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THE DOUBLE ROMAN DOMATIC NUMBER OF A DIGRAPH

Lutz Volkmann

Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

Abstract

A *double Roman dominating function* on a digraph D with vertex set $V(D)$ is defined in [G. Hao, X. Chen and L. Volkmann, *Double Roman domination in digraphs*, Bull. Malays. Math. Sci. Soc. (2017).] as a function $f : V(D) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one in-neighbor u with $f(u) \geq 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(D)$ is called a *double Roman dominating family* (of functions) on D. The maximum number of functions in a double Roman dominating family on D is the *double Roman domatic number* of D, denoted by $d_{dR}(D)$. We initiate the study of the double Roman domatic number, and we present different sharp bounds on $d_{dR}(D)$. In addition, we determine the double Roman domatic number of some classes of digraphs.

Keywords: digraph, double Roman domination, double Roman domatic number.

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1. Terminology and Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [6]. Specifically, let D be a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) with vertex set $V(D) = V$ and arc set $A(D) = A$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. For two different vertices $u, v \in V(D)$, we use uv to denote the arc with tail u and head v, and we also call v an *out-neighbor* of u and u an *in-neighbor* of v. For $v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of v, denoted by $N_D^+(v) = N^+(v)$ and $N_D^-(v) = N^-(v)$, are the sets of out-neighbors and in-neighbors of v, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are the sets $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$, respectively. The *out-degree* and *in-degree* of a vertex v are defined by $d_D^+(v) = d^+(v) = |N^+(v)|$ and $d_D^-(v) = d^-(v) = |N^-(v)|$. The *maximum out-degree*, *maximum in-degree*, *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\Delta^+(D) = \Delta^+$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$, respectively. A digraph D is r*-out-regular* when $\Delta^+(D) = \delta^+(D) = r$ and r*-in-regular* when $\Delta^-(D) = \delta^-(D) = r$. If D is r-out-regular and r-in-regular, then D is called r*-regular.* The *underlying graph* of a digraph D is the graph obtained by replacing each arc uv or symmetric pairs uv, vu of arcs by the edge uv. A digraph D is *connected* if the underlying graph of D is connected. If X is a nonempty subset of the vertex set $V(D)$ of a digraph D, then $D[X]$ is the subdigraph of D *induced* by X. A digraph D is *bipartite* if its underlying graph is bipartite. Let K_n^* be the complete digraph of order n, C_n the oriented cycle of order n and $K_{p,q}^*$ the complete bipartite digraph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.

In this paper we continue the study of double Roman dominating functions and double Roman domatic numbers in graphs and digraphs (see, for example, [1–5, 7, 9, 11]). Inspired by an idea of the work [4], we defined in [5] the double Roman domination number of a digraph as follows. A *double Roman dominating function* (DRD function) on a digraph D is a function $f : V(D) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one in-neighbor u with $f(u) \geq 2$. The *double Roman domination number* $\gamma_{dR}(D)$ equals the minimum weight of a double Roman dominating function on D, and a double Roman dominating function of D with weight $\gamma_{dR}(D)$ is called a $\gamma_{dR}(D)$*-function* of D.

A set $\{f_1, f_2, \dots, f_d\}$ of distinct double Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(D)$ is called a *double Roman dominating family* (of functions) on D. The maximum number of functions in a double Roman dominating family (DRD family) on D is the *double Roman domatic number* of D, denoted by $d_{dR}(D)$. The double Roman domatic number is well-defined and $d_{dR}(D) \geq 1$ for each digraph D since the set consisting of any DRD function forms a DRD family on D.

Our purpose in this work is to initiate the study of the double Roman domatic number of a digraph. We first present basic properties and sharp bounds for the double Roman domatic number of a digraph. In addition, we determine the double Roman domatic number of some classes of digraphs.

2. Properties of the Double Roman Domatic Number

In this section we present basic properties and bounds on the double Roman domatic number.

Theorem 1. *If D is a digraph of order n, then*

$$\gamma_{dR}(D) \cdot d_{dR}(D) \leq 3n.$$

Moreover, if we have the equality $\gamma_{dR}(D) \cdot d_{dR}(D) = 3n$, then for each DRD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_{dR}(D)$, each f_i is a $\gamma_{dR}(D)$-function and $\sum_{i=1}^{d} f_i(v) = 3$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a DRD family on D with $d = d_{dR}(D)$, and let $v \in V(G)$. Then

$$\begin{aligned} d \cdot \gamma_{dR}(D) &= \sum_{i=1}^{d} \gamma_{dR}(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 3 = 3n. \end{aligned}$$

If $\gamma_{dR}(D) \cdot d_{dR}(D) = 3n$, then the two inequalities occuring in the proof become equalities. Hence for the DRD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} f_i(v) = \gamma_{dR}(D)$. Thus each f_i is a $\gamma_{dR}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 3$ for each $v \in V(D)$. ■

Theorem 2. *If D is a digraph, then $d_{dR}(D) \leq \delta^-(D) + 1$.*

Proof. If $d_{dR}(D) = 1$, then clearly $d_{dR}(D) \leq \delta^-(D) + 1$. Assume next that $d_{dR}(D) \geq 2$, and let $\{f_1, f_2, \ldots, f_d\}$ be a DRD family on D such that $d = d_{dR}(D)$. Assume that v is a vertex of minimum in-degree. Since $\sum_{x \in N^-[v]} f_i(x) = 2$ holds for at most one index $i \in \{1, 2, \ldots, d\}$, we deduce that

$$3d - 1 \leq \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N^-[v]} 3 = 3(\delta^-(D) + 1).$$

This implies $d \leq \delta^-(D) + 4/3$ and thus $d_{dR}(D) \leq \delta^-(D) + 1$. ■

Corollary 3. *Let D be a digraph of order n. Then $d_{dR}(D) \leq n$, and if $\delta^-(D) = 0$, then $d_{dR}(D) = 1$.*

Example 4. Let p, n be integers with $1 \le p \le n-1$. Let H be the digraph of order n with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $H[\{v_1, v_2, \ldots, v_p\}]$ is isomorphic to the complete digraph K_p^*, there exist all arcs from $\{v_1, v_2, \ldots, v_p\}$ to $\{v_{p+1}, v_{p+2}, \ldots, v_n\}$ and all arcs from v_{p+1} to $\{v_1, v_2, \ldots, v_p\}$. Then $\delta^-(H) = p$ and thus $d_{dR}(H) \le p+1$ according to Theorem 2. Define the functions $f_i : V(H) \to \{0,1,2,3\}$ by $f_i(v_i) = 3$ and $f_i(x) = 0$ for $x \in V(H) \setminus \{v_i\}$ for $1 \le i \le p$ and $f_{p+1}(v_{p+1}) = f_{p+1}(v_{p+2}) = \cdots = f_{p+1}(v_n) = 3$ and $f_{p+1}(v_i) = 0$ for $1 \le i \le p$. Then $f_1, f_2, \ldots, f_{p+1}$ are DRD functions on H such that $f_1(x) + f_2(x) + \cdots + f_{p+1}(x) = 3$ for each $x \in V(H)$. Therefore $\{f_1, f_2, \ldots, f_{p+1}\}$ is a double Roman dominating family on H and thus $d_{dR}(H) \ge p+1$ and so $d_{dR}(H) = p+1 = \delta^-(H)+1$. This example demonstrates that Theorem 2 is sharp.

Theorem 5. *If D is a bipartite digraph with $\delta^-(D) \ge 1$, then $d_{dR}(D) \ge 2$.*

Proof. Let X, Y be a bipartition of D. Define the functions $f, g : V(D) \to \{0,1,2,3\}$ by $f(x) = 3$ for $x \in X$ and $f(y) = 0$ for $y \in Y$ and $g(x) = 0$ for $x \in X$ and $g(y) = 3$ for $y \in Y$. Since $\delta^-(D) \ge 1$, we observe that f and g are DRD functions on D such that $f(v) + g(v) = 3$ for each vertex $v \in V(D)$. Thus $\{f, g\}$ is a double Roman dominating family on D and so $d_{dR}(D) \ge 2$. ■

Theorems 2 and 5 imply the next result immediately.

Corollary 6. *If C_n is an oriented cycle of even order, then $d_{dR}(C_n) = 2$.*

Following an idea of Zelinka [10], we prove a lower bound for the double Roman domatic number.

Theorem 7. If D is a digraph of order n, then

$$d_{dR}(D) \ge \left\lfloor \frac{n}{n - \delta^-(D)} \right\rfloor .$$

Proof. Let $S \subseteq V(D)$ with $|S| \ge n - \delta^-(D)$. If $v \in V(D) \setminus S$, then $|N^-[v]| \ge 1 + \delta^-(D)$ implies $N^-(v) \cap S \ne \emptyset$. Thus the function $f : V(D) \to \{0,1,2,3\}$ with $f(x) = 3$ for $x \in S$ and $f(x) = 0$ for $x \in V(D) \setminus S$ is a DRD function on D. Hence one can take any $\lfloor n/(n - \delta^-(D)) \rfloor$ disjoint subsets of $V(D)$, each of cardinality $n - \delta^-(D)$. Each of these subsets is a DRD function on D, and this leads to the desired result. ■

Corollary 8. *Let D be a digraph of order $n \ge 2$. Then $d_{dR}(D) = n$ if and only if D is isomorphic to the complete digraph K_n^*.*

Proof. If D is isomorphic to the complete digraph K_n^*, then Theorem 7 implies that $d_{dR}(K_n^*) \geq n$. Applying Theorem 2, we obtain $d_{dR}(K_n^*) = n$.

Conversely, assume that $d_{dR}(D) = n$. If D is not isomorphic to the complete digraph K_n^*, then $\delta^-(D) \leq n-2$, and Theorem 2 leads to the contradiction $n = d_{dR}(D) \leq n-1$. ∎

Proposition 9. *Let D be a digraph of order $n \geq 2$. If D has $1 \leq p \leq n$ vertices of out-degree $n-1$, then $d_{dR}(D) \geq p$.*

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices of out-degree $n-1$. Define $f_i : V(D) \to \{0,1,2,3\}$ by $f_i(v_i) = 3$ and $f_i(x) = 0$ for $x \neq v_i$ for $1 \leq i \leq p$. Then $f_1, f_2, \ldots, f_p$ are DRD functions on D such that $f_1(x) + f_2(x) + \cdots + f_p(x) \leq 3$ for each $x \in V(D)$. Therefore $\{f_1, f_2, \ldots, f_p\}$ is a double Roman dominating family on D and thus $d_{dR}(D) \geq p$. ∎

Corollary 8 shows that Proposition 9 is sharp for $p = n$. The next example will demonstrate that Proposition 9 is also sharp for each p with $1 \leq p \leq n-1$.

Example 10. Let p, n be integers with $1 \leq p \leq n-1$. Let Q be the digraph of order n with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $Q[\{v_1, v_2, \ldots, v_p\}]$ is isomorphic to the complete digraph K_p^* and there exist all arcs from $\{v_1, v_2, \ldots, v_p\}$ to $\{v_{p+1}, v_{p+2}, \ldots, v_n\}$. Then $\delta^-(Q) = p-1$ and thus $d_{dR}(Q) \leq p$ according to Theorem 2. Define the function $f_i : V(H) \to \{0,1,2,3\}$ by $f_i(v_i) = 3$ and $f_i(x) = 0$ for $x \in V(H) \setminus \{v_i\}$ for $1 \leq i \leq p$. Then $f_1, f_2, \ldots, f_p$ are DRD functions on Q such that $f_1(x) + f_2(x) + \cdots + f_p(x) \leq 3$ for each $x \in V(Q)$. Therefore $\{f_1, f_2, \ldots, f_p\}$ is a double Roman dominating family on Q and thus $d_{dR}(Q) \geq p$ and so $d_{dR}(Q) = p$.

Theorem 11. *Let D be a digraph of order $n \geq 2$ and let k be an integer with $2 \leq k \leq n$. If $\Delta^+(D) \leq (n-k)/(k-1)$, then $d_{dR}(D) \leq n/k$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a DRD family on D with $d = d_{dR}(D)$. According to [5], we can assume, without loss of generality, that no vertex of f_i is assigned the value 1. In [5], the authors show this for $\gamma_{dR}(D)$-functions, however, the same proof works for each DRD function. Since $\Delta^+(D) \leq (n-k)/(k-1)$, we observe that $f_i(x) \geq 2$ for at least k different vertices for each $i \in \{1, 2, \ldots, d\}$. Because of $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(D)$, we deduce the desired result that $d_{dR}(D) \leq n/k$. ∎

Example 12. If D is an $(n-2)$-regular digraph of order $n \geq 2$, then $d_{dR}(D) = \lfloor n/2 \rfloor$.

Proof. Applying Theorem 7, we deduce that $d_{dR}(D) \geq \lfloor n/2 \rfloor$. In addition, Theorem 11 implies for $k = 2$ that $d_{dR}(D) \leq \lfloor n/2 \rfloor$ and thus $d_{dR}(D) = \lfloor n/2 \rfloor$. ∎

Example 12 shows that Theorem 11 is sharp for $k = 2$.

Example 13. Let $p \geq 3$ be an integer. If $K^*_{p,p}$ is the complete bipartite digraph, then $d_{dR}(K^*_{p,p}) = p$.

Proof. Since $p \geq 3$, it is straightforward to verify that $\gamma_{dR}(K^*_{p,p}) = 6$. Thus Theorem 1 implies that $d_{dR}(K^*_{p,p}) \leq p$. Let now $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_p\}$ be a bipartition of $K^*_{p,p}$. Define $f_i : V(K_{p,p}) \to \{0, 1, 2, 3\}$ by $f_i(u_i) = f_i(v_i) = 3$ and $f_i(u_j) = f_i(v_j) = 0$ for $1 \leq i, j \leq p$ and $i \neq j$. Then f_i is a DRD function on $K^*_{p,p}$ for $1 \leq i \leq p$ such that $f_1(x) + f_2(x) + \cdots + f_p(x) = 3$ for each $x \in V(K^*_{p,p})$. Therefore $\{f_1, f_2, \ldots, f_p\}$ is a double Roman dominating family on $K^*_{p,p}$ and thus $d_{dR}(K^*_{p,p}) \geq p$. This yields to $d_{dR}(K^*_{p,p}) = p$. ■

Example 13 demonstrates that Theorem 1 is sharp, and that Theorem 11 is sharp for $k = 2$.

Example 14. If C_n is an oriented cycle of odd order n, then $d_{dR}(C_n) = 1$.

Proof. Let $k = (n+1)/2$ in Theorem 11. Then $\Delta^+(C_n) = 1 = (n-k)/(k-1)$ and therefore Theorem 11 implies that $d_{dR}(C_n) \leq n/k = (2n)/(n+1) < 2$. Thus $d_{dR}(C_n) = 1$. ■

3. Nordhaus-Gaddum Type Results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph or digraph and its complement. In their classical paper [8], Nordhaus and Gaddum discussed this problem for the chromatic number of graphs. We establish such inequalities for the double Roman domatic number of digraphs.

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. As an application of Theorem 2 we will prove the following Nordhaus-Gaddum type result.

Theorem 15. *If D is a digraph of order n, then*

$$d_{dR}(D) + d_{dR}(\overline{D}) \leq n + 1.$$

If $d_{dR}(D) + d_{dR}(\overline{D}) = n + 1$, then D is in-regular.

Proof. Since $\delta^-(\overline{D}) = n - 1 - \Delta^-(D)$, Theorem 2 implies that

$$\begin{aligned} d_{dR}(D) + d_{dR}(\overline{D}) &\leq (\delta^-(D) + 1) + (\delta^-(\overline{D}) + 1) \\ &= \delta^-(D) + 1 + (n - \Delta^-(D) - 1) + 1 \leq n + 1, \end{aligned}$$

and this is the desired bound. If D is not in-regular, then $\Delta^-(D)-\delta^-(D) \geq 1$, and thus the inequality chain above leads to the better bound $d_{dR}(D) + d_{dR}(\overline{D}) \leq n$. ■

Corollary 8 leads to $d_{dR}(K_n^*) + d_{dR}(\overline{K_n^*}) = n + 1$, and therefore equality in Theorem 15. For some special digraphs we can improve Theorem 15.

Corollary 16. *Let D be a digraph of order $n \geq 3$. If $\Delta^+(D) \leq n-2$ and $\Delta^+(\overline{D}) \leq n-2$, then*

$$d_{dR}(D) + d_{dR}(\overline{D}) \leq n,$$

and if n is odd, then

$$d_{dR}(D) + d_{dR}(\overline{D}) \leq n-1.$$

Proof. It follows from Theorem 11 for $k = 2$ that $d_{dR}(D) \leq n/2$ and $d_{dR}(\overline{D}) \leq n/2$. Therefore $d_{dR}(D) + d_{dR}(\overline{D}) \leq n$ and if n is odd, then $d_{dR}(D) + d_{dR}(\overline{D}) \leq n-1$. ■

Example 17. If $D = K_{p,p}^*$ for $p \geq 3$, then it follows from Corollary 8 and Example 13 that $d_{dR}(K_{p,p}^*) + d_{dR}(\overline{K_{p,p}^*}) = 2p = n(K_{p,p}^*)$. This example demonstrates that Corollary 16 is sharp for n even.

Example 18. Let $n = 2r + 1$ for an integer $r \geq 1$. We define the *circulant tournament* $T(n)$ of order n as follows. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of $T(n)$, and for each i, the arcs go from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n. Note that $T(n)$ is r-regular. Applying Theorem 11 for $k = 2$, we deduce that $d_{dR}(T(n)) \leq r$.

Now define the function $f_i : V(T(n)) \to \{0, 1, 2, 3\}$ by $f_i(u_i) = f_i(u_{i+r+1}) = 3$ for $1 \leq i \leq r$ and $f_i(x) = 0$ otherwise. Then f_i is a DRD function on $T(n)$ for $1 \leq i \leq r$ such that $f_1(x) + f_2(x) + \cdots + f_r(x) \leq 3$ for each $x \in V(T(n))$. Therefore $\{f_1, f_2, \ldots, f_r\}$ is a double Roman dominating family on $T(n)$ and thus $d_{dR}(T(n)) \geq r$ and so $d_{dR}(T(n)) = r$.

Since $\overline{T(n)}$ is also a circulant tournament, we observe that $d_{dR}(\overline{T(n)}) = r$ and thus $d_{dR}(T(n)) + d_{dR}(\overline{T(n)}) = 2r = n - 1$. This example shows that Corollary 16 is sharp for n odd too.

4. BOUNDS ON $\gamma_{dR}(D) + d_{dR}(D)$

In this section we make use of the following known results.

Proposition 19 [5]**.** *If D is a connected digraph of order $n \geq 4$, then $\gamma_{dR}(D) \leq 2n - 2$.*

Proposition 20 [5]**.** *Let D be a connected digraph of order $n \geq 2$. Then $\gamma_{dR}(D) = 3$ if and only if $\Delta^+(D) = n - 1$.*

The upper bound on the product $\gamma_{dR}(D) \cdot d_{dR}(D) \leq 3n$ in Theorem 1 leads to upper bounds on the sum of these two parameters.

Theorem 21. *If D is a connected digraph of order $n \geq 5$, then*

$$\gamma_{dR}(D) + d_{dR}(D) \leq 2n - 1.$$

Proof. Let $d = d_{dR}(D)$. If $d = 1$, then it follows from Proposition 19 that $\gamma_{dR}(D) + d_{dR}(D) \leq (2n-2) + 1 = 2n - 1$.

Let now $d \geq 2$. According to Corollary 3, we have $2 \leq d \leq n$. Theorem 1 implies that

$$\gamma_{dR}(D) + d_{dR}(D) \leq \frac{3n}{d_{dR}(D)} + d_{dR}(D).$$

Using these bounds and the fact that the function $g(x) = x + (3n)/x$ is decreasing for $2 \leq x \leq \sqrt{3n}$ and increasing for $\sqrt{3n} \leq x \leq n$, we deduce that

$$\begin{aligned}\gamma_{dR}(D) + d_{dR}(D) &\leq \frac{3n}{d_{dR}(D)} + d_{dR}(D) \\ &\leq \max\left\{\frac{3n}{2} + 2, 3 + n\right\} = \frac{3n}{2} + 2.\end{aligned}$$

Since $n \geq 5$, we obtain

$$\gamma_{dR}(D) + d_{dR}(D) \leq \left\lfloor \frac{3n}{2} \right\rfloor + 2 \leq 2n - 1,$$

and the proof is complete. ■

Since $\gamma_{dR}(C_4) + d_{dR}(C_4) = 8$, $\gamma_{dR}(C_3) + d_{dR}(C_3) = 6$ and $\gamma_{dR}(C_2) + d_{dR}(C_2) = 5$, we observe that Theorem 21 is not valid for $2 \leq n \leq 4$ in general.

Example 22. Let H be the digraph of order $n \geq 5$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_2v_1, v_3v_1, \ldots, v_nv_1\}$. Then $\gamma_{dR}(H) = 2(n-1)$ and $d_{dR}(H) = 1$ and thus $\gamma_{dR}(H) + d_{dR}(H) = 2n - 1$. This example shows that Theorem 21 is sharp.

Theorem 23. *If D is a bipartite digraph of order n with $\delta^-(D) \geq 1$, then*

$$\gamma_{dR}(D) + d_{dR}(D) \leq \frac{3n}{2} + 2.$$

Proof. According to Corollary 3 and Theorem 5, we have $2 \leq d_{dR}(D) \leq n$. Now we obtain the desired bound analogously to the second part of the proof of Theorem 21. ■

Example 24. If H_1 is isomorphic to pC_2 with an integer $p \geq 1$, then $\gamma_{dR}(H_1) = (3n)/2$ and $d_{dR}(H_1) = 2$ with $n = 2p$. Thus $\gamma_{dR}(H_1) + d_{dR}(H_1) = (3n)/2 + 2$.

If H_2 is isomorphic to pC_4 with an integer $p \geq 1$, then $\gamma_{dR}(H_2) = (3n)/2$ and $d_{dR}(H_2) = 2$ with $n = 4p$. Thus $\gamma_{dR}(H_2) + d_{dR}(H_2) = (3n)/2 + 2$.

These examples show that Theorem 23 is sharp.

Theorem 25. *If D is a digraph of order $n \geq 2$, then*

$$\gamma_{dR}(D) + d_{dR}(D) \geq 4,$$

with equality if and only if D contains a vertex v with $d_D^+(v) = n-1$ and $d_D^-(v) = 0$.

Proof. Since $\gamma_{dR}(D) \geq 3$ and $d_{dR}(D) \geq 1$, the lower bound is immediate.

If there exists a vertex v with $d_D^+(v) = n-1$ and $d_D^-(v) = 0$ then $\gamma_{dR}(D) = 3$ by Proposition 20 and $d_{dR}(D) = 1$ by Corollary 3 and thus $\gamma_{dR}(D) + d_{dR}(D) = 4$.

Conversely, assume that $\gamma_{dR}(D) + d_{dR}(D) = 4$. Then the bounds $\gamma_{dR}(D) \geq 3$ and $d_{dR}(D) \geq 1$ lead to $\gamma_{dR}(D) = 3$ and $d_{dR}(D) = 1$. Therefore Proposition 20 implies that $\Delta^+(D) = n-1$. Let v be a vertex with $d_D^+(v) = n-1$. Now we will show that $d_D^-(v) = 0$. Suppose that $d_D^-(v) \geq 1$. Then there exists an arc wv for a vertex $w \in V(D)\setminus\{v\}$. Define the functions $f, g : V(D) \to \{0, 1, 2, 3\}$ by $f(v) = 3$ and $f(x) = 0$ for $x \in V(D) \setminus \{v\}$ and $g(v) = 0$ and $g(x) = 3$ for $x \in V(D) \setminus \{v\}$. We observe that f and g are DRD functions on D such that $f(x) + g(x) = 3$ for each vertex $x \in V(D)$. Thus $\{f, g\}$ is a double Roman dominating family on D and so $d_{dR}(D) \geq 2$. This yields to the contradiction $\gamma_{dR}(D) + d_{dR}(D) \geq 5$. Thus $d_D^-(v) = 0$, and the proof is complete. ∎

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