \) or for any \( x \in B_2 \), \( x_{k+2} \to x \to x_0 \), then \( D \) is Hamiltonian.

**Proof.** If \( D - V(P) \) is strong, then, by Theorem 10, we are done. If \( D - V(P) \) is not strong, then let \( D_1, D_2, \ldots, D_t \) be strong components of \( D - V(P) \), where \( t \geq 2 \). Since \( D \) is strong, there exist \( x \in V(D_1) \) and \( y \in V(D_t) \) such that \( (V(P), x) \neq \emptyset \) and \( (y, V(P)) \neq \emptyset \). By the hypothesis of this theorem and Lemmas 4 and 5, \( x_{k+2} \to x \) and \( y \to x_0 \). It is easy to see that there exists a Hamiltonian path \( R \) from \( x \) to \( y \) in \( D - V(P) \). So \( x_{k+2}Rx_0x_1 \cdots x_{k+2} \) is a Hamiltonian cycle of \( D \).
Suppose $D - V(P)$ is not strong and there exists a vertex $u \in B_2$ such that $u \mapsto x_{k+2}$, we may construct some $k$-quasi-transitive digraphs such that they are not Hamiltonian. For example, let $V(D) \setminus V(P) = \{u, v\}$ and $u \rightarrow v$, $\{x_{k-1}, x_k, x_{k+1}, x_{k+2}\} \rightarrow v \rightarrow \{x_0, x_1, x_2, x_3\}$ and $x_{k+1} \rightarrow u \rightarrow \{x_0, x_1, \ldots, x_k, x_{k+2}\}$. It is not difficult to see that $D$ contains no Hamiltonian cycle.

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References


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