HAMiltonian cycle problem in strong k-quasi-transitive digraphs with large diameter

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Abstract

Let $k$ be an integer with $k \geq 2$. A digraph is $k$-quasi-transitive, if for any path $x_0x_1 \ldots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. Let $D$ be a strong $k$-quasi-transitive digraph with even $k \geq 4$ and diameter at least $k + 2$. It has been shown that $D$ has a Hamiltonian path. However, the Hamiltonian cycle problem in $D$ is still open. In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for $D$ to be Hamiltonian.

Keywords: quasi-transitive digraph, $k$-quasi-transitive digraph, Hamiltonian cycle.

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1. Terminology and Introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \to y$ if $xy \in A(D)$, and also, we will write $xy$ if $x \to y$ or $y \to x$. For disjoint subsets $X$ and $Y$ of $V(D)$, $X \to Y$ means that every vertex of $X$ dominates every vertex of $Y$, $X \Rightarrow Y$ means that there is no arc from $Y$ to $X$ and $X \Rightarrow Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. For subsets $X, Y$ of $V(D)$, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $X = \{x\}$, then we write $(x, Y)$ instead of $(\{x\}, Y)$. Likewise, if $Y = \{y\}$, then we write $(X, y)$ instead of $(X, \{y\})$. Let
Let $x$ and $y$ be two vertices of $V(D)$. The distance from $x$ to $y$ in $D$, denoted $d(x, y)$, is the minimum length of an $(x, y)$-path, if $y$ is reachable from $x$, and otherwise $d(x, y) = \infty$. The distance from a set $X$ to a set $Y$ of vertices in $D$ is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The diameter of $D$ is $\text{diam}(D) = d(V(D), V(D))$. Clearly, $D$ has finite diameter if and only if it is strong.

Let $P = v_1v_2 \cdots v_p$ be a path or a cycle of $D$. For $i \neq j$, $v_i, v_j \in V(P)$ we denote by $P[v_i, v_j]$ the subpath of $P$ from $v_i$ to $v_j$. Let $Q = u_1u_2 \cdots u_q$ be a vertex-disjoint path or cycle with $P$ in $D$. If there exist $v_i \in V(P)$ and $u_j \in V(Q)$ such that $v_iu_j \in A(D)$, then we will use $P[v_i, v_j]Q[u_j, u_q]$ to denote the path $v_1v_2 \cdots v_iu_ju_{j+1} \cdots u_q$.

A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, $x_0$ and $x_2$ are adjacent. The concept of $k$-quasi-transitive digraphs was introduced in [2] as a generalization of quasi-transitive digraphs. A digraph is $k$-quasi-transitive, if for any path $x_0x_1 \cdots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. The $k$-quasi-transitive digraphs have been studied in [2–7].

In [7], Wang and Zhang showed that a strong $k$-quasi-transitive digraph $D$ with even $k \geq 4$ and $\text{diam}(D) \geq k + 2$ has a Hamiltonian path and proposed the following problem. Let $k$ be an even integer with $k \geq 4$. Is it true that every strong $k$-quasi-transitive digraph with diameter at least $k + 2$ is Hamiltonian?

In this paper, we shall show that $D$ may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for it to be Hamiltonian.

2. Main Results

For the rest of this paper, let $k$ be an even integer with $k \geq 4$ and $D$ denote a strong $k$-quasi-transitive digraph with $\text{diam}(D) \geq k + 2$. There exist two vertices $u, v$ such that $d(u, v) = k + 2$ in $D$. Let $P = x_0x_1 \cdots x_{k+2}$ denote a shortest $(u, v)$-path in $D$, where $u = x_0$ and $v = x_{k+2}$.

**Theorem 1** [7]. The subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \rightarrow x_i$ for $1 \leq i + 1 < j \leq k + 2$.

**Lemma 2** [5]. Let $k$ be an integer with $k \geq 2$ and $D$ be a strong $k$-quasi-transitive digraph. Suppose that $C = x_0x_1 \cdots x_{n-1}x_0$ is a cycle of length $n$ with $n \geq k$ in $D$. Then for any $x \in V(D) \setminus V(C)$, $x$ and $C$ are adjacent.

By Theorem 1, $x_{k+2} \rightarrow x_0$. So $x_0x_1 \cdots x_{k+2}x_0$ is a cycle of length $k + 3$. By Lemma 2, every vertex of $V(D) \setminus V(P)$ is adjacent to $P$. Hence we can divide
$V(D) \setminus V(P)$ into three subsets:

$$O = \{x \in V(D) \setminus V(P) : V(P) \Rightarrow x\},$$

$$I = \{x \in V(D) \setminus V(P) : x \Rightarrow V(P)\},$$

and

$$B = V(D) \setminus (V(P) \cup O \cup I).$$

One of $I, O$ and $B$ may be empty.

**Theorem 3** [7]. The subdigraph induced by $V(D) \setminus V(P)$ is a semicomplete digraph.

**Lemma 4** [7]. For any $x \in B$, either $x$ is adjacent to every vertex of $V(P)$ or
\[\{x_k, x_{k+1}, x_k, x_{k-1}\} \rightarrow x \rightarrow \{x_0, x_1, x_2, x_3\}.\] In particular, if $k = 4$, then $x$ is adjacent to every vertex of $V(P)$.

From the proof of Lemma 2.11 in [7], we have the following result.

**Lemma 5** [7]. $V(P) \rightarrow O$ and $I \rightarrow V(P)$.

By Theorems 1 and 3 and Lemmas 4 and 5, a strong 4-quasi-transitive digraph $D$ with $\text{diam}(D) \geq 6$ is in fact a semicomplete digraph. It is well known that a strong semicomplete digraph is Hamiltonian. Hence, for the rest of this paper, we consider the case $k \geq 6$.

**Lemma 6.** Let $H$ be a digraph and $u, v \in V(H)$ such that $d(u, v) = n$ with $n \geq 4$. Let $Q = x_0, x_1, \ldots, x_n$ be a shortest $(u, v)$-path in $H$. If $H[V(Q)]$ is a semicomplete digraph, then, for any $x_i, x_j \in V(Q)$ with $0 \leq i < j \leq n$, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-1\}$ in $H[V(Q)]$.

**Proof.** We prove the result by induction on $n$. For $n = 4$, it is not difficult to check that the result is true. Suppose $n \geq 5$. Assume $j - i = n$. It must be $j = n$ and $i = 0$. Then the length of the path $x_nP[x_2, x_0]x_0$ is $p$ with $p \in \{2, 3, \ldots, n-1\}$. Now assume $1 \leq j - i \leq n - 1$. Then $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$ or $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, assume that $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$. By induction, there exists a path of length $p$ from $x_j$ to $x_i$ with $p \in \{2, 3, \ldots, n-2\}$. Now we only need to show that there exists a path of length $n - 1$ from $x_j$ to $x_i$. If $j - i = 1$, then $P[x_j, x_{n-1}]P[x_0, x_i]$ is the desired path. If $j - i = 2$, then $P[x_j, x_n]P[x_0, x_i]$ is the desired path. If $3 \leq j - i \leq n - 1$, then $P[x_j, x_n]P[x_{i+2}, x_{j-1}]P[x_0, x_i]$ is the desired path.

By Lemma 6, we can obtain the following lemma.
Lemma 7. For any \( x \in V(D) \setminus V(P) \) and \( x_i \in V(P) \), if \( x \rightarrow x_i \), then \( x \) and every vertex of \( \{x_0, x_1, \ldots, x_{i-1}\} \) are adjacent; if \( x_i \rightarrow x \), then \( x \) and every vertex of \( \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \) are adjacent.

Proof. If \( x \rightarrow x_i \), then for any \( x_j \in \{x_0, x_1, \ldots, x_{i-1}\} \), by Lemma 6, there exists a path \( Q \) of length \( k-1 \) from \( x_i \) to \( x_j \). Then the path \( xQ \) implies \( xx_j \). If \( x_i \rightarrow x \), then for any \( x_j \in \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\} \), by Lemma 6, there exists a path \( R \) of length \( k-1 \) from \( x_j \) to \( x_i \). Then the path \( Rx \) implies \( xx_j \).

Using Lemma 7, Lemma 4 can be improved to the following result.

Lemma 8. For any \( x \in B \), either \( x \) and every vertex of \( V(P) \) are adjacent or there exist two vertices \( x_i, x_s \in V(P) \) with \( 4 \leq t+1 < s \leq k-1 \) such that \( \{x_s, \ldots, x_{k+2}\} \rightarrow x \rightarrow \{x_0, \ldots, x_t\} \).

Proof. If \( x \) and every vertex of \( V(P) \) are adjacent, then we are done. Suppose not. By the definition of \( B \), \( (x, V(P)) \neq \emptyset \) and \( (V(P), x) \neq \emptyset \). Take \( t = \max\{i : x \rightarrow x_i\} \) and \( s = \min\{j : x_j \rightarrow x\} \). By Lemma 7, \( x \) and every vertex \( \{x_0, \ldots, x_t\} \cup \{x_s, \ldots, x_{k+2}\} \) are adjacent. Moreover, since \( x \) and some vertex of \( V(P) \) are not adjacent, we can conclude \( s > t+1 \) and \( \{x_s, \ldots, x_{k+2}\} \rightarrow x \rightarrow \{x_0, \ldots, x_t\} \). By Lemma 4, \( t \geq 3 \) and \( s \leq k-1 \).

Lemma 9. Let \( Q = z_0z_1 \cdots z_n \) be a path of length \( n \) with \( 1 \leq n \leq k-1 \) in \( D - V(P) \). For some \( x_i \in V(P) \), if \( z_n \rightarrow x_i \), then \( z_0 \) and \( x_{i+(k-n-1)} \) are adjacent; if \( x_i \rightarrow z_0 \), then \( z_n \) and \( x_{i-(k-n-1)} \) are adjacent, where the subscripts are taken modulo \( k+3 \).

Proof. Using the definition of \( k \)-quasi-transitive digraphs, the proof is easy and so we omit it.

According to Lemma 8, we can divide \( B \) into two subsets. Let \( B_1 = \{x \in B : x \) and some vertex of \( V(P) \) are not adjacent} \) and \( B_2 = \{x \in B : x \) and every vertex of \( V(P) \) are adjacent\} \). Now we consider the arcs among \( I, O, B_1 \) and \( B_2 \). First we show \( I \rightarrow O \). Let \( x \in I \) and \( y \in O \) be arbitrary. By Lemma 5, \( x \rightarrow V(P) \) and \( V(P) \rightarrow y \). If \( y \rightarrow x \), then the path \( yx_{k+2}x_0 \) contradicts \( d(x_0, x_{k+2}) = k+2 \geq 8 \). Thus \( I \rightarrow O \). Let \( z \in B_1 \) be arbitrary. By the definition of \( B_1 \), \( z \) and some vertex of \( V(P) \) are not adjacent, say \( x_{n_0} \). By Lemma 8, \( 3 < n_0 < k-1 \). It is not difficult to see that \( I \rightarrow B_1 \) and \( B_1 \rightarrow O \), otherwise, by Lemma 9, \( I \rightarrow V(P) \) and \( V(P) \rightarrow O \), \( z \) and every vertex of \( V(P) \) are adjacent.

Since \( D \) is strong, \((B_2, I) \neq \emptyset \) and \((O, B_2) \neq \emptyset \). Let \( B_2' = \{u \in B_2 : (u, I) \neq \emptyset \} \) and \( B_2'' = \{v \in B_2 : (O, v) \neq \emptyset \} \). Let \( u \in B_2' \) and \( v \in B_2'' \) be two arbitrary vertices. By the definition of \( B_2' \), there exist \( x_i \in V(P) \) and \( x' \in I \) such that \( x_i \rightarrow u \rightarrow x' \). Then the path \( x_iu x_{k+2} \) implies that \( d(x_i, x_{k+2}) \leq 3 \). Hence \( i \geq k-1 \), which means \( u \rightarrow \{x_0, x_1, \ldots, x_{k-2}\} \). By the definition of \( B_2'' \), there
exist \( x_j \in V(P) \) and \( y' \in O \) such that \( y' \to v \to x_j \). Then the path \( x_0 y' v x_j \) implies that \( d(x_0, x_j) \leq 3 \). Hence \( j \leq 3 \), which means \( \{x_4, x_5, \ldots, x_{k+2}\} \to v \). Note that \( k - 2 \geq 4 \). Thus \( u \neq v \), which implies \( B_2'^n \cap B_2'' = \emptyset \). Hence \( B_2' \to O \) and \( I \to B_2'' \). If \( z \to u \), then considering the path \( zuv' \), by Lemma 9, \( z \) and every vertex of \( V(P) \) are adjacent, a contradiction. Hence \( B_1' \to B_2'' \). If \( v \to z \), then considering the path \( y'uvz \), by Lemma 9, \( z \) and every vertex of \( V(P) \) are adjacent, a contradiction. Hence \( B_1' \to B_2'' \). If \( k = 6 \), denote the path \( R = x_{k+2} x_{n_0} \). If \( k \geq 7 \), by Lemma 6, there exists a path of length \( k - 5 \) from \( x_{k+2} \) to \( x_{n_0} \), denote it by \( R \). If \( v \to u \), then \( y'vux'R \) implies \( z v x u \), a contradiction. Hence \( B_1' \to B_2'' \).

**Theorem 10.** If \( D - V(P) \) is strong, then \( D \) is Hamiltonian.

**Proof.** By Lemma 3, \( D - V(P) \) is a semicomplete digraph. Hence \( D - V(P) \) contains a Hamiltonian cycle, denote it by \( H = y_0 y_1 \cdots y_m y_0 \). Clearly, if there exists a pair of arcs \( x_i x_{i+1} \in A(P) \) and \( y_j y_{j+1} \in A(H) \) such that \( x_i \to y_{j+1} \) and \( y_j \to x_{i+1} \), then \( D \) contains a Hamiltonian cycle \( x_i H[y_{j+1}, y_j]P[x_{i+1}, x_i] \). Next we shall find out such a pair of arcs. Suppose \( O \neq \emptyset \). Since \( D \) is strong, \( B \cup I \neq \emptyset \) and there exists \( y_j \in V(H) \) such that \( y_j \in B \cup I \) and \( y_{j+1} \in O \). There exists \( x_i \in V(P) \) such that \( y_j \to x_i \). Then \( y_j y_{j+1} \) and \( x_{i-1} x_i \) are the desired arcs. Now assume \( O = \emptyset \). Analogously, assume \( I = \emptyset \) and so \( V(D) \setminus V(P) = B \). If \( B_1 = \emptyset \), then \( D \) is semicomplete and so \( D \) is Hamiltonian. Now assume that \( B_1 \neq \emptyset \). If \( |V(H)| = 1 \), then \( y_0 \in B_1 \) and \( x_{k+2} y_0 y_0 x_1 \cdots x_{k+2} \) is a Hamiltonian cycle of \( D \). Assume \( |V(H)| \geq 2 \). If there exist two consecutive vertices \( y_j, y_{j+1} \in B_1 \), then \( y_j y_{j+1} \) and \( x_{k+2} x_0 \) are the desired arcs. Assume there is no such a pair of arcs. So there exists a pair of vertices \( y_j, y_{j+1} \in V(H) \) such that \( y_j \in B_2 \) and \( y_{j+1} \in B_1 \). If \( y_j \to x_0 \), then \( y_j y_{j+1} \) and \( x_{k+2} x_0 \) are the desired arcs. Assume \( x_0 \to y_j \). If \( |V(H)| = 2 \), then \( x_0 y_j y_{j+1} x_1 x_2 \cdots x_{k+2} x_0 \) is a Hamiltonian cycle of \( D \). Assume \( |V(H)| \geq 3 \). According to the above argument, \( y_{j+2} \in B_2 \). If \( y_{j+2} \to x_{k+2} \), then \( x_0 y_j y_{j+1} y_{j+2} x_{k+2} \) is a path of length 4 from \( x_0 \) to \( x_{k+2} \), a contradiction to \( d(x_0, x_{k+2}) \geq 8 \). Thus \( x_{k+2} \to y_{j+2} \). Then \( y_{j+1} y_{j+2} \) and \( x_{k+2} x_0 \) are the desired arcs.

**Theorem 11.** If \( B_2 = \emptyset \) or for any \( x \in B_2 \), \( x_{k+2} \to x \to x_0 \), then \( D \) is Hamiltonian.

**Proof.** If \( D - V(P) \) is strong, then, by Theorem 10, we are done. If \( D - V(P) \) is not strong, then let \( D_1, D_2, \ldots, D_t \) be strong components of \( D - V(P) \), where \( t \geq 2 \). Since \( D \) is strong, there exist \( x \in V(D_1) \) and \( y \in V(D_t) \) such that \( (V(P), x) \neq \emptyset \) and \( (y, V(P)) \neq \emptyset \). By the hypothesis of this theorem and Lemmas 4 and 5, \( x_{k+2} \to x \) and \( y \to x_0 \). It is easy to see that there exists a Hamiltonian path \( R \) from \( x \) to \( y \) in \( D - V(P) \). So \( x_{k+2} R x_0 x_1 \cdots x_{k+2} \) is a Hamiltonian cycle of \( D \).
Suppose $D - V(P)$ is not strong and there exists a vertex $u \in B_2$ such that $u \xrightarrow{} x_{k+2}$, we may construct some $k$-quasi-transitive digraphs such that they are not Hamiltonian. For example, let $V(D) \setminus V(P) = \{u, v\}$ and $u \xrightarrow{} v$, $\{x_k, x_{k+1}, x_{k+2}\} \xrightarrow{} v \rightarrow \{x_0, x_1, x_2, x_3\}$ and $x_{k+1} \rightarrow u \rightarrow \{x_0, x_1, \ldots, x_k, x_{k+2}\}$. It is not difficult to see that $D$ contains no Hamiltonian cycle.

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References


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