HAMILTONIAN CYCLE PROBLEM IN STRONG
k-QUASI-TRANSITIVE DIGRAPHS
WITH LARGE DIAMETER

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Abstract

Let \( k \) be an integer with \( k \geq 2 \). A digraph is \( k \)-quasi-transitive, if for any path \( x_0x_1 \ldots x_k \) of length \( k \), \( x_0 \) and \( x_k \) are adjacent. Let \( D \) be a strong \( k \)-quasi-transitive digraph with even \( k \geq 4 \) and diameter at least \( k + 2 \). It has been shown that \( D \) has a Hamiltonian path. However, the Hamiltonian cycle problem in \( D \) is still open. In this paper, we shall show that \( D \) may contain no Hamiltonian cycle with \( k \geq 6 \) and give the sufficient condition for \( D \) to be Hamiltonian.

Keywords: quasi-transitive digraph, \( k \)-quasi-transitive digraph, Hamiltonian cycle.

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1. Terminology and Introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let \( D \) be a digraph with vertex set \( V(D) \) and arc set \( A(D) \). For any \( x, y \in V(D) \), we will write \( x \rightarrow y \) if \( xy \in A(D) \), and also, we will write \( \overline{xy} \) if \( x \rightarrow y \) or \( y \rightarrow x \). For disjoint subsets \( X \) and \( Y \) of \( V(D) \), \( X \rightarrow Y \) means that every vertex of \( X \) dominates every vertex of \( Y \), \( X \Rightarrow Y \) means that there is no arc from \( Y \) to \( X \) and \( X \Rightarrow Y \) means that both of \( X \rightarrow Y \) and \( X \Rightarrow Y \) hold. For subsets \( X, Y \) of \( V(D) \), we define \((X,Y) = \{xy \in A(D) : x \in X, y \in Y\} \). If \( X = \{x\} \), then we write \((x,Y)\) instead of \((\{x\},Y)\). Likewise, if \( Y = \{y\} \), then we write \( (X,y) \) instead of \((X,\{y\})\). Let
$D'$ be a subdigraph of $D$ and $x \in V(D) \setminus V(D')$. We say that $x$ and $D'$ are adjacent if $x$ and some vertex of $D'$ are adjacent. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of $D$ induced by the vertex set $S$.

Let $x$ and $y$ be two vertices of $V(D)$. The distance from $x$ to $y$ in $D$, denoted $d(x, y)$, is the minimum length of an $(x, y)$-path, if $y$ is reachable from $x$, and otherwise $d(x, y) = \infty$. The distance from a set $X$ to a set $Y$ of vertices in $D$ is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The diameter of $D$ is $\text{diam}(D) = d(V(D), V(D))$. Clearly, $D$ has finite diameter if and only if it is strong.

Let $P = v_1v_2\cdots v_p$ be a path or a cycle of $D$. For $i \neq j$, $v_i, v_j \in V(P)$ we denote by $P[v_i, v_j]$ the subpath of $P$ from $v_i$ to $v_j$. Let $Q = u_1u_2\cdots u_q$ be a vertex-disjoint path or cycle with $P$ in $D$. If there exist $v_i \in V(P)$ and $u_j \in V(Q)$ such that $v_iu_j \in A(D)$, then we will use $P[v_i, v_j]Q[u_j, u_q]$ to denote the path $v_1v_2\cdots v_iu_ju_{j+1}\cdots u_q$.

A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, $x_0$ and $x_2$ are adjacent. The concept of $k$-quasi-transitive digraphs was introduced in [2] as a generalization of quasi-transitive digraphs. A digraph is $k$-quasi-transitive, if for any path $x_0x_1\cdots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. The $k$-quasi-transitive digraphs have been studied in [2–7].

In [7], Wang and Zhang showed that a strong $k$-quasi-transitive digraph $D$ with even $k \geq 4$ and $\text{diam}(D) \geq k + 2$ has a Hamiltonian path and proposed the following problem. Let $k$ be an even integer with $k \geq 4$. Is it true that every strong $k$-quasi-transitive digraph with diameter at least $k + 2$ is Hamiltonian?

In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for it to be Hamiltonian.

2. Main Results

For the rest of this paper, let $k$ be an even integer with $k \geq 4$ and $D$ denote a strong $k$-quasi-transitive digraph with $\text{diam}(D) \geq k + 2$. There exist two vertices $u, v$ such that $d(u, v) = k + 2$ in $D$. Let $P = x_0x_1\cdots x_{k+2}$ denote a shortest $(u, v)$-path in $D$, where $u = x_0$ and $v = x_{k+2}$.

Theorem 1 [7]. The subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq k + 2$.

Lemma 2 [5]. Let $k$ be an integer with $k \geq 2$ and $D$ be a strong $k$-quasi-transitive digraph. Suppose that $C = x_0x_1\cdots x_{n-1}x_0$ is a cycle of length $n$ with $n \geq k$ in $D$. Then for any $x \in V(D) \setminus V(C)$, $x$ and $C$ are adjacent.

By Theorem 1, $x_{k+2} \to x_0$. So $x_0x_1\cdots x_{k+2}x_0$ is a cycle of length $k + 3$. By Lemma 2, every vertex of $V(D) \setminus V(P)$ is adjacent to $P$. Hence we can divide
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$V(D) \setminus V(P)$ into three subsets:

$$O = \{x \in V(D) \setminus V(P) : V(P) \Rightarrow x\},$$

$$I = \{x \in V(D) \setminus V(P) : x \Rightarrow V(P)\},$$

and

$$B = V(D) \setminus (V(P) \cup O \cup I).$$

One of $I, O$ and $B$ may be empty.

**Theorem 3** [7]. The subdigraph induced by $V(D) \setminus V(P)$ is a semicomplete digraph.

**Lemma 4** [7]. For any $x \in B$, either $x$ is adjacent to every vertex of $V(P)$ or $\{x_{k+2}, x_{k+1}, x_k, x_{k-1}\} \Rightarrow x \Rightarrow \{x_0, x_1, x_2, x_3\}$. In particular, if $k = 4$, then $x$ is adjacent to every vertex of $V(P)$.

From the proof of Lemma 2.11 in [7], we have the following result.

**Lemma 5** [7]. $V(P) \mapsto O$ and $I \mapsto V(P)$.

By Theorems 1 and 3 and Lemmas 4 and 5, a strong 4-quasi-transitive digraph $D$ with $\text{diam}(D) \geq 6$ is in fact a semicomplete digraph. It is well known that a strong semicomplete digraph is Hamiltonian. Hence, for the rest of this paper, we consider the case $k \geq 6$.

**Lemma 6.** Let $H$ be a digraph and $u, v \in V(H)$ such that $d(u, v) = n$ with $n \geq 4$. Let $Q = x_0x_1 \cdots x_n$ be a shortest $(u, v)$-path in $H$. If $H[V(Q)]$ is a semicomplete digraph, then, for any $x_i, x_j \in V(Q)$ with $0 \leq i < j \leq n$, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-1\}$ in $H[V(Q)]$.

**Proof.** We prove the result by induction on $n$. For $n = 4$, it is not difficult to check that the result is true. Suppose $n \geq 5$. Assume $j - i = n$. It must be $j = n$ and $i = 0$. Then the length of the path $x_nP[x_2, x_0]x_0$ is $p$ with $p \in \{2, 3, \ldots, n-1\}$. Now assume $1 \leq j - i \leq n - 1$. Then $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$ or $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, assume that $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$. By induction, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-2\}$. Now we only need to show that there exists a path of length $n - 1$ from $x_j$ to $x_i$. If $j - i = 1$, then $P[x_j, x_{n-1}]P[x_0, x_i]$ is the desired path. If $j - i = 2$, then $P[x_j, x_n]P[x_0, x_i]$ is the desired path. If $3 \leq j - i \leq n - 1$, then $P[x_j, x_n]P[x_{i+2}, x_{j-1}]P[x_0, x_i]$ is the desired path.

By Lemma 6, we can obtain the following lemma.
Lemma 7. For any \( x \in V(D) \setminus V(P) \) and \( x_i \in V(P) \), if \( x \rightarrow x_i \), then \( x \) and every vertex of \( \{x_0, x_1, \ldots, x_{i-1}\} \) are adjacent; if \( x_i \rightarrow x \), then \( x \) and every vertex of \( \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \) are adjacent.

Proof. If \( x \rightarrow x_i \), then for any \( x_j \in \{x_0, x_1, \ldots, x_{i-1}\} \), by Lemma 6, there exists a path \( Q \) of length \( k-1 \) from \( x_i \) to \( x_j \). Then the path \( xQ \) implies \( x \rightarrow x_j \). If \( x_i \rightarrow x \), then for any \( x_j \in \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \), by Lemma 6, there exists a path \( R \) of length \( k-1 \) from \( x_j \) to \( x_i \). Then the path \( Rx \) implies \( x \rightarrow x_j \).

Using Lemma 7, Lemma 4 can be improved to the following result.

Lemma 8. For any \( x \in B \), either \( x \) and every vertex of \( V(P) \) are adjacent or there exist two vertices \( x_1, x_k \in V(P) \) with \( 4 \leq t + 1 < s \leq k - 1 \) such that \( \{x_s, \ldots, x_{k+2}\} \rightarrow x \rightarrow \{x_0, \ldots, x_t\} \).

Proof. If \( x \) and every vertex of \( V(P) \) are adjacent, then we are done. Suppose not. By the definition of \( B \), \( x, V(P) \neq \emptyset \) and \( (V(P), x) \neq \emptyset \). Take \( t = \max \{i : x \rightarrow x_i\} \) and \( s = \min \{j : x_j \rightarrow x\} \). By Lemma 7, \( x \) and every vertex \( \{x_0, \ldots, x_t\} \cup \{x_s, \ldots, x_{k+2}\} \) are adjacent. Moreover, since \( x \) and some vertex of \( V(P) \) are not adjacent, we can conclude \( s > t + 1 \) and \( \{x_s, \ldots, x_{k+2}\} \rightarrow x \rightarrow \{x_0, \ldots, x_t\} \). By Lemma 4, \( t \geq 3 \) and \( s \leq k - 1 \).

Lemma 9. Let \( Q = z_0z_1 \cdots z_n \) be a path of length \( n \) with \( 1 \leq n \leq k - 1 \) in \( D - V(P) \). For some \( x_i \in V(P) \), if \( z_n \rightarrow x_i \), then \( z_0 \) and \( x_{i+(k-n-1)} \) are adjacent; if \( x_i \rightarrow z_0 \), then \( z_n \) and \( x_{i-(k-n-1)} \) are adjacent, where the subscripts are taken modulo \( k + 3 \).

Proof. Using the definition of \( k \)-quasi-transitive digraphs, the proof is easy and so we omit it.

According to Lemma 8, we can divide \( B \) into two subsets. Let \( B_1 = \{x \in B : x \) and some vertex of \( V(P) \) are not adjacent\} and \( B_2 = \{x \in B : x \) and every vertex of \( V(P) \) are adjacent\}. Now we consider the arcs among \( I, O, B_1 \) and \( B_2 \). First we show \( I \rightarrow O \). Let \( x \in I \) and \( y \in O \) be arbitrary. By Lemma 5, \( x \rightarrow V(P) \) and \( V(P) \rightarrow y \). If \( y \rightarrow x \), then the path \( x_0yxk+2 \) contradicts \( d(x_0, x_{k+2}) = k + 2 \geq 8 \). Thus \( I \rightarrow O \). Let \( z \in B_1 \) be arbitrary. By the definition of \( B_1 \), \( z \) and some vertex of \( V(P) \) are not adjacent, say \( x_{n_0} \). By Lemma 8, \( 3 < n_0 < k - 1 \). It is not difficult to see that \( I \rightarrow B_1 \) and \( B_1 \rightarrow O \), otherwise, by Lemma 9, \( I \rightarrow V(P) \) and \( V(P) \rightarrow O \). \( z \) and every vertex of \( V(P) \) are adjacent. Since \( D \) is strong, \( (B_2, I) \neq \emptyset \) and \( (O, B_2) \neq \emptyset \). Let \( B'_2 = \{u \in B_2 : (u, I) \neq \emptyset\} \) and \( B''_2 = \{v \in B_2 : (O, v) \neq \emptyset\} \). Let \( u \in B'_2 \) and \( v \in B''_2 \) be two arbitrary vertices. By the definition of \( B'_2 \), there exist \( x_i \in V(P) \) and \( x' \in I \) such that \( x_i \rightarrow u \rightarrow x' \). Then the path \( x_iux'x_{k+2} \) implies that \( d(x_i, x_{k+2}) \leq 3 \). Hence \( i \geq k - 1 \), which means \( u \rightarrow \{x_0, x_1, \ldots, x_{k-2}\} \). By the definition of \( B''_2 \), there
exist $x_j \in V(P)$ and $y' \in O$ such that $y' \to v \to x_j$. Then the path $x_0y'vx_j$ implies that $d(x_0, x_j) \leq 3$. Hence $j \leq 3$, which means $\{x_4, x_5, \ldots, x_{k+2}\} \to v$. Note that $k - 2 \geq 4$. Thus $u \neq v$, which implies $B'_2 \cap B''_2 = \emptyset$. Hence $B'_2 \to O$ and $I \to B''_2$. If $z \to u$, then considering the path $zux'$, by Lemma 9, $z$ and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B'_2 \to B_1$. If $v \to z$, then considering the path $y'vux$, by Lemma 9, $z$ and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B_1 \to B''_2$. If $k = 6$, denote the path $R = x_{k+2}x_{n_0}$. If $k \geq 7$, by Lemma 6, there exists a path of length $k - 5$ from $x_{k+2}$ to $x_{n_0}$, denote it by $R$. If $v \to u$, then $zy'vux'R$ implies $\overline{xx_{n_0}}$, a contradiction. Hence $B'_2 \to B''_2$.

**Theorem 10.** If $D - V(P)$ is strong, then $D$ is Hamiltonian.

**Proof.** By Lemma 3, $D - V(P)$ is a semicomplete digraph. Hence $D - V(P)$ contains a Hamiltonian cycle, denote it by $H = y_0y_1 \cdots y_{m-1}y_0$. Clearly, if there exists a pair of arcs $x_i, x_{i+1} \in A(P)$ and $y_jy_{j+1} \in A(H)$ such that $x_i \to y_{j+1}$ and $y_j \to x_{i+1}$, then $D$ contains a Hamiltonian cycle $x_iH[y_j+1, y_j]P[x_{i+1}, x_i]$. Next we shall find out such a pair of arcs. Suppose $O \neq \emptyset$. Since $D$ is strong, $B \cup I \neq \emptyset$ and there exists $y_j \in V(H)$ such that $y_j \in B \cup I$ and $y_{j+1} \in O$. There exists $x_i \in V(P)$ such that $y_j \to x_i$. Then $y_jy_{j+1}$ and $x_{i-1}x_i$ are the desired arcs. If $O = \emptyset$. Analogously, assume $I = \emptyset$ and so $V(D) \setminus V(P) = B$. If $B_1 = \emptyset$, then $D$ is semicomplete and so $D$ is Hamiltonian. Now assume $B_1 \neq \emptyset$. If $|V(H)| = 1$, then $y_0 \in B_1$ and $x_{k+2}y_0x_0x_1 \cdots x_{k+2}$ is a Hamiltonian cycle of $D$. Assume $|V(H)| \geq 2$. If there exist two consecutive vertices $y_j, y_{j+1} \in B_1$, then $y_jy_{j+1}$ and $x_{k+2}x_0$ are the desired arcs. Assume there is no such a pair of arcs. So there exists a pair of vertices $y_j, y_{j+1} \in V(H)$ such that $y_j \in B_2$ and $y_{j+1} \in B_1$. If $y_j \to x_0$, then $y_jy_{j+1}$ and $x_{k+2}x_0$ are the desired arcs. Assume $x_0 \to y_j$. If $|V(H)| = 2$, then $x_0y_jy_{j+1}x_1x_2 \cdots x_{k+2}x_0$ is a Hamiltonian cycle of $D$. Assume $|V(H)| \geq 3$. According to the above argument, $y_{j+2} \in B_2$. If $y_{j+2} \to x_{k+2}$, then $x_0y_jy_{j+1}y_{j+2}x_{k+2}x_0$ is a path of length 4 from $x_0$ to $x_{k+2}$, a contradiction to $d(x_0, x_{k+2}) \geq 8$. Thus $x_{k+2} \to y_{j+2}$. Then $y_{j+1}y_{j+2}$ and $x_{k+2}x_0$ are the desired arcs.

**Theorem 11.** If $B_2 = \emptyset$ or for any $x \in B_2$, $x_{k+2} \to x \to x_0$, then $D$ is Hamiltonian.

**Proof.** If $D - V(P)$ is strong, then, by Theorem 10, we are done. If $D - V(P)$ is not strong, then let $D_1, D_2, \ldots, D_t$ be strong components of $D - V(P)$, where $t \geq 2$. Since $D$ is strong, there exist $x \in V(D_1)$ and $y \in V(D_t)$ such that $(V(P), x) \neq \emptyset$ and $(y, V(P)) \neq \emptyset$. By the hypothesis of this theorem and Lemmas 4 and 5, $x_{k+2} \to x$ and $y \to x_0$. It is easy to see that there exists a Hamiltonian path $R$ from $x$ to $y$ in $D - V(P)$. So $x_{k+2}R_{x_0x_1} \cdots x_{k+2}$ is a Hamiltonian cycle of $D$. \hfill \blacksquare
Suppose $D - V(P)$ is not strong and there exists a vertex $u \in B_2$ such that $u \mapsto x_{k+2}$, we may construct some $k$-quasi-transitive digraphs such that they are not Hamiltonian. For example, let $V(D) \setminus V(P) = \{u, v\}$ and $u \rightarrow v$, \{$x_{k-1}, x_k, x_{k+1}, x_{k+2}\} \rightarrow v \rightarrow \{x_0, x_1, x_2, x_3\}$ and $x_{k+1} \rightarrow u \rightarrow \{x_0, x_1, \ldots, x_k, x_{k+2}\}$. It is not difficult to see that $D$ contains no Hamiltonian cycle.

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References


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