ON LOCAL ANTIMAGIC CHROMATIC NUMBER OF CYCLE-RELATED JOIN GRAPHS

GEE-CHOON LAU

Faculty of Computer & Mathematical Sciences
Universiti Teknologi MARA (Segamat Campus)
85000, Johor, Malaysia
e-mail: geeclau@yahoo.com

WAI-CHEE SHIU

Department of Mathematics, Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong, Hong Kong, P.R. China
e-mail: wcshiu@hkbu.edu.hk

AND

HO-KUEN NG

Department of Mathematics, San José State University
San José CA 95192 USA
e-mail: ho-kuen.ng@sjsu.edu

Abstract

An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \rightarrow \{1, \ldots, |E|\}$ such that for any pair of adjacent vertices $x$ and $y$, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with $e$ ranging over all the edges incident to $x$. The local antimagic chromatic number of $G$, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of $G$. In this paper, several sufficient conditions for $\chi_{la}(H) \leq \chi_{la}(G)$ are obtained, where $H$ is obtained from $G$ with a certain edge deleted or added. We then determined the exact value of the local antimagic chromatic number of many cycle-related join graphs.

Keywords: local antimagic labeling, local antimagic chromatic number, cycle, join graphs.

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1. Introduction

A connected graph $G = (V, E)$ is said to be local antimagic if it admits a local antimagic edge labeling, i.e., a bijection $f : E \rightarrow \{1, \ldots, |E|\}$ such that the induced vertex labeling $f^+ : V \rightarrow \mathbb{Z}$ given by $f^+(u) = \sum f(e)$ (with $e$ ranging over all the edges incident to $u$) has the property that any two adjacent vertices have distinct induced vertex labels (see [1, 2]). Thus, $f^+$ is a coloring of $G$. Clearly, the order of $G$ must be at least 3. The vertex label $f^+(u)$ is called the induced color of $u$ under $f$ (the color of $u$, for short, if no ambiguity occurs). The number of distinct induced colors under $f$ is denoted by $c(f)$, and is called the color number of $f$. The local antimagic chromatic number of $G$, denoted by $\chi_{la}(G)$, is $\min \{c(f) : f$ is a local antimagic labeling of $G\}$.

Let $O_n = \overline{K_n}$ be the empty graph of order $n \geq 1$. For any graph $G$, the join graph $H = G \vee O_n$ is defined by $V(H) = V(G) \cup \{v_j : 1 \leq j \leq n\}$ and $E(H) = E(G) \cup \{uv_j : u \in V(G), 1 \leq j \leq n\}$. In [1, Theorem 2.16], it was claimed that for any $G$ with order $n \geq 4$,

$$\chi_{la}(G) + 1 \leq \chi_{la}(G \vee O_2) \leq \begin{cases} \chi_{la}(G) + 1 & \text{if } n \text{ is even,} \\ \chi_{la}(G) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

In [4], Lau et al. showed that there exists a graph $G$ order $n \geq 3$ such that $\chi_{la}(G \vee O_2) - \chi_{la}(G) = 3 - n \leq 0$. This implies that the above lower bound is invalid. They then showed that $\chi_{la}(G + O_n) \geq \chi(G) + 1$ and the bound is sharp. Several sufficient conditions for the following conjecture to hold were also given.

Conjecture 1.1. For $n \geq 1$, $\chi_{la}(G \vee O_n) \geq \chi_{la}(G) + 1$ if and only if $\chi(G) = \chi_{la}(G)$.

Let $G - e$ (or $G + e$) be the graph $G$ with an edge $e$ deleted (or added). As a natural extension, we have obtained in this paper several sufficient conditions for $\chi_{la}(G - e) \leq \chi_{la}(G)$ (or $\chi_{la}(G + e) \leq \chi_{la}(G)$). We then determine the exact value of the local antimagic chromatic number of many cycle related join graphs. We shall use the notation $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$, for integers $a \leq b$. Unless stated otherwise, all graphs considered in this paper are simple, undirected, connected and of order at least 3. Thus $\chi_{la}(G) \geq 2$ for any graph $G$. Interested readers may refer to Yu et al. [7] for local antimagic labeling of subcubic graphs without isolated edges.

For $m, n \geq 2$, it is well known that a magic $(m, n)$-rectangle exists if and only if $m \equiv n \pmod{2}$ and $(m, n) \neq (2, 2)$ (see [3, 6]). Let $a_{i,j}$ be the $(i, j)$-entry of a magic $(m, n)$-rectangle with row constant $n(mn + 1)/2$ and column constant $m(mn + 1)/2$. 


2. Bounds on Graphs with an Edge Deleted or Added

Observe that $K_t$, $t \geq 3$, is a complete $t$-partite graph with $\chi_{la}(K_t) = t$. The contrapositive of the following lemma gives a sufficient condition for a bipartite graph $G$ to have $\chi_{la}(G) \geq 3$.

**Lemma 2.1.** Let $G$ be a graph of size $q$. Suppose there is a local antimagic labeling of $G$ inducing a 2-coloring of $G$ with colors $x$ and $y$, where $x < y$. Let $X$ and $Y$ be the numbers of vertices of colors $x$ and $y$, respectively. Then $G$ is a bipartite graph whose sizes of parts are $X$ and $Y$ with $X > Y$, and

$$xX = yY = \frac{q(q + 1)}{2}.$$

**Proof.** Clearly $G$ is bipartite. Each edge is incident with one vertex of color $x$ and one vertex of color $y$. Hence we have the equation (1). Since $x < y$, $X > Y$. This completes the proof. ■

**Lemma 2.2.** Suppose $G$ is a $d$-regular graph of size $q$. If $f$ is a local antimagic labeling of $G$, then $g = q + 1 - f$ is also a local antimagic labeling of $G$ with $c(f) = c(g)$. Moreover, suppose $c(f) = \chi_{la}(G)$ and if $f(uv) = 1$ or $f(uv) = q$, then $\chi_{la}(G - uv) \leq \chi_{la}(G)$.

**Proof.** Let $x, y \in V(G)$. Here, $g^+(x) = d(q + 1) - f^+(x)$ and $g^+(y) = d(q + 1) - f^+(y)$. Therefore, $f^+(x) = f^+(y)$ if and only if $g^+(x) = g^+(y)$. Thus, $g$ is also a local antimagic labeling of $G$ with $c(g) = c(f)$.

If $f(uv) = q$, then we may consider $g = q + 1 - f$. So without loss of generality, we may assume that $f(uv) = 1$. Define $h : E(G - uv) \to [1, |E(G)| - 1]$ such that $h(e) = f(e) - 1$ for $e \neq uv$. So, $h^+(x) = f^+(x) - d$ for each vertex $x$ of $G - uv$. Therefore, $f^+(x) = f^+(y)$ if and only if $h^+(x) = h^+(y)$. Thus, $h$ is also a local antimagic labeling of $G$ with $c(h) = c(f)$. Consequently, $\chi_{la}(G - uv) \leq \chi_{la}(G)$. ■

Note that if $G$ is a regular edge-transitive graph, then $\chi_{la}(G - e) \leq \chi_{la}(G)$.

**Lemma 2.3.** Suppose $G$ is a graph of size $q$ and $f$ is a local antimagic labeling of $G$. For any $x, y \in V(G)$, if

(i) $f^+(x) = f^+(y)$ implies that $\deg(x) = \deg(y)$, and

(ii) $f^+(x) \neq f^+(y)$ implies that $(q + 1)(\deg(x) - \deg(y)) \neq f^+(x) - f^+(y)$,

then $g = q + 1 - f$ is also a local antimagic labeling of $G$ with $c(f) = c(g)$.

**Proof.** For any $x, y \in V(G)$, we have $g^+(x) = \deg(x)(q + 1) - f^+(x)$ and $g^+(y) = \deg(y)(q + 1) - f^+(y)$. Here $g^+(x) - g^+(y) = (q + 1)(\deg(x) - \deg(y)) - (f^+(x) - f^+(y))$. If $f^+(x) = f^+(y)$, then condition (i) implies that $g^+(x) = g^+(y)$. If $f^+(x) \neq f^+(y)$, then condition (ii) implies that $g^+(x) \neq g^+(y)$. Thus, $g$ is also a local antimagic labeling of $G$ with $c(g) = c(f)$. ■
For \( t \geq 2 \), consider the following conditions for a graph \( G \).

(i) \( \chi_{la}(G) = t \) and \( f \) is a local antimagic labeling of \( G \) that induces a \( t \)-independent partition \( \bigcup_{i=1}^{t} V_i \) of \( V(G) \).

(ii) For each \( x \in V_k \), \( 1 \leq k \leq t \), \( \deg(x) = d_k \) satisfying \( f^+(x) - d_a \neq f^+(y) - d_b \), where \( x \in V_a \) and \( y \in V_b \) for \( 1 \leq a \neq b \leq t \).

(iii) There exist two non-adjacent vertices \( u, v \) with \( u \in V_i, v \in V_j \) for some \( 1 \leq i \neq j \leq t \) such that

(a) \( |V_i| = |V_j| = 1 \) and \( \deg(x) = d_k \) for \( x \in V_k \), \( 1 \leq k \leq t \); or

(b) \( |V_i| = 1, |V_j| \geq 2 \) and \( \deg(x) = d_k \) for \( x \in V_k \), \( 1 \leq k \leq t \) except that \( \deg(v) = d_j - 1 \); or

(c) \( |V_i| \geq 2, |V_j| \geq 2 \) and \( \deg(x) = d_k \) for \( x \in V_k \), \( 1 \leq k \leq t \) except that \( \deg(u) = d_i - 1 \), \( \deg(v) = d_j - 1 \), each satisfying \( f^+(x) + d_a \neq f^+(y) + d_b \), where \( x \in V_a \) and \( y \in V_b \) for \( 1 \leq a \neq b \leq t \).

**Lemma 2.4.** Let \( H \) be obtained from \( G \) with an edge \( e \) deleted. If \( G \) satisfies conditions (i) and (ii) and \( f(e) = 1 \), then \( \chi(H) \leq \chi_{la}(H) \leq t \).

**Proof.** By definition, we have the lower bound. Define \( g : E(H) \rightarrow [1, |E(H)|] \) such that \( g(e') = f(e') - 1 \) for each \( e' \in E(H) \). Observe that \( g \) is a bijection with \( g^+(x) = f^+(x) - d_k \) for each \( x \in V_k \), \( 1 \leq k \leq t \). Thus, \( g^+(x) = g^+(y) \) if and only if \( x, y \in V_k \), \( 1 \leq k \leq t \). Therefore, \( g \) is a local antimagic labeling of \( H \) with \( c(g) = c(f) \). Thus, \( \chi_{la}(H) \leq t \). \( \blacksquare \)

**Lemma 2.5.** Suppose \( uv \not\in E(G) \). Let \( H \) be obtained from \( G \) with an edge \( uv \) added. If \( G \) satisfies conditions (i) and (iii), then \( \chi(H) \leq \chi_{la}(H) \leq t \).

**Proof.** By definition, we have the lower bound. Define \( g : E(H) \rightarrow [1, |E(H)|] \) such that \( g(uv) = 1 \) and \( g(e) = f(e) + 1 \) for \( e \in E(G) \). Observe that \( g \) is a bijection with \( g^+(x) = f^+(x) + d_k \) for each \( x \in V_k \), \( 1 \leq k \leq t \). Thus, \( g^+(x) = g^+(y) \) if and only if \( x, y \in V_k \), \( 1 \leq k \leq t \). Therefore, \( g \) is a local antimagic labeling of \( H \) with \( c(g) = c(f) \). Thus, \( \chi_{la}(H) \leq t \). \( \blacksquare \)

In [1, Theorem 2.11], the authors showed that for any two distinct integers \( m, n \geq 2 \), \( \chi_{la}(K_{m,n}) = 2 \) if and only if \( m \equiv n \) (mod 2). Let \( K^-_{m,n} \) be the graph \( K_{m,n} \) with an edge deleted. From the proof of [1, Theorem 2.11] and by Lemma 2.4, the following result is obvious.

**Corollary 2.6.** For any two distinct integers \( m, n \geq 2 \) and \( m \equiv n \) (mod 2), \( \chi_{la}(K^-_{m,n}) = 2 \).
3. Cycle-Related Join Graphs

Consider the join graph $C_m \vee O_n$ with $V(C_m) = \{u_i : 1 \leq i \leq m\}$, $V(O_n) = \{v_j : 1 \leq j \leq n\}$ and $E(C_m \vee O_n) = \{u_iu_{i+1} : 1 \leq i \leq m\} \cup \{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$, where $u_{m+1} = u_1$. Let $e_i = u_iu_{i+1}$ for $1 \leq i \leq m$. So $e_m = u_mu_1$. We shall keep these notations in this section unless stated otherwise.

**Theorem 3.1.** For odd $m, n \geq 3$, $\chi_{la}(C_m \vee O_n) = 4$.

**Proof.** Define an edge labeling $f : E(C_m \vee O_n) \to [1, mn + m]$ such that $f(e_{2i-1}) = i$ $(1 \leq i \leq (m + 1)/2)$ and $f(e_{2i}) = m + 1 - i$ $(1 \leq i \leq (m - 1)/2)$ and that $f(u_iv_j)$ is the $(i, j)$-entry of a magic $(m,n)$-rectangle containing integers in $[m+1, mn+m]$ with row sum constant $n(mn+1)/2 + mn$ and column sum constant $m(mn+1)/2 + m^2$. One can check that

(i) $f^+(v_j) = m(mn+1)/2 + m^2$,
(ii) $f^+(u_1) = n(mn+1)/2 + mn + (m+3)/2$,
(iii) $f^+(u_i) = n(mn+1)/2 + mn + m + 1$ for even $i$, and
(iv) $f^+(u_i) = n(mn+1)/2 + mn + m + 2$ for odd $i \geq 3$.

Suppose $m \leq n$. We have $mn(mn+1)/2 + m^2 < n(mn+1)/2 + mn + (m+3)/2 < n(mn+1)/2 + mn + m + 1 < n(mn+1)/2 + mn + m + 2$. So, $\chi_{la}(G) \leq 4$.

Suppose $m > n$. We have $m(mn+1)/2 + m^2 = n(mn+1)/2 + mn + (m-n)(mn+1)/2 > n(mn+1)/2 + mn + m + 2$. So, $\chi_{la}(G) \leq 4$.

Since $\chi_{la}(G) \geq \chi(G) = 4$, we have $\chi_{la}(G) = 4$.

**Corollary 3.2.** For odd $m, n \geq 3$, if $H = (C_m \vee O_n) - e$ where $e \notin E(C_m)$, then $\chi_{la}(H) = 4$.

**Proof.** Note that $G = C_m \vee O_n$ has size $mn + m$ and every vertex belonging to $C_m$ (or $O_n$) has degree $n + 2$ (or $m$). Let $f$ be the local antimagic labeling as defined in the proof of Theorem 3.1. We can check that $f$ satisfies the conditions of Lemma 2.3. Therefore, $g = mn + m + 1 - f$ is also a local antimagic labeling of $G$ with $c(g) = 4$ such that $g(e) = 1$ for an edge $e \notin E(C_m)$. It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we have $4 = \chi(H) \leq \chi_{la}(H) \leq 4$. Thus, the result holds.

**Theorem 3.3.** For $m \geq 2$ and $n \geq 1$, $\chi_{la}(C_{2m} \vee O_{2n}) = 3$.

**Proof.** Let $G = C_{2m} \vee O_{2n}$. Define an edge labeling $f : E(G) \to [1, 4mn + 2m]$ such that $f(e_h) = h$ for $1 \leq h \leq 2m$ and $f(u_kv_k)$ is given below, for $1 \leq h \leq 2m$ and $1 \leq k \leq 2n$.

We define $f(u_1v_1) = 2m + 1$ and $f(u_{2i-1}v_1) = 4m - 2i + 3$ for $2 \leq i \leq m$. For $1 \leq i \leq m$, define
(i) \( f(u_{2i-1}v_2) = 6m - 2i + 1 \);
(ii) \( f(u_{2i-1}v_{2j-1}) = 2m(j - 1) + 2i \) and \( f(u_{2i-1}v_{2j}) = 2m(2n - j - 1) - 2i + 2 \), for \( 2 \leq j \leq n \),
(iii) \( f(u_{2i}v_1) = 2m(2n + 1) - 2i + 2 \) and \( f(u_{2i}v_2) = 4mn - 2i + 2 \),
(iv) \( f(u_{2i}v_{2j-1}) = 2m(2n - j + 3) - 2i + 1 \) and \( f(u_{2i}v_{2j}) = 2m(j + 1) + 2i - 1 \), for \( 2 \leq j \leq n \).

One may check that \( f \) is a bijection. Observe that
(i) \( f(u_{2i-1}v_1) + f(u_{2i-1}v_2) = 10m - 4i + 4 \) and \( f(u_{2i}v_1) + f(u_{2i}v_2) = 8mn + 2m - 4i + 4 \) for \( 1 \leq i \leq m \),
(ii) \( f(u_{2i-1}v_{2j-1}) + f(u_{2i}v_{2j}) = 4m(n + 2) \) for \( 1 \leq i \leq m \) and \( 2 \leq j \leq n \),
(iii) \( f(u_{2i-1}v_{2j-1}) + f(u_{2i-1}v_{2j}) = 4mn + 2 \) for \( 1 \leq i \leq m \) and \( 2 \leq j \leq n \).

Thus
\[
\begin{align*}
f^+(u_1) &= f(e_1) + f(e_{2m}) + f(u_{1}v_1) + f(u_{1}v_2) + \sum_{j=2}^{n}(4mn + 2) \\
&= 4mn^2 - 4mn + 2n + 10m - 1;
\end{align*}
\[
\begin{align*}
f^+(u_{2i-1}) &= f(e_{2i-2}) + f(e_{2i-1}) + (10m - 4i + 4) + \sum_{j=2}^{n}(4mn + 2) \\
&= (4i - 3) + (10m - 4i + 4) + (4mn + 2)(n - 1) \\
&= 4mn^2 - 4mn + 2n + 10m - 1 \text{ if } 2 \leq i \leq m;
\end{align*}
\[
\begin{align*}
f^+(u_{2i}) &= f(e_{2i-1}) + f(e_{2i}) + (8mn + 2m - 4i + 4) + \sum_{j=2}^{n}4m(n + 2) \\
&= (8mn + 2m + 3) + 4m(n + 2)(n - 1) \\
&= 4mn^2 + 12mn - 6m + 3 \text{ if } 1 \leq i \leq m,
\end{align*}
\[
\begin{align*}
f^+(v_1) &= (2m + 1) + \sum_{i=2}^{m}(4m - 2i + 3) + \sum_{i=1}^{m}(4mn + 2m - 2i + 2) \\
&= 4m^2n + 4m^2 + m;
\end{align*}
\[
\begin{align*}
f^+(v_2) &= \sum_{i=1}^{m}(4mn + 6m - 4i + 3) = 4m^2n + 4m^2 + m;
\end{align*}
\[
\begin{align*}
f^+(v_k) &= \sum_{i=1}^{m}(4mn + 4m + 1) = 4m^2n + 4m^2 + m \text{ if } 3 \leq k \leq 2n.
\end{align*}

Now, let \( g_1 = f^+(u_{2i-1}) = 4mn^2 - 4mn + 2n + 10m - 1 \), \( g_2 = f^+(u_{2i}) = 4mn^2 + 12mn - 6m + 3 \), and \( g_3 = f^+(v_j) = 4m^2n + 4m^2 + m \). Clearly, \( g_1 < g_2 \).
Suppose \( n \geq m \). We have \( g_2 - g_3 = 4mn(n - m) + m(12n - 4m - 7) + 6 > 0 \).
Suppose \( m > n \). \( g_3 - g_2 = 4mn(m - n - 2) + m(4m - 4n + 7) - 3 \). When \( m - n \geq 2 \), clearly \( g_3 > g_2 \). For \( m - n = 1 \), \( g_3 - g_2 = -4m^2 + 15m - 3 \neq 0 \).
We now consider \( g_3 - g_1 = 2n[2m(m - n) - 1] + m(4n + 4m - 9) + 1 \). If \( m \geq n \), then \( g_3 - g_1 \geq 2n(m - 1) + m(2n + 4m - 9) + 1 > 0 \). Suppose \( n > m \).
Now \( g_1 - g_3 = 4mn(n - m - 2) + 4m(n - m) + 2n + 9m - 1 > 0 \) when \( n - m \geq 2 \).
When \( n - m = 1 \), \( g_1 - g_3 = -4m^2 + 11m + 1 \neq 0 \).
Thus, \( \chi_{la}(G) \leq 3 \). Since \( \chi_{la}(G) \geq \chi(G) = 3 \), we have \( \chi_{la}(G) = 3 \).

**Corollary 3.4.** For \( m \geq 2 \), \( n \geq 1 \), if \( H = (C_{2m} \lor O_{2n}) - e \), then \( \chi_{la}(H) = 3 \), where \( e \) is an edge of \( C_{2m} \lor O_{2n} \).

**Proof.** Note that \( G = C_{2m} \lor O_{2n} \) has size \( 4mn + 2m \) where every vertex belonging to \( C_{2m} \) (or \( O_{2n} \)) has degree \( 2n + 2 \) (or \( 2m \)). Let \( f \) be the local antimagic labeling as defined in the proof of Theorem 3.3. Suppose \( e \in E(C_{2m}) \). It is straightforward to check that \( f \) satisfies the conditions of Lemma 2.4. Thus, we have \( 3 = \chi(H) \leq \chi_{la}(H) \leq 3 \). Suppose \( e \notin E(C_{2m}) \). We can check that \( f \) satisfies the conditions of Lemma 2.3. Therefore, \( g = 4mn + 2m + 1 - f \) is also a local antimagic labeling of \( G \) with \( \chi(g) = 3 \) such that \( g(e) = 1 \). It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we have \( 3 = \chi(H) \leq \chi_{la}(H) \leq 3 \). Thus, the result holds.

Since for odd \( m, n \geq 3 \), \( \chi_{la}(C_m \lor O_n) = \chi_{la}(C_m) + 1 = \chi(C_m) + 1 \), and for even \( n \geq 2 \), \( \chi_{la}(C_m \lor O_n) = \chi_{la}(C_m) = \chi(C_m) + 1 \), Theorems 3.1 and 3.3 provide further evidence that Conjecture 1.1 holds.

Note that \( C_m \lor O_1 = W_m \), the wheel graph of order \( m + 1 \geq 4 \). In [4, Theorem 3.1], the authors proved that \( \chi_{la}(W_m) = 3 \) if \( m \equiv 0 \mod 4 \). In [1, Theorem 2.14], the authors proved that \( \chi_{la}(W_m) = 3 \) if \( m \equiv 2 \mod 4 \), and \( \chi_{la}(W_m) = 4 \) if \( m \) is odd. We note that for \( m \equiv 1 \mod 4 \), the defined local antimagic labeling \( f \) (or \( f_3 \) in the proof) has three errors that should be corrected as \( f(v_i) = (8m + 5 - i)/4 \) for \( i \equiv 1 \mod 4 \), \( i \neq 1 \); \( f(v_i) = (7m + 4 - i)/4 \) for \( i \equiv 3 \mod 4 \); and \( f^+(v_i) = (11m + 13)/4 \) for odd \( i \neq 1 \). Moreover, for \( m \equiv 3 \mod 4 \), the induced vertex label for \( v_i \), \( i \neq 1 \) is odd, should be \( 9(m + 1)/4 \).

**Theorem 3.5.**

\[
\chi_{la}(W_4 - e) = \begin{cases} 
3 & \text{if } e \notin E(C_4), \\
4 & \text{otherwise}.
\end{cases}
\]

**Proof.** The graph in Figure 1 shows that \( W_4 - e \) admits a local antimagic labeling \( f \) with \( c(f) = 3 \) so that \( \chi_{la}(W_4 - e) = 3 \) if \( e \notin E(C_4) \).

Suppose \( e \in E(C_4) \). Without loss of generality we may assume that \( e = u_4u_1 \).
Suppose there were a local antimagic labeling \( f \) of \( W_4 - e \) with \( c(f) = 3 \). Then
\[ f^+(v_1) = c, \ f^+(u_1) = f^+(u_3) = a \ \text{and} \ f^+(u_2) = f^+(u_4) = b, \ \text{where} \ a, b, c \ \text{are distinct.} \]

![Figure 1. \( W_4 - e \).](image)

Clearly

\[ (2) \quad 28 = \sum_{i=1}^{7} i = 2a + f(v_1u_2) + f(v_1u_4) = 2b + f(v_1u_1) + f(v_1u_3). \]

Thus, \( f(v_1u_2) \equiv f(v_1u_4) \pmod{2} \) and \( f(v_1u_1) \equiv f(v_1u_3) \pmod{2} \).

It is easy to check that \( \{f(v_1u_2), f(u_2u_3), f(u_3u_4)\} \neq \{2, 4, 6\} \). So we may assume that \( f(v_1u_1) \) and \( f(v_1u_3) \) are odd, and \( f(v_1u_2) \) and \( f(v_1u_4) \) are even.

Under these conditions and from (2) we have \( 9 \leq a \leq 11 \) and \( 8 \leq b \leq 12 \).

1. Suppose \( a = 9 \). Then \( f(v_1u_2) + f(v_1u_4) = 10 \) and hence \( \{f(v_1u_2), f(v_1u_4)\} = \{4, 6\} \). This implies that \( f(u_1u_2) = 2 \) and \( f(v_1u_1) = 7 \). If \( f(v_1u_2) = 4 \) and \( f(v_1u_4) = 6 \), then \( f(u_2u_3) = f(u_3u_4) \) which is impossible. Thus \( f(v_1u_2) = 6 \) and \( f(v_1u_4) = 4 \). This implies that \( 9 \leq 2 + 6 + f(u_2u_3) = b = 4 + f(u_3u_4) \leq 9 \).

Hence \( b = 9 = a \) which is a contradiction.

2. Suppose \( a = 10 \). We have \( \{f(v_1u_1), f(u_1u_2)\} = \{3, 7\} \) and \( \{f(v_1u_3), f(u_2u_3), f(u_3u_4)\} = \{1, 4, 5\} \). Since \( f(v_1u_2) + f(v_1u_4) = 8, \ \{f(v_1u_2), f(v_1u_4)\} = \{2, 6\} \).

Since \( b \geq 8, f(v_1u_4) = 6 \). Hence \( f(v_1u_2) = 2 \). Since \( a \neq b, f(u_3u_4) = 5 \) and hence \( f(u_2u_3) = 4 \). Now \( f^+(u_2) \neq b = 11 \), which is a contradiction.

3. Suppose \( a = 11 \). We have \( f(v_1u_2) + f(v_1u_4) = 6 \). This implies that \( \\{f(v_1u_2), f(v_1u_4)\} = \{2, 4\} \).

Since \( 4 \) is occupied and \( f(v_1u_1) + f(u_1u_2) = 11, f(v_1u_1) = 5 \) and \( f(u_1u_2) = 6 \). Also we have \( \{f(v_1u_3), f(u_2u_3), f(u_3u_4)\} = \{1, 3, 7\} \).

Since \( b \geq 8, f(u_3u_4) = 7 \). Since \( b \neq a, f(v_1u_4) = 2 \). Now \( b = 9 \) and \( f^+(u_2) \geq 10 \) which yields a contradiction.

As a conclusion, \( \chi_{la}(W_4 - e) \geq 4 \). Note that from the discussion above, we have obtained a local antimagic labeling \( g \) for \( W_4 - e \) with \( c(g) = 4 \).

\[ \blacksquare \]

**Theorem 3.6.** Let \( e \) be an edge of \( W_m \). For even \( m \geq 6 \), \( \chi_{la}(W_m - e) = 3 \).
Proof. Consider $m = 6$. In Figure 2, we have the local antimagic labelings $f$ with $c(f) = 3$ for the two cases of $W_6 - e$.

Thus, $\chi_{la}(W_6 - e) = 3$.

Consider $m \geq 8$. We have two cases.

Case (a) $e \in E(C_m)$. By [4, Theorem 3.1] and [1, Theorem 2.14] and the proofs, we have $\chi_{la}(W_m) = 3$ such that the corresponding local antimagic labeling $f$ has $f(u_1u_2) = 1$. By symmetry we may let $e = u_1u_2$. By Lemma 2.4, we get $\chi_{la}(W_m - e) \leq 3$. Since $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$, $\chi_{la}(W_m - e) = 3$.

Case (b) $e \notin E(C_m)$. For $m = 8$, the graph in Figure 3(a) shows that $W_8 - e$ admits a local antimagic labeling $g$ with $c(g) = 3$. Thus, $\chi_{la}(W_8 - e) = 3$.

Consider $m \geq 10$. By [4, Theorem 3.1] and [1, Theorem 2.14] and the proofs, we know that $W_m$ admits a local antimagic labeling $f$ with $f(v_1u_2) = 2m$ if $m \equiv 0 \pmod{4}$, and $f(v_1u_4) = 2m$ if $m \equiv 2 \pmod{4}$. By symmetry we may let $e = v_1u_2$ if $m \equiv 0 \pmod{4}$, and $e = v_1u_4$ if $m \equiv 2 \pmod{4}$. It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we get $\chi_{la}(W_m - e) \leq 3$. Since $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$, $\chi_{la}(W_m - e) = 3$.

\[\chi_{la}(W_m - e) = \begin{cases} 3 & \text{for } m = 3, 5, 7; \\ 4 & \text{otherwise.} \end{cases}\]

Theorem 3.7. Suppose $m \geq 3$ is odd. If $e \notin E(C_m)$, then

\[\chi_{la}(W_m - e) = \begin{cases} 3 & \text{for } m = 3, 5, 7; \\ 4 & \text{otherwise.} \end{cases}\]
If $e \in E(C_m)$, then $3 \leq \chi_{la}(W_m - e) \leq 4$.

**Proof.** Suppose $e \notin E(C_m)$. Note that $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$. Suppose $e \notin E(C_m)$. Note that $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$. Without loss of generality, assume $e = v_1u_{2k+1}$. Thus, we must have $f^+(v_1) = f^+(u_{2k+1}) \neq f^+(u_1) = f^+(u_3) = \cdots = f^+(u_{2k-1}) \neq f^+(u_2) = f^+(u_4) = f^+(u_{2k})$. Thus, $k(2k + 1) \leq f^+(v_1) = f^+(u_{2k+1}) \leq 8k + 1$ giving us $1 \leq k \leq 3$. Thus, $\chi_{la}(W_m - e) \geq 4$ for $m \geq 9$. For $m = 3$, $W_3 - e \cong K_{1,1,2}$. The labeling is obvious. For $m = 5$, the labeling in Figure 3(b) shows that $\chi_{la}(W_5 - v_1u_5) = 3$. For $m = 7$, the labeling in Figure 3(c) shows that $\chi_{la}(W_7 - v_1u_7) = 3$.

Consider $m \geq 9$. By [1, Theorem 2.14] and the proof, we know that $W_m$ admits a local antimagic labeling $f$ with $c(f) = 4$. Moreover, $f(v_1u_5) = 2m$ if $m \equiv 1 \pmod{4}$, and $f(v_1u_2) = 2m$ if $m \equiv 3 \pmod{4}$. It is straightforward to check the conditions of Lemmas 2.3 and 2.4. By Lemma 2.3, we know $W_m$ admits a local antimagic labeling $g$ with $g(v_1u_5) = 1$ if $m \equiv 1 \pmod{4}$, and $g(v_1u_2) = 1$ if $m \equiv 3 \pmod{4}$. By Lemma 2.4, we get $\chi_{la}(W_m - e) = 4$.

Suppose $e \in E(C_m)$. By [1, Theorem 2.14] and the proof, together with Lemma 2.4, we know that $\chi_{la}(W_m - e) \leq 4$.

**Theorem 3.8.** For odd $m, n \geq 3$, $\chi_{la}(C_m \vee C_n) = 6$.

**Proof.** Since $C_m \vee C_n$ and $C_n \vee C_m$ are isomorphic, we may assume that $n \leq m$. Suppose $V(C_m \vee C_n) = V(C_m \vee O_n)$ and $E(C_m \vee C_n) = E(C_m \vee O_n) \cup \{e'_j = v_jv_{j+1} : 1 \leq j \leq n\}$ as in Theorem 3.1, where $v_{n+1} = v_1$. Let $f$ be the local antimagic labeling of $C_m \vee O_n$ defined in the proof of Theorem 3.1. Define an edge labeling $g : E(C_m \vee C_n) \to [1, m + mn + n]$ such that $g(e) = f(e)$ for $e \in E(C_m \vee O_n)$ and $g(e'_j) = m + mn + f(e_j)$. One may check that $g$ is a bijection. Moreover,

(i) $g^+(u_1) = g_5 = (mn+1)/2 + mn + (m+3)/2$,
(ii) $g^+(u_i) = g_2 = (mn+1)/2 + mn + m + 1$ for even $i$,
(iii) $g^+(u_i) = g_3 = (mn+1)/2 + mn + m + 2$ for odd $i \geq 3$,
(iv) $g^+(v_1) = g_4 = (mn+1)/2 + m^2 + 2(m+mn) + (n+3)/2$,
(v) $g^+(v_j) = g_5 = (mn+1)/2 + m^2 + 2(m+mn) + n + 1$ for even $j$, and
(vi) $g^+(v_j) = g_6 = (mn+1)/2 + m^2 + 2(m+mn) + n + 2$ for odd $j \geq 3$.

Clearly $g_k < g_{k+1}$ for $1 \leq k \leq 5$. Thus, $\chi_{la}(C_m \vee C_n) \leq 6$. Since $\chi_{la}(C_m \vee C_n) \geq \chi(C_m \vee C_n) = \chi(C_m) + \chi(C_n) = 6$, we have $\chi_{la}(C_m \vee C_n) = 6$.

In [5], Haslegrave proved that every connected graph $G \neq K_2$ admits a local antimagic labeling which implies that $\chi_{la}(K_n) = n$ for all $n \geq 3$. We now consider the join graph $C_m \vee K_n$ with $V(C_m \vee K_n) = V(C_m \vee O_n)$ and
$E(C_m \lor K_n) = E(C_m \lor O_n) \cup \{v_iv_j : 1 \leq i < j \leq n\}$. In [1], the authors showed that $\chi_{la}(C_m \lor K_1) = 4$ for odd $m \geq 3$.

**Theorem 3.9.** For odd $m, n \geq 3$, $\chi_{la}(C_m \lor K_n) = n + 3$.

**Proof.** Let $f$ be the local antimagic labeling of $C_m \lor O_n$ defined in the proof of Theorem 3.1. Let $h : E(K_n) \to [1, n(n - 1)/2]$ be a local antimagic labeling of $K_n$. Note that $h^+(v_j)$ are distinct for $1 \leq j \leq n$. Define an edge labeling $g : E(C_m \lor K_n) \to [1, mn + m + n(n - 1)/2]$ such that $g(e) = f(e)$ for $e \in E(C_m \lor O_n)$ and $g(e) = h(e) + mn + m$ for $e \in E(K_n)$. Note that $g^+(v_j) = f^+(v_j) + h^+(v_j) + (n - 1)(mn + n)$. Since $f^+(v_j)$ are the same and $h^+(v_j)$ are distinct, $g^+(v_j)$ are distinct for $1 \leq j \leq n$.

Moreover,

(i) $g^+(u_1) = n(mn + 1)/2 + mn + (m + 3)/2$,
(ii) $g^+(u_i) = n(mn + 1)/2 + mn + m + 1$ for even $i$,
(iii) $g^+(u_i) = n(mn + 1)/2 + mn + m + 2$ for odd $i \geq 3$, and
(iv) $g^+(v_j) = f^+(v_j) + h^+(v_j) + (n - 1)(mn + n) \geq m(mn + 1)/2 + m^2 + (n - 1)(mn + m + n(n - 1)/2).

It is easy to show that $g^+(v_j) > g^+(u_i)$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Thus, $\chi_{la}(C_m \lor K_n) \leq n + 3$. Since $\chi_{la}(C_m \lor K_n) \geq \chi(C_m \lor K_n) = n + 3$, the theorem holds.

**Theorem 3.10.** For $m \geq 2, n \geq 1$, $\chi_{la}(C_{2m} \lor K_{2n}) = 2n + 2$.

**Proof.** Let $f$ be the local antimagic labeling of $C_{2m} \lor O_{2n}$ defined in the proof of Theorem 3.3.

Suppose $n = 1$. Define an edge labeling $g : E(C_{2m} \lor K_2) \to [1, 6m + 1]$ such that $g(e) = f(e)$ for $e \in E(C_{2m} \lor O_2)$ and $g(v_1v_2) = 6m + 1$. We now swap the labels of $g(u_1v_1) = 2m + 1$ and $g(u_1v_2) = 6m - 1$ to get $g^+(u_{2i-1}) = 10m + 1$ and $g^+(u_{2i}) = 10m + 3$ for $1 \leq i \leq m$ and $g^+(v_1) = 8m^2 + 11m - 1$ and $g^+(v_2) = 8m^2 + 3m + 3$. Thus, $\chi_{la}(C_{2m} \lor K_2) \leq 4$.

Now, consider $n \geq 2$. Let $h : E(K_{2n}) \to [1, n(2n - 1)]$ be a local antimagic labeling of $K_{2n}$. Note that $h^+(v_j)$ are distinct for $1 \leq j \leq 2n$. Define an edge labeling $g : E(C_{2m} \lor K_{2n}) \to [1, 4mn + 2m + n(2n - 1)]$ such that $g(e) = f(e)$ for $e \in E(C_{2m} \lor O_{2n})$ and $g(e) = h(e) + 4mn + 2m$ for $e \in E(K_{2n})$.

By the same argument as in the proof of Theorem 3.9, we obtain that $g^+(v_j)$ are distinct for $1 \leq j \leq 2n$.

From Theorem 3.3 we have $g^+(u_{2i}) = 4mn^2 - 4mn + 2n + 10m - 1 < g^+(u_{2i-1}) = 4mn^2 + 12mn - 6m + 3$ for $1 \leq i \leq m$. Moreover, $g^+(v_j) = f^+(v_j) + h^+(v_j) + (2n - 1)(4mn + 2m) \geq 4m^2n + 4m^2 + m + (2n - 1)(4mn + 2m) + n(2n - 1)$ for each $j$. Clearly $g^+(v_j) > g^+(u_{2i-1})$ for $1 \leq i \leq m$ and $1 \leq j \leq 2n$. 

Thus, \( \chi_{la}(C_{2m} \lor K_{2n}) \leq 2n + 2 \). Since \( \chi_{la}(C_{2m} \lor K_{2n}) \geq \chi(C_{2m} \lor K_{2n}) = 2n + 2 \), the theorem holds.

**Conjecture 3.11.** For \( n \geq 2 \), \( \chi_{la}(G \lor K_n) \geq \chi_{la}(G) + n \) if and only if \( \chi_{la}(G) = \chi(G) \).

For \( n \geq 2 \), let \( M_{2n} \) be the Möbius ladder obtained from \( C_{2n} = u_1 u_2 \cdots u_n v_1 v_2 \cdots v_n u_1 \) by adding the edges \( u_i v_i, 1 \leq i \leq n \).

**Theorem 3.12.** For odd \( n \geq 3 \), \( \chi_{la}(M_{2n}) = 3 \).

**Proof.** Note that \( M_{2n} \) has size \( 3n \), and is bipartite with parts of the same size. Thus, by Lemma 2.1, \( \chi_{la}(M_{2n}) \geq 3 \).

Suppose \( n = 3 \), we get a local antimagic labeling by assigning the edges \( u_1 u_2, u_2 u_3, u_3 v_1, v_1 v_2, v_2 v_3, v_3 u_1, u_1 u_2, u_2 v_2, u_3 v_3 \) by \( 1, 5, 4, 8, 6, 7, 3, 9, 2 \), respectively. Clearly, the induced vertex coloring has three distinct colors, namely \( 11, 15, 23 \).

Suppose \( n \geq 5 \). Define a bijection \( f : E(M_{2n}) \rightarrow [1, 3n] \) such that \( f(u_1 v_n) = \frac{3(n+1)}{2} \), \( f(u_n v_1) = n, f(v_1 v_2) = n + 1 \) and that

\[
\begin{align*}
(\text{i}) & \quad f(u_i u_{i+1}) = i \text{ for odd } i \in [1, n-2], \\
(\text{ii}) & \quad f(u_i u_{i+1}) = \frac{3n+3-i}{2} \text{ for even } i \in [2, n-1], \\
(\text{iii}) & \quad f(v_i v_{i+1}) = i \text{ for even } i \in [2, n-1], \\
(\text{iv}) & \quad f(v_i v_{i+1}) = 2n - \frac{i-3}{2} \text{ for odd } i \in [3, n-2], \\
(\text{v}) & \quad f(u_i v_i) = \frac{5n+i-1}{2} \text{ for odd } i \in [1, n], \\
(\text{vi}) & \quad f(u_i v_i) = 3n + 1 - \frac{i}{2} \text{ for even } i \in [2, n-1].
\end{align*}
\]

One can verify that \( f^+(u_i) = f^+(v_j) = \frac{9n+3}{2} \) for even \( i \in [2, n-1] \) and odd \( j \in [1, n] \); \( f^+(u_i) = f^+(v_j) = 4n + 3 \) for odd \( i \in [1, n] \) and \( f^+(v_j) = 5n + 3 \) for even \( j \in [4, n-1] \). Therefore, \( \chi_{la}(M_{2n}) \leq 3 \). Hence, the theorem holds.

**Corollary 3.13.** For odd \( n \geq 3 \), \( \chi_{la}(M_{2n} - e) = 3 \).

**Proof.** By Lemma 2.1, we know that \( \chi_{la}(M_{2n} - e) \geq 3 \). Note that there are two possible graphs obtained by deleting an edge from \( M_{2n} \) (if \( n > 3 \)), but using Lemma 2.2 with reference to the smallest label deals with one, and the largest label deals with the other. Therefore, we have \( \chi_{la}(M_{2n} - e) \leq 3 \). Thus, \( \chi_{la}(M_{2n} - e) = 3 \).

Note that \( M_4 = K_4 \) with \( \chi_{la}(M_4) = 4 \).

**Conjecture 3.14.** For even \( n \geq 4 \), \( \chi_{la}(M_{2n}) = 4 \).

**Theorem 3.15.** For \( n \geq 1 \), \( \chi_{la}(M_6 \lor O_{2n}) = 3 \).
Proof. Let \( V(M_6 \lor O_{2n}) = \{u_i : 1 \leq i \leq 6\} \cup \{v_j : 1 \leq j \leq 2n\} \) and \( E(M_6 \lor O_{2n}) = \{u_iu_{i+1} : 1 \leq i \leq 5\} \cup \{u_1u_6, u_1u_4, u_2u_5, u_3u_6\} \cup \{u_iv_j : 1 \leq i \leq 6, 1 \leq j \leq 2n\} \). Define a bijection \( g : E(M_6 \lor O_{2n}) \rightarrow [1, 12n + 9] \) such that \( g(u_1u_2) = 1, g(u_2u_3) = 3, g(u_3u_4) = 4, g(u_4u_5) = 2, g(u_5u_6) = 8, g(u_1u_4) = 9, g(u_2u_5) = 7, g(u_3u_6) = 6 \) and \( g(u_iv_j) = f(u_iv_j) + 3 \) for \( 1 \leq i \leq 6, 1 \leq j \leq 2n \),

where \( f \) is the function as defined in the proof of Theorem 3.3 by taking \( m = 3 \).

One can easily check that \( g^+(u_1) = 1, g^+(u_i) = g^+(u_1) \) for \( i = 2, 4, 6 \), we also have \( g^+(u_i) = 12n^2 + 42n - 7 \), whereas \( g^+(v_j) = 36n + 57 \) for \( 1 \leq j \leq 2n \). Clearly, \( g \) is a local antimagic labeling with \( c(g) = 3 \). Therefore, \( \chi_{la}(M_6 \lor O_{2n}) \leq 3. \) Since \( M_6 \) is bipartite, we have \( \chi_{la}(M_6 \lor O_{2n}) \geq \chi(M_6 \lor O_{2n}) = \chi(M_6) + \chi(O_{2n}) = 3. \) Thus, \( \chi_{la}(M_6 \lor O_{2n}) = 3. \)

Corollary 3.16. For \( n \geq 1 \), \( \chi_{la}((M_6 \lor O_{2n}) - e) = 3. \)

Proof. Let \( G = (M_6 \lor O_{2n}) - e \). We note that \( \chi_{la}(G) \geq \chi(G) = 3 \). Since \( M_6 \) is edge-transitive, we only need to consider (i) \( e \not\in E(M_6) \), and (ii) \( e \in E(M_6) \).

In (i), it is straightforward to check the conditions of Lemma 2.3. By Lemma 2.3, we know \( M_6 \lor O_{2n} \) admits a local antimagic labeling \( h = 12n + 10 - g \) with \( c(h) = c(g) = 3 \), where \( g \) is as defined in the proof of Theorem 3.15. Now,

\[
h^+(u_i) = \begin{cases} 
12n^2 + 60n - 7 & \text{if } i = 1, 3, 5, \\
12n^2 + 14n + 37 & \text{if } i = 2, 4, 6,
\end{cases}
\]

\( h^+(v_j) = 36n + 3 \) for \( 1 \leq j \leq 2n \), and \( h(uv) = 1 \) for an edge \( uv \not\in E(M_6) \). It is straightforward to check the condition of Lemma 2.4. By Lemma 2.4, we have \( \chi_{la}(G) = 3. \)

In (ii), it is straightforward to check the condition of Lemma 2.4. By Lemma 2.4, we have \( \chi_{la}(G) = 3. \)

For \( m \geq 3, n \geq 1 \), let \( G(m, n) \) be the graph obtained from \( C_m \lor O_n \) by deleting the edges \( u_mv_j, 1 \leq j \leq n \). We can also view \( G(m, n) \) as the graph obtained from \( C_{m-1} \lor O_n \) by subdividing one of the cycle edges. Note that \( G(m, 1) \) is the graph \( W_m \) with a spoke deleted. By Theorems 3.5 and 3.6, we have \( \chi_{la}(G(2m, 1)) = 3 \) for \( m \geq 2 \). Moreover, by Theorem 3.7, we have determined the value of \( \chi_{la}(G(2m + 1, 1)) \) for \( m \geq 1 \).

Theorem 3.17. For \( n \geq 1, \chi_{la}(G(4, n)) = 3. \)

Proof. When \( n = 1 \), we have proved the result in Theorem 3.5. So we may assume that \( n \geq 2 \). Since \( \chi(G(4, n)) \geq 3 \), it suffices to provide a local antimagic labeling \( f \) for \( G(4, n) \) with \( c(f) = 3. \)
For $n = 4k - 1$, $k \geq 1$, the labeling matrix of $G(4, 3)$ under $f$ is given below.

<table>
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<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
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<td>*</td>
<td>9</td>
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<td>*</td>
<td>*</td>
<td>*</td>
<td>* 19</td>
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The following tables are the first 4 rows of the labeling matrix of $G(4, 4k - 1)$ under $f$, where $k \geq 3$.

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</tr>
<tr>
<td>$u_4$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>18k + 1</td>
</tr>
</tbody>
</table>

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k + 1$, i.e., the $v_j$-column sum, for $1 \leq j \leq 4k - 1$. This labeling can be applied to $k = 2$ (the block-columns for $v_{2k+1}$ to $v_{4k-4}$ do not appear). The following shows the assignment for $G(4, 7)$.

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$f^+(u_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>21</td>
<td>*</td>
<td>12</td>
<td>16</td>
<td>15</td>
<td>18</td>
<td>17</td>
<td>2</td>
<td>10</td>
<td>9</td>
<td>120</td>
</tr>
<tr>
<td>$u_2$</td>
<td>21</td>
<td>8</td>
<td>*</td>
<td>*</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>3</td>
<td>24</td>
<td>23</td>
<td>22</td>
</tr>
<tr>
<td>$u_3$</td>
<td>*</td>
<td>8</td>
<td>25</td>
<td>20</td>
<td>19</td>
<td>14</td>
<td>13</td>
<td>11</td>
<td>4</td>
<td>6</td>
<td>120</td>
</tr>
<tr>
<td>$u_4$</td>
<td>12</td>
<td>25</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>37</td>
</tr>
</tbody>
</table>

For $n = 4k + 1$, $k \geq 1$, the labeling matrix for $G(4, 5)$ is given next.
Similarly, we show the first 4 rows of the labeling matrix of $G(4, 4k+1)$ under $f$, where $k \geq 3$.

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_{k-2}$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$v_k$</th>
<th>$v_{k+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>19</td>
<td>22</td>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>22</td>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>22</td>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>25</td>
</tr>
</tbody>
</table>

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k + 10$, for $1 \leq j \leq 4k + 1$. This labeling can be applied to $k = 2$ (the block-columns for $v_1$ to $v_{2k-4}$ do not appear). The following shows the assignment for $G(4, 9)$.

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_9$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$f^+(u_i)$</th>
<th>$f^+(v_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>18k + 10</td>
<td>18k + 10</td>
</tr>
<tr>
<td>25</td>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>28</td>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>31</td>
<td>26</td>
<td>31</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>10</td>
<td>3</td>
<td>22</td>
<td>22</td>
</tr>
</tbody>
</table>
It is easy to check that $f^+(u_4) = f^+(v_j) = 18k + 13$, for $1 \leq j \leq 4k + 2$. This labeling can be applied to $k = 0$. The following shows the assignment for $G(4, 2)$.

<table>
<thead>
<tr>
<th>$v_{3k+1}$ $v_{3k+2}$ $\cdots$ $v_{4k+1}$</th>
<th>$v_{4k+2}$</th>
<th>$f^+(u_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$ $5k + 3$ $5k + 2$ $\cdots$ $4k + 4$</td>
<td>$4k + 3$</td>
<td>32$k^2 + 55k + 23$</td>
</tr>
<tr>
<td>$u_2$ $2k + 2$ $2k + 4$ $\cdots$ $4k$</td>
<td>$10k + 8$</td>
<td>$8k^2 + 36k + 25$</td>
</tr>
<tr>
<td>$u_3$ $11k + 8$ $11k + 7$ $\cdots$ $10k + 9$</td>
<td>$4k + 2$</td>
<td>$32k^2 + 55k + 23$</td>
</tr>
<tr>
<td>$u_4$ $\ast$ $\ast$ $\ast$ $\ast$</td>
<td>$18k + 13$</td>
<td></td>
</tr>
</tbody>
</table>

For $n = 4k$, the following tables are the first 4 rows of the labeling matrix of $G(4, 4k)$ under $f$, where $k \geq 2$.

<table>
<thead>
<tr>
<th>$v_{k-1}$</th>
<th>$v_{k-2}$</th>
<th>$\cdots$</th>
<th>$v_1$</th>
<th>$v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$ $\ast$ $10k + 3$ $\ast$ $12k + 4$</td>
<td>$10k + 2$ $10k + 1$ $\cdots$ $9k + 4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_2$ $10k + 3$ $\ast$ $6k + 2$ $\ast$</td>
<td>$1$ $3$ $\cdots$ $2k - 3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_3$ $\ast$ $6k + 2$ $\ast$ $6k + 1$</td>
<td>$8k + 2$ $8k + 1$ $\cdots$ $7k + 4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_4$ $12k + 4$ $\ast$ $6k + 1$ $\ast$</td>
<td>$\ast$ $\ast$ $\cdots$ $\ast$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v_{j}$</th>
<th>$v_{j+1}$ $\cdots$ $v_{j+2}$ $\cdots$ $v_{j+1}$</th>
<th>$f^+(u_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$ $7k + 3$ $7k + 2$ $\cdots$ $6k + 8$</td>
<td>$6k + 4$</td>
<td>$12k + 3$ $12k + 2$ $\cdots$ $11k + 5$</td>
</tr>
<tr>
<td>$u_2$ $2k - 1$ $2k + 1$ $\cdots$ $4k - 5$</td>
<td>$4k - 3$</td>
<td>$2$ $4$ $\cdots$ $2k - 2$</td>
</tr>
<tr>
<td>$u_3$ $9k + 3$ $9k + 2$ $\cdots$ $8k + 5$</td>
<td>$8k + 4$</td>
<td>$8k + 3$ $6k$ $6k - 1$ $\cdots$ $5k + 2$</td>
</tr>
<tr>
<td>$u_4$ $\ast$ $\ast$ $\cdots$ $\ast$</td>
<td>$\ast$</td>
<td>$\ast$ $\ast$ $\cdots$ $\ast$</td>
</tr>
</tbody>
</table>

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k + 5$, for $1 \leq j \leq 4k$. Again, this labeling can be applied to $k = 1$. The following shows the assignment for
\( G(4, 4) \).

\[
\begin{array}{c|cccc|cccc|c}
 & u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 & f^+(u_i) \\
\hline
u_1 & * & 13 & * & 16 & 10 & 9 & 6 & 4 & 58 \\
u_2 & 13 & * & 8 & * & 1 & 3 & 2 & 14 & 41 \\
u_3 & * & 8 & * & 7 & 12 & 11 & 15 & 5 & 58 \\
u_4 & 16 & * & 7 & * & * & * & * & * & 23 \\
\hline
f^+(v_j) & * & * & * & * & 23 & 23 & 23 & 23 & \\
\end{array}
\]

Since \( f^+(u_1) = f^+(u_3) \neq f^+(u_2) \neq f^+(u_4) = f^+(v_j) \), \( 1 \leq j \leq n \), we have \( c(f) = 3 \). The proof is complete.

Note that \( P_3 \vee O_{n+1} \) can be obtained from \( G(4, n) \) by adding the edge \( u_2u_4 \).

By Lemma 2.5, the following is obtained.

**Corollary 3.18.** If \( G \equiv P_3 \vee O_{n+1} \), then \( \chi_{la}(G) = 3 \).

**Problem 3.19.** Determine \( \chi_{la}(P_m \vee O_n) \) for \( m \geq 4, n \geq 2 \).

**Theorem 3.20.** For (i) \( m \geq 3, n \geq 4 \), (ii) \( m \geq 21, n = 3 \), and (iii) \( m \geq 4, n = 2, \chi_{la}(G(2m, 2n - 1)) = 4 \).

**Proof.** Note that \( \chi_{la}(G(2m, 2n - 1)) \geq \chi(G(2m, 2n - 1)) = 3 \). Suppose \( f \) is a local antimagic labeling of \( G(2m, 2n - 1) \) with \( c(f) = 3 \). We may have (I) \( a = f^+(u_{2i-1}), 1 \leq i \leq m; \ b = f^+(v_j) = f^+(u_{2m}), 1 \leq j \leq 2n - 1; \ c = f^+(u_{2i}), 1 \leq i < m \); or (II) \( a = f^+(u_{2i-1}), 1 \leq i \leq m; \ b = f^+(v_j), 1 \leq j \leq 2n - 1; \ c = f^+(u_{2i}), 1 \leq i \leq m \). Here \( a, b, c \) are distinct. Now, every \( v_j \) is adjacent to \( 2m - 1 \) vertices of \( C_{2m} \).

For (I), \( \sum_{j=1}^{2n-1} f^+(v_j) \geq 1 + 2 + \cdots + (2n - 1)(2m - 1) = (2n - 1)(2m - 1)(2mn - m - n + 1) \). So,

\[
(3) \quad (2m - 1)(2mn - m - n + 1) \leq b = f^+(u_{2m}) \leq 8mn - 4n + 1
\]

giving \( n \leq \frac{(2m-1)(m-1)+1}{2(2m-1)(2m-2)} \). By simple calculus, we have \( n \leq \frac{11}{3} \). When \( n = 2 \), we get \( m = 3 \). This is not a case.

For (II), there are exactly \((2n - 1)(m - 1) + 2m - 2 = 2mn + m - 2n - 1\) edges incident to the vertices \( u_{2i} \) for \( 1 \leq i \leq m - 1 \). Each label of these edges contributes to the sum \( \sum_{i=1}^{m-1} f^+(u_{2i}) \) exactly once. Thus, \( (m - 1)c \geq \frac{1}{2}(2mn + m - 2n - 1)(2mn + m - 2n) \). Therefore, we will get

\[
(4) \quad (2n + 1)(2mn + m - 2n) \leq 2c = 2f^+(u_{2m}) \leq 16mn - 8n + 2.
\]

However, if \( n \geq 5 \) and \( m \geq 3 \), \( (2n + 1)(2mn + m - 2n) \geq 11(2mn + m - 2n) \geq 16mn + 18n + 11m - 22n = 16mn - 4n + 11m \), contradicting (4). When \( n = 4 \), we
get $m = 2$, contradicting $m \geq 3$. When $n = 3$, we get $2 \leq m \leq 20$, contradicting $m \geq 21$. So, $\chi_{la}(G(2m, 2n-1)) \geq 4$ under each of the given condition.

Define $f : E(G(2m, 2n-1)) \to [1, 4mn - 2n + 1]$ such that $f(u_{2m}u_1) = (2m-1)(2n-1)+1$, $f(u_{2m}u_{2i+1}) = (2m-1)(2n-1)+i+1$ for $1 \leq i \leq m-1$, $f(u_{2m-1}u_{2i}) = (2m-1)(2n-1)+2m+i-1$ for $1 \leq i \leq m$ and $f(u_iu_j) = a_{i,j}$, $1 \leq i \leq 2m-1, 1 \leq j \leq 2n-1$, where $a_{i,j}$ is the $(i, j)$-entry of a $(2m-1, 2n-1)$-magic rectangle with constant row sum $(2n-1)(2mn - m - n + 1)$ and constant column sum $(2m-1)(2mn - m - n + 1)$. One may check that $f$ is a bijection with $g_1 = f^+(v_j) = (2m-1)(2mn - m - n + 1)$ for $1 \leq j \leq 2n-1$, $g_2 = f^+(u_{2i}) = (2n-1)(2mn - m - n + 1)+2(2m-1)(2n-1)+2m+2 = (2n+3)(2mn - m - n + 1)+2m$ for $1 \leq i \leq m-1$, $g_3 = f^+(u_{2m-1}) = (2m-1)(2mn - m - n + 1)+2(2m-1)(2n-1)+2m+1 = (2n+3)(2mn - m - n + 1)+2m-1$ for $1 \leq i \leq m$ and $g_4 = f^+(u_{2m}) = 2(2m-1)(2n-1)+m+2 = 4(2mn - m - n + 1)+m$. Clearly, $g_2 > g_3 > g_4$. It is routine to verify that $g_1 \neq g_2, g_3, g_4$. Thus, $\chi_{la}(G(2m, 2n-1)) \leq 4$. The theorem holds.

**Example 3.21.** The following are labelings that give $\chi_{la}(G(5, 2)) = \chi_{la}(G(6, 2)) = \chi_{la}(G(6, 3)) = 3.$

![Graphs](image)

Note that $G(5, 2)$ and $G(6, 2)$ are two graphs we have not considered before.

**Problem 3.22.** For $m \geq 5$, find $\chi_{la}(G(m, n))$ for $G(m, n)$ not being a graph in Theorem 3.20 and Example 3.21.

Little is known about bipartite graphs $G$ with $\chi_{la}(G) = 2$ (see [1, Theorems 2.11 and 2.12]). For $m \geq 2, i \geq 1$, let $B(n_1, n_2, \ldots, n_m)$ be the union of $K_{2, n_i}$ with bipartition $(X_i, Y_i)$, where $X_i = \{x_{i-1}, x_i\}$, $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}\}$ and $x_m = x_0$.

It is known from [1, Theorem 2.8 and Theorem 2.12] that $\chi_{la}(B(1^{[n_i]})) = \chi_{la}(C_{2m}) = 3$ and $\chi_{la}(B(2^{[n_i]})) = \chi_{la}(K_{2, 2n}) = 2$ for $n \geq 2$. The following theorem gives another family of bipartite graphs with $\chi_{la}$ equal to 2.

**Theorem 3.23.** Suppose $m \geq 3$ and $n \geq 2$. We have $\chi_{la}(B(n^{[m]})) = 2$ if $n$ is even or both $m$ and $n$ are odd; $2 \leq \chi_{la}(B(n^{[m]})) \leq 3$ for odd $n$ and even $m$. 
Proof. First note that the edges in each $K_{2,n}$ are $x_{i-1}y_{i,j}$ and $x_iy_{i,j}$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Suppose $n \geq 2$ is even. Define a bijection $f : E(G) \to [1, 2mn]$ such that

$$f(x_{i-1}y_{i,j}) = \begin{cases} (i - 1)n + j & \text{for odd } j \in [1, n - 1]; \\ (2m - i + 1)n - j + 1 & \text{for even } j \in [2, n], \end{cases}$$

$$f(x_iy_{i,j}) = \begin{cases} (2m - i + 1)n - (j - 1) & \text{for odd } j \in [1, n - 1]; \\ (i - 1)n + j & \text{for even } j \in [2, n], \end{cases}$$

where $1 \leq i \leq m$.

Recall that $x_m = x_0$. It is easy to verify that $f^+(y_{i,j}) = 2mn + 1$ and $f^+(x_i) = 2mn^2 + n$ for $1 \leq i \leq m, 1 \leq j \leq n$. Hence, $\chi_{la}(G) \leq 2$. Since $\chi_{la}(G) \geq \chi(G) = 2$, we have $\chi_{la}(G) = 2$ for even $n \geq 2$.

Suppose $n$ is odd and $m$ is odd. Let $A$ be a magic $(m, n)$-rectangle. For $1 \leq i \leq m$, let $(f(x_1y_{1,1}), \ldots, f(x_iy_{i,n}))$ be the $i$-th row of $A$ and let $f(x_{i-1}y_{i,j}) = 2mn + 1 - f(x_iy_{i,j})$ for $1 \leq j \leq n$. Clearly $f^+(y_{i,j}) = 2mn + 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Since the row sum of $A$ is $n(mn + 1)/2$, $f^+(x_i) = n(2mn + 1)$ for $1 \leq i \leq m$. Here, $\chi_{la}(G) \leq 2$ and hence $\chi_{la}(G) = 2$.

Suppose $n$ is odd and $m$ is even. Define a bijection $f : E(G) \to [1, 2mn]$.

$$f(x_{i-1}y_{i,j}) = (i - 1)n + j;$$

$$f(x_iy_{i,j}) = (2m - i + 1)n - j + 1,$$

where $1 \leq i \leq m$.

It is easy to verify that $f^+(y_{i,j}) = 2mn + 1$, $f^+(x_0) = n(mn + n + 1)$ and $f^+(x_i) = n(2mn + n + 1)$ for $1 \leq i \leq m - 1$. Thus, $\chi_{la}(G) \leq 3$.

Example 3.24. The following is a local antimagic labeling according to the construction described in the proof above, which induces a 2-coloring for $B(3^{[3]})$.

$$A = \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}.$$

<table>
<thead>
<tr>
<th>$y_{1,1}$</th>
<th>$y_{1,2}$</th>
<th>$y_{1,3}$</th>
<th>$y_{2,1}$</th>
<th>$y_{2,2}$</th>
<th>$y_{2,3}$</th>
<th>$y_{3,1}$</th>
<th>$y_{3,2}$</th>
<th>$y_{3,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$x_2$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>9</td>
<td>5</td>
<td>1</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>$x_3$</td>
<td>17</td>
<td>12</td>
<td>13</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

It is clear that each row sum is 57 and each column sum is 19.
Example 3.25. The following is a local antimagic labeling inducing a 2-coloring for $B(3^{[4]})$.

<table>
<thead>
<tr>
<th></th>
<th>$y_{1,1}$</th>
<th>$y_{1,2}$</th>
<th>$y_{1,3}$</th>
<th>$y_{2,1}$</th>
<th>$y_{2,2}$</th>
<th>$y_{2,3}$</th>
<th>$y_{3,1}$</th>
<th>$y_{3,2}$</th>
<th>$y_{3,3}$</th>
<th>$y_{4,1}$</th>
<th>$y_{4,2}$</th>
<th>$y_{4,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>5</td>
<td>17</td>
<td>23</td>
<td>19</td>
<td>10</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$x_2$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>2</td>
<td>6</td>
<td>15</td>
<td>22</td>
<td>16</td>
<td>14</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$x_3$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>3</td>
<td>9</td>
<td>11</td>
<td>21</td>
<td>18</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>24</td>
<td>20</td>
<td>8</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is easy to see that the row sum is always 75 and the column sum is always 25.

Problem 3.26. Determine $\chi_{la}(B(n_1, n_2, \ldots, n_m))$ for $B(n_1, n_2, \ldots, n_m) \neq B(n^{[m]})$.

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References


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