HAMILTONIAN CYCLE PROBLEM IN STRONG $k$-QUASI-TRANSITIVE DIGRAPHS
WITH LARGE DIAMETER

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Abstract

Let $k$ be an integer with $k \geq 2$. A digraph is $k$-quasi-transitive, if for any path $x_0x_1\ldots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. Let $D$ be a strong $k$-quasi-transitive digraph with even $k \geq 4$ and diameter at least $k + 2$. It has been shown that $D$ has a Hamiltonian path. However, the Hamiltonian cycle problem in $D$ is still open. In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for $D$ to be Hamiltonian.

Keywords: quasi-transitive digraph, $k$-quasi-transitive digraph, Hamiltonian cycle.

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1. Terminology and Introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $x \to y$ or $y \to x$. For disjoint subsets $X$ and $Y$ of $V(D)$, $X \to Y$ means that every vertex of $X$ dominates every vertex of $Y$, $X \Rightarrow Y$ means that there is no arc from $Y$ to $X$ and $X \Rightarrow Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. For subsets $X, Y$ of $V(D)$, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $X = \{x\}$, then we write $(x, Y)$ instead of $(\{x\}, Y)$. Likewise, if $Y = \{y\}$, then we write $(X, y)$ instead of $(X, \{y\})$. Let
$D'$ be a subdigraph of $D$ and $x \in V(D) \setminus V(D')$. We say that $x$ and $D'$ are adjacent if $x$ and some vertex of $D'$ are adjacent. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of $D$ induced by the vertex set $S$.

Let $x$ and $y$ be two vertices of $V(D)$. The distance from $x$ to $y$ in $D$, denoted $d(x, y)$, is the minimum length of an $(x, y)$-path, if $y$ is reachable from $x$, and otherwise $d(x, y) = \infty$. The distance from a set $X$ to a set $Y$ of vertices in $D$ is $d(X, Y) = \max \{d(x, y) : x \in X, y \in Y\}$. The diameter of $D$ is $\text{diam}(D) = d(V(D), V(D))$. Clearly, $D$ has finite diameter if and only if it is strong.

Let $P = v_1v_2 \cdots v_p$ be a path or a cycle of $D$. For $i \neq j$, $v_i, v_j \in V(P)$ we denote by $P[v_i, v_j]$ the subpath of $P$ from $v_i$ to $v_j$. Let $Q = u_1u_2 \cdots u_q$ be a vertex-disjoint path or cycle with $P$ in $D$. If there exist $v_i \in V(P)$ and $u_j \in V(Q)$ such that $v_iu_j \in A(D)$, then we will use $P[v_i, v_j]Q[u_j, u_q]$ to denote the path $v_1v_2 \cdots v_iu_ju_{j+1} \cdots u_q$.

A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, $x_0$ and $x_2$ are adjacent. The concept of $k$-quasi-transitive digraphs was introduced in [2] as a generalization of quasi-transitive digraphs. A digraph is $k$-quasi-transitive, if for any path $x_0x_1 \cdots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. The $k$-quasi-transitive digraphs have been studied in [2–7].

In [7], Wang and Zhang showed that a strong $k$-quasi-transitive digraph $D$ with even $k \geq 4$ and $\text{diam}(D) \geq k + 2$ has a Hamiltonian path and proposed the following problem. Let $k$ be an even integer with $k \geq 4$. Is it true that every strong $k$-quasi-transitive digraph with diameter at least $k + 2$ is Hamiltonian?

In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for it to be Hamiltonian.

## 2. Main Results

For the rest of this paper, let $k$ be an even integer with $k \geq 4$ and $D$ denote a strong $k$-quasi-transitive digraph with $\text{diam}(D) \geq k + 2$. There exist two vertices $u, v$ such that $d(u, v) = k + 2$ in $D$. Let $P = x_0x_1 \cdots x_{k+2}$ denote a shortest $(u, v)$-path in $D$, where $u = x_0$ and $v = x_{k+2}$.

**Theorem 1** [7]. The subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \rightarrow x_i$ for $1 \leq i + 1 < j \leq k + 2$.

**Lemma 2** [5]. Let $k$ be an integer with $k \geq 2$ and $D$ be a strong $k$-quasi-transitive digraph. Suppose that $C = x_0x_1 \cdots x_{n-1}x_0$ is a cycle of length $n$ with $n \geq k$ in $D$. Then for any $x \in V(D) \setminus V(C)$, $x$ and $C$ are adjacent.

By Theorem 1, $x_{k+2} \rightarrow x_0$. So $x_0x_1 \cdots x_{k+2}x_0$ is a cycle of length $k + 3$. By Lemma 2, every vertex of $V(D) \setminus V(P)$ is adjacent to $P$. Hence we can divide
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$V(D) \setminus V(P)$ into three subsets:

$$ O = \{ x \in V(D) \setminus V(P) : V(P) \Rightarrow x \}, $$

$$ I = \{ x \in V(D) \setminus V(P) : x \Rightarrow V(P) \}, $$

and

$$ B = V(D) \setminus (V(P) \cup O \cup I). $$

One of $I, O$ and $B$ may be empty.

**Theorem 3** [7]. The subdigraph induced by $V(D) \setminus V(P)$ is a semicomplete digraph.

**Lemma 4** [7]. For any $x \in B$, either $x$ is adjacent to every vertex of $V(P)$ or $\{ x_{k+2}, x_{k+1}, x_k, x_{k-1} \} \Rightarrow x \Rightarrow \{ x_0, x_1, x_2, x_3 \}$. In particular, if $k = 4$, then $x$ is adjacent to every vertex of $V(P)$.

From the proof of Lemma 2.11 in [7], we have the following result.

**Lemma 5** [7]. $V(P) \Rightarrow O$ and $I \Rightarrow V(P)$.

By Theorems 1 and 3 and Lemmas 4 and 5, a strong 4-quasi-transitive digraph $D$ with $\text{diam}(D) \geq 6$ is in fact a semicomplete digraph. It is well known that a strong semicomplete digraph is Hamiltonian. Hence, for the rest of this paper, we consider the case $k \geq 6$.

**Lemma 6.** Let $H$ be a digraph and $u, v \in V(H)$ such that $d(u, v) = n$ with $n \geq 4$. Let $Q = x_0x_1 \cdots x_n$ be a shortest $(u, v)$-path in $H$. If $H[Q]$ is a semicomplete digraph, then, for any $x_i, x_j \in V(Q)$ with $0 \leq i < j \leq n$, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-1\}$ in $H[Q]$.

**Proof.** We prove the result by induction on $n$. For $n = 4$, it is not difficult to check that the result is true. Suppose $n \geq 5$. Assume $j - i = n$. It must be $j = n$ and $i = 0$. Then the length of the path $x_nP[x_2, x_p]x_0$ is $p$ with $p \in \{2, 3, \ldots, n-1\}$. Now assume $1 \leq j - i \leq n - 1$. Then $x_i, x_j \in \{ x_0, x_1, \ldots, x_{n-1} \}$ or $x_i, x_j \in \{ x_1, x_2, \ldots, x_n \}$. Without loss of generality, assume that $x_i, x_j \in \{ x_0, x_1, \ldots, x_{n-1} \}$. By induction, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-2\}$. Now we only need to show that there exists a path of length $n - 1$ from $x_j$ to $x_i$. If $j - i = 1$, then $P[x_j, x_{n-1}]P[x_0, x_i]$ is the desired path. If $j - i = 2$, then $P[x_j, x_n]P[x_0, x_i]$ is the desired path. If $3 \leq j - i \leq n - 1$, then $P[x_j, x_n]P[x_{i+2}, x_{j-1}]P[x_0, x_i]$ is the desired path.

By Lemma 6, we can obtain the following lemma.
Lemma 7. For any \( x \in V(D) \setminus V(P) \) and \( x_i \in V(P) \), if \( x \to x_i \), then \( x \) and every vertex of \( \{x_0, x_1, \ldots, x_{i-1}\} \) are adjacent; if \( x_i \to x \), then \( x \) and every vertex of \( \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \) are adjacent.

**Proof.** If \( x \to x_i \), then for any \( x_j \in \{x_0, x_1, \ldots, x_{i-1}\} \), by Lemma 6, there exists a path \( Q \) of length \( k-1 \) from \( x_i \) to \( x_j \). Then the path \( xQ \) implies \( x \to x_i \). If \( x_i \to x \), then for any \( x_j \in \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \), by Lemma 6, there exists a path \( R \) of length \( k-1 \) from \( x_j \) to \( x_i \). Then the path \( Rx \) implies \( x \to x_i \). □

Using Lemma 7, Lemma 4 can be improved to the following result.

Lemma 8. For any \( x \in B \), either \( x \) and every vertex of \( V(P) \) are adjacent or there exist two vertices \( x_1, x_s \in V(P) \) with \( 4 \leq t+1 < s \leq k-1 \) such that \( \{x_s, \ldots, x_{k+2}\} \to x \to \{x_0, \ldots, x_t\} \).

**Proof.** If \( x \) and every vertex of \( V(P) \) are adjacent, then we are done. Suppose not. By the definition of \( B \), \( (x, V(P)) \neq \emptyset \) and \( (V(P), x) \neq \emptyset \). Take \( t = \max\{i: x \to x_i\} \) and \( s = \min\{j: x_j \to x\} \). By Lemma 7, \( x \) and every vertex \( \{x_0, \ldots, x_t\} \cup \{x_s, \ldots, x_{k+2}\} \) are adjacent. Moreover, since \( x \) and some vertex of \( V(P) \) are not adjacent, we can conclude \( s > t+1 \) and \( \{x_s, \ldots, x_{k+2}\} \to x \to \{x_0, \ldots, x_t\} \). By Lemma 4, \( t \geq 3 \) and \( s \leq k-1 \).

Lemma 9. Let \( Q = z_0z_1 \cdots z_n \) be a path of length \( n \) with \( 1 \leq n \leq k-1 \) in \( D-V(P) \). For some \( x_i \in V(P) \), if \( z_n \to x_i \), then \( z_0 \) and \( x_{i+(k-n-1)} \) are adjacent; if \( x_i \to z_0 \), then \( z_n \) and \( x_{i-(k-n-1)} \) are adjacent, where the subscripts are taken modulo \( k+3 \).

**Proof.** Using the definition of \( k \)-quasi-transitive digraphs, the proof is easy and so we omit it. □

According to Lemma 8, we can divide \( B \) into two subsets. Let \( B_1 = \{x \in B: x \) and some vertex of \( V(P) \) are not adjacent\} and \( B_2 = \{x \in B: x \) and every vertex of \( V(P) \) are adjacent\}. Now we consider the arcs among \( I, O, B_1 \) and \( B_2 \). First we show \( I \to O \). Let \( x \in I \) and \( y \in O \) be arbitrary. By Lemma 5, \( x \to V(P) \) and \( V(P) \) \( \to \) \( y \). If \( y \to x \), then the path \( x_0x_1x_{k+2} \) contradicts \( d(x_0, x_{k+2}) = k+2 \geq 8 \). Thus \( I \to O \). Let \( z \in B_1 \) be arbitrary. By the definition of \( B_1 \), \( z \) and some vertex of \( V(P) \) are not adjacent, say \( x_{n_0} \). By Lemma 8, \( 3 < n_0 < k-1 \). It is not difficult to see that \( I \to B_1 \) and \( B_1 \to O \), otherwise, by Lemma 9, \( I \to V(P) \) and \( V(P) \to O \). \( z \) and every vertex of \( V(P) \) are adjacent. Since \( D \) is strong, \( (B_2, I) \neq \emptyset \) and \( (O, B_2) \neq \emptyset \). Let \( B'_2 = \{u \in B_2: (u, I) \neq \emptyset \} \) and \( B''_2 = \{v \in B_2: (O, v) \neq \emptyset \} \). Let \( u \in B'_2 \) and \( v \in B''_2 \) be two arbitrary vertices. By the definition of \( B'_2 \), there exist \( x_i \in V(P) \) and \( x' \in I \) such that \( x_i \to u \to x' \). Then the path \( x_iux_{k+2} \) implies that \( d(x_i, x_{k+2}) \leq 3 \). Hence \( i \geq k-1 \), which means \( u \to \{x_0, x_1, \ldots, x_{k-2}\} \). By the definition of \( B'_2 \), there
exist $x_j \in V(P)$ and $y' \in O$ such that $y' \rightarrow v \rightarrow x_j$. Then the path $x_0y'vx_j$ implies that $d(x_0, x_j) \leq 3$. Hence $j \leq 3$, which means $\{x_4, x_5, \ldots, x_{k+2}\} \rightarrow v$. Note that $k - 2 \geq 4$. Thus $u \neq v$, which implies $B_2' \cap B_2'' = \emptyset$. Hence $B_2' \rightarrow O$ and $I \rightarrow B_2''$. If $z \rightarrow u$, then considering the path $zuw'$, by Lemma 9, $z$ and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B_2' \rightarrow B_1$. If $v \rightarrow z$, then considering the path $ywz$, by Lemma 9, $z$ and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B_1 \rightarrow B_2''$. If $k = 6$, denote the path $R = x_{k+2}x_{n_0}$. If $k \geq 7$, by Lemma 6, there exists a path of length $k - 5$ from $x_{k+2}$ to $x_{n_0}$, denote it by $R$. If $v \rightarrow u$, then $zyvwux'R$ implies $\overline{zx_{n_0}}$, a contradiction. Hence $B_2' \rightarrow B_2''$.

**Theorem 10.** If $D - V(P)$ is strong, then $D$ is Hamiltonian.

**Proof.** By Lemma 3, $D - V(P)$ is a semicomplete digraph. Hence $D - V(P)$ contains a Hamiltonian cycle, denote it by $H = y_0y_1 \cdots y_{m}y_0$. Clearly, if there exists a pair of arcs $x_i, x_{i+1} \in A(P)$ and $y_jy_{j+1} \in A(H)$ such that $x_i \rightarrow y_{j+1}$ and $y_j \rightarrow x_{i+1}$, then $D$ contains a Hamiltonian cycle $x_iH[y_{j+1}, y_j]P[x_{i+1}, x_i]$. Next we shall find out such a pair of arcs. Suppose $O \neq \emptyset$. Since $D$ is strong, $B \cup I \neq \emptyset$ and there exists $y_j \in V(H)$ such that $y_j \in B \cup I$ and $y_{j+1} \in O$. There exists $x_i \in V(P)$ such that $y_j \rightarrow x_i$. Then $y_jy_{j+1}$ and $x_{i-1}x_i$ are the desired arcs. Now assume $O = \emptyset$. Analogously, assume $I = \emptyset$ and so $V(D) \setminus V(P) = B$. If $B = \emptyset$, then $D$ is semicomplete and so $D$ is Hamiltonian. Now assume that $B_1 \neq \emptyset$. If $|V(H)| = 1$, then $y_0 \in B_1$ and $x_{k+2}y_0x_0x_1 \cdots x_{k+2}$ is a Hamiltonian cycle of $D$. Assume $|V(H)| \geq 2$. If there exist two consecutive vertices $y_j, y_{j+1} \in B_1$, then $y_jy_{j+1}$ and $x_{k+2}x_0$ are the desired arcs. Assume there is no such a pair of arcs. So there exists a pair of vertices $y_j, y_{j+1} \in V(H)$ such that $y_j \in B_2$ and $y_{j+1} \in B_1$. If $y_j \rightarrow x_0$, then $y_jy_{j+1}$ and $x_{k+2}x_0$ are the desired arcs. Assume $x_0 \rightarrow y_j$. If $|V(H)| = 2$, then $x_0y_jy_{j+1}x_1x_2 \cdots x_{k+2}x_0$ is a Hamiltonian cycle of $D$. Assume $|V(H)| \geq 3$. According to the above argument, $y_{j+2} \in B_2$. If $y_{j+2} \rightarrow x_{k+2}$, then $x_0y_jy_{j+1}x_1x_2 \cdots x_{k+2}x_0$ is a path of length 4 from $x_0$ to $x_{k+2}$, a contradiction to $d(x_0, x_{k+2}) \geq 8$. Thus $x_{k+2} \rightarrow y_{j+2}$. Then $y_{j+1}y_{j+2}$ and $x_{k+2}x_0$ are the desired arcs.

**Theorem 11.** If $B_2 = \emptyset$ or for any $x \in B_2$, $x_{k+2} \rightarrow x \rightarrow x_0$, then $D$ is Hamiltonian.

**Proof.** If $D - V(P)$ is strong, then, by Theorem 10, we are done. If $D - V(P)$ is not strong, then let $D_1, D_2, \ldots, D_t$ be strong components of $D - V(P)$, where $t \geq 2$. Since $D$ is strong, there exist $x \in V(D_1)$ and $y \in V(D_t)$ such that $(V(P), x) \neq \emptyset$ and $(y, V(P)) \neq \emptyset$. By the hypothesis of this theorem and Lemmas 4 and 5, $x_{k+2} \rightarrow x$ and $y \rightarrow x_0$. It is easy to see that there exists a Hamiltonian path $R$ from $x$ to $y$ in $D - V(P)$. So $x_{k+2}Rx_0x_1 \cdots x_{k+2}$ is a Hamiltonian cycle of $D$. 

Suppose $D - V(P)$ is not strong and there exists a vertex $u \in B_2$ such that $u \rightarrow x_{k+2}$, we may construct some $k$-quasi-transitive digraphs such that they are not Hamiltonian. For example, let $V(D) \setminus V(P) = \{u, v\}$ and $u \rightarrow v, \{x_{k-1}, x_k, x_{k+1}, x_{k+2}\} \rightarrow v \rightarrow \{x_0, x_1, x_2, x_3\}$ and $x_{k+1} \rightarrow u \rightarrow \{x_0, x_1, \ldots, x_k, x_{k+2}\}$. It is not difficult to see that $D$ contains no Hamiltonian cycle.

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References


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