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LIGHT MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO 6-VERTICES [1]

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Abstract

In 1940, Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P_5}$ of 3-polytopes with minimum degree 5.

Given a 3-polytope P, by $w(P)$ denote the minimum of the degree-sum (weight) of the neighborhoods of 5-vertices (minor 5-stars) in P.

In 1996, Jendrol' and Madaras showed that if a polytope P in $\mathbf{P_5}$ is allowed to have a 5-vertex adjacent to four 5-vertices, then $w(P)$ can be arbitrarily large.

For each P in $\mathbf{P_5}$ without vertices of degree 6 and 5-vertices adjacent to four 5-vertices, it follows from Lebesgue's Theorem that $w(P) \leq 68$. Recently, this bound was lowered to $w(P) \leq 55$ by Borodin, Ivanova, and Jensen and then to $w(P) \leq 51$ by Borodin and Ivanova.

In this note, we prove that every such polytope P satisfies $w(P) \leq 44$, which bound is sharp.

Keywords: planar map, planar graph, 3-polytope, structural properties, 5-star, weight, height.

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1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a 3-*polytope*) is denoted by $d(x)$. A k-*vertex* is a vertex v with $d(v) = k$. A k^+-*vertex* (k^--*vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. A 3-polytope with minimum degree 5 is denoted by P_5, and the set of such 3-polytopes is $\mathbf{P_5}$.

The *weight* of a subgraph S of P_5 is the degree sum of the vertices of S in P_5, and the *height* of S is the maximum degree of the vertices of S in P_5. A k-star, a star with k rays, is *minor* if its center v has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor.

By $w(S_k)$ and $h(S_k)$ we denote the minimum weight and height, respectively, of minor k-stars in a given 3-polytope P_5.

In 1904, Wernicke [13] proved that every P_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [9] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, in attempts to solve the Four Color Problem, Lebesgue [12, p. 36] gave an approximate description of the neighborhoods of 5-vertices in P_5s. In particular, this description implies the results in [9, 13] and shows that there is a 5-vertex with three 7^--neighbors.

The bounds $w(S_1) \leq 11$ (Wernicke [13]) and $w(S_2) \leq 17$ (Franklin [9]) are tight. It was proved by Lebesgue [12] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [10] to the sharp bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [10] gave a precise description of minor 3-stars in P_5s.

Lebesgue [12] proved $w(S_4) \leq 31$, which was strengthened by Borodin and Woodall [8] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$ and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). Recently, Borodin and Ivanova [1] obtained a precise description of 4-stars in P_5s.

The more general problem of precisely describing 5-stars at 5-vertices in P_5s inspired by Lebesgue's Theorem is still widely open.

Jendrol' and Madaras [10] showed that if a polytope P_5 has a 5-vertex adjacent to four 5-vertices, called a *minor* $(5,5,5,5,\infty)$-*star*, then $h(S_5)$ and hence $w(S_5)$ can be arbitrarily large. Therefore, in what follows we consider P_5s without minor $(5,5,5,5,\infty)$-stars.

Recently, precise upper bounds for the height and weight of minor 5-stars have been obtained for some restricted subclasses in $\mathbf{P_5}$. A lot of earlier results on the structure of stars in 3-polytopes can be found in [11].

For every P_5 having no vertices of degree from 6 to 9, Lebegue's bounds $h(S_5) \leq 14$ and $w(S_5) \leq 44$ were improved by Borodin and Ivanova [3] to the sharp bounds $h(S_5) \leq 12$ and $w(S_5) \leq 42$.

For each P_5 with no 6- to 8-vertices, it follows from Lebesgue's Theorem that $h(S_5) \leq 17$ and $w(S_5) \leq 46$, which bounds were improved in Borodin, Ivanova and Nikiforov [7] to the best possible bounds $h(S_5) \leq 12$ and $w(S_5) \leq 42$.

Under the absence of 6- and 7-vertices, Lebegue's bound $h(S_5) \leq 23$ was improved by Borodin *et al.* [5] to the sharp bound $h(S_5) \leq 14$.

For each P_5 with no 6-vertices, it follows from Lebesgue's Theorem that $h(S_5) \leq 41$. This bound was lowered to $h(S_5) \leq 28$ by Borodin, Ivanova, and Jensen [4], then to $h(S_5) \leq 23$ in Borodin-Ivanova [2], and finally to the tight bound $h(S_5) \leq 17$ by Borodin, Ivanova, and Nikiforov [6].

As for the minimum weight of minor 5-stars in P_5s under the absence of 6-vertices, Lebesgue's bound $w(S_5) \leq 68$ was lowered to $w(S_5) \leq 55$ by Borodin, Ivanova, and Jensen [4] and then to $w(S_5) \leq 51$ in Borodin-Ivanova [2]. The purpose of this paper is to prove the following fact.

Theorem 1. *Every* 3*-polytope with minimum degree* 5 *and neither* 6*-vertices nor minor* $(5, 5, 5, 5, \infty)$*-stars has a minor* 5*-star with weight at most* 44*, which bound is best possible.*

We note that a light minor 5-star ensured by Theorem 1 has height at most $44 - 4 \times 5 - 7 = 17$. The tightness of the bounds 44 and 17 is confirmed by a construction in [6].

2. PROOF OF THEOREM 1

Discharging.

Suppose that a 3-polytope P'_5 is a counterexample to the main statement of Theorem 1. Thus each minor 5-star in P'_5 has weight at least 45 and at most three 5-vertices.

Let P_5 be a counterexample with the maximum number of edges on the same set of vertices as P'_5.

Remark 2. P_5 has no 4^+-face with two nonconsecutive 7^+-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with greater number of edges.

Let V, E, and F be the sets of vertices, edges, and faces of P_5. Euler's formula $|V| - |E| + |F| = 2$ implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have a negative initial charge.

Using the properties of P_5 as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12.

The *final charge* $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R11 below (see Figure 1).

For a vertex v, let $v_1, \dots, v_{d(v)}$ be the vertices adjacent to v in a fixed cyclic order. If f is a face, then $v_1, \dots, v_{d(f)}$ are the vertices incident with f in the same cyclic order.

If d is an integer with $8 \le d \le 15$, then we put $\xi_d = \frac{d-6}{d}$.

A vertex is *simplicial* if it is completely surrounded by 3-faces. A simplicial 5-vertex v is *helpful* if $d(v_1) \ge 12$, $d(v_2) = d(v_4) = 5$, $d(v_3) = 7$, and $d(v_5) \ge 12$ (see Figure 1, R10). A simplicial 5-vertex v is *strong* if $d(v_1) = d(v_2) = 5$, $7 \le d(v_3) \le 11$, and $7 \le d(v_5) \le 11$ (so $d(v_4) \ge 45 - 2 \times 11 - 3 \times 5 \ge 8$) (see Figure 1, R11).

R1. Each 4^+-face gives $\frac{1}{2}$ to each incident 5-vertex.

R2. If a 5-vertex v is incident with precisely one 4^+-face, then v receives $\frac{1}{2}$ from each adjacent 16^+-vertex.

R3. A simplicial 5-vertex v with at least two 12^+-neighbors receives $\frac{1}{2}$ from each adjacent 16^+-vertex.

R4. A simplicial 5-vertex v with $d(v_4) \ne 5$, $d(v_5) \ge 16$, and no other 12^+-neighbors receives the following charge from v_5:

(a) if $d(v_1) \ne 5$, then 1, and

(b) if $d(v_1) = 5$, then $\frac{3}{4}$ provided that $d(v_5) \le 17$ or $\frac{5}{6}$ otherwise.

R5. A simplicial 5-vertex v with $d(v_5) \ge 18$, $d(v_1) = d(v_4) = 5$, and $7 \le d(v_2) \le d(v_3) \le 11$ receives $\frac{2}{3}$ from v_5.

R6. A simplicial 5-vertex v with $16 \le d(v_5) \le 17$, $d(v_1) = d(v_4) = 5$, and $7 \le d(v_2) \le d(v_3) \le 11$ receives from v_5:

(n) $\frac{5}{8}$ if neither v_2 nor v_3 is a 7-vertex having six simplicial 5-neighbors ("normally"), and

(e) $\frac{2}{3}$ otherwise ("as an exception").

R7. A simplicial 5-vertex v with $d(v_5) \ge 16$, $d(v_1) = d(v_2) = d(v_4) = 5$, and $7 \le d(v_3) \le 11$ receives the following charge from v_5.

(n) If v_1 is not simplicial or v_2 is not strong (that is "normal"), then $\frac{3}{4}$ if $d(v_5) \le 17$ or $\frac{5}{6}$ otherwise.

(e) If v_1 is simplicial and v_2 is strong (which is "an exception"), then $\frac{5}{8}$ if $d(v_5) \le 17$ or $\frac{2}{3}$ otherwise.

R8. A d-vertex v with $8 \leq d(v) \leq 15$ gives its 5-neighbor v_2:
(a) ξ_d if $d(v_1) = d(v_3) = 5$,
(b) $\frac{3\xi_d}{2}$ if $d(v_1) = 5$ and $d(v_3) \neq 5$, and
(c) $2\xi_d$ if $d(v_1) \neq 5$ and $d(v_3) \neq 5$.

R9. A 7-vertex v gives each adjacent simplicial 5-vertex:
(n) $\frac{1}{5}$ "as a norm", that is if v has at most five simplicial 5-neighbors, or
(e) $\frac{1}{6}$ "as an exception".

R10. A 7-vertex v receives $\frac{1}{6}$ from each helpful 5-neighbor.

R11. A strong 5-vertex gives $\frac{1}{6}$ to each 5-neighbor.

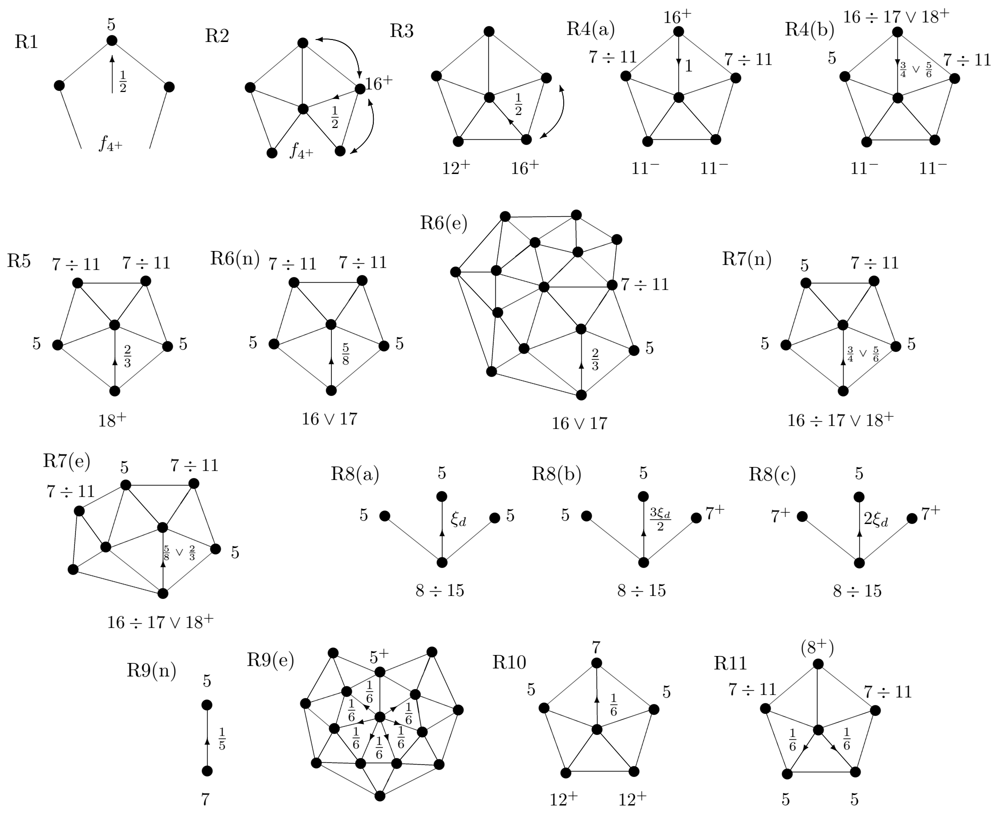

Figure 1. Rules of discharging.

Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$.

If f is a 4^+-face, then $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0$ by R1.

Now suppose $v \in V$.

Case 1. $d(v) \geq 18$. We know that v gives one of the charges in $\left\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$ to each adjacent 5-vertex incident with at least four 3-faces by R2–R7. Since $d(v) - 6 \geq \frac{2d(v)}{3}$, it suffices to average these donations so that each 5^+-neighbor will receive at most $\frac{2}{3}$ from v.

To this end, we first switch $\frac{1}{6}$ from 1 given to a 5-neighbor v_k by R4(a) to each of the 7^+-neighbors v_{k-1} and v_{k+1} (hereafter, addition modulo $d(v)$). As a result, the averaged donation from v to v_k becomes $1 - 2 \times \frac{1}{6} = \frac{2}{3}$.

Next, if $\frac{5}{6}$ is given to a 5-neighbor v_k by R4(b), then we switch $\frac{1}{6}$ to its common 7^+-neighbor with v.

Finally, the donation of $\frac{5}{6}$ by R7(n) happens to a simplicial 5-neighbor v_k of v having cyclic neighbors $v_{k-1}, x_k, y_k, v_{k+1}$ with $7 \leq d(x_k) \leq 11$ and 5-neighbors v_{k-1}, y_k, v_{k+1}, where either v_{k+1} is not simplicial or y_k is not strong.

If v_{k+1} is not simplicial, then we switch $\frac{1}{6}$ from v_k to v_{k+1} and note that the latter receives at most $\frac{1}{2}$ from v by R2.

From now on suppose that v_{k+1} is simplicial, and let z_k be the vertex conjugated with v_k with respect to the edge $y_k v_{k+1}$. Since v_k receives $\frac{5}{6}$ by R7(n) by our assumption, it follows that $d(z_k) \notin \{7, \ldots, 11\}$, for otherwise y_k is strong since it has the fifth neighbor of degree at least $w(S_5) - 3 \times 5 - 2 \times 11 = 8$ and is simplicial in view of Remark 2.

If $d(z_k) \geq 12$, then we switch $\frac{1}{6}$ from v_k to v_{k+1}, where v_{k+1} this time receives $\frac{1}{2}$ by R3. Note that v_{k+2} receives $\frac{1}{2}$ by R2 or R3, which implies that $\frac{1}{6}$ is switched to v_{k+1} only once.

It remains to assume that $d(z_k) = 5$. This implies that $d(v_{k+2}) \geq 7$ since v_{k+1} cannot have four 5-neighbors. Here, we switch $\frac{1}{6}$ from v_k to v_{k+2}. (Of course, v_{k+1} also switches $\frac{1}{6}$ from its $\frac{5}{6}$ obtained by R4(b) to v_{k+2}, as said above.)

It is not hard to see that no 5-vertex v_{k+1} can receive $\frac{1}{6}$ in the course of our averaging both from v_k and v_{k+2} since then v_{k+1} would have four 5-neighbors, which is impossible.

As a result, the averaged donation of v to each 5-neighbor becomes at most $1 - 2 \times \frac{1}{6} = \frac{5}{6} - \frac{1}{6} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ and that to each 7^+-neighbor is at most $0 + 4 \times \frac{1}{6} = \frac{2}{3}$, as desired.

Case 2. $16 \leq d(v) \leq 17$. We now show that the neighbors of v receive from v by R2–R7 at most $\frac{5}{8}$ on the average, which implies that $\mu'(v) \geq d(v) - 6 - \frac{5d(v)}{8} = \frac{3(d(v)-16)}{8} \geq 0$. We proceed similarly to Case 1 with a 5-vertex v_k getting more than $\frac{5}{8}$ from v by R4, R6(e) or R7(n).

If v_k is as in R4(a), then we shift $\frac{1}{4}$ from 1 obtained by v_k to each of the 7^+-vertices v_{k-1} and v_{k+1}. In R4(b), we shift $\frac{1}{8}$ from $\frac{3}{4}$ to a unique 7^+-vertex in $\{v_{k-1}, v_{k+1}\}$.

Now consider R6(e), which has no analogues in Case 1. By symmetry, we can assume that v_{k+1} lies in a common 3-face with v_k and a 7-vertex having six simplicial 5-neighbors. Now $d(v_{k+2}) \geq 7$ as v_{k+1} cannot have three 5-neighbors in addition to a 7-neighbor and a 17^{-}-neighbor since $w(S_5) \geq 45$ by assumption. Recall that v_{k+1} receives at most $\frac{3}{4}$ by R4(b), R3, or R7(n) and that $\frac{1}{8}$ was already switched from v_{k+1} to v_{k+2} in the previous paragraph. Here, we also switch $\frac{1}{8}$ from $\frac{2}{3}$ received by v_k to v_{k+2}.

In the situation of R7(n), let v_{k+1} lie in a 3-face incident with three 5-vertices. Arguing as in Case 1, we see that either v_{k+1} receives $\frac{1}{2}$ from v, in which case we switch $\frac{1}{8}$ from v_k to v_{k+1}, or we have $d(v_{k+2}) \geq 7$, in which case we switch $\frac{1}{8}$ from v_k to v_{k+2}.

As a result of this averaging, each 5-neighbor of v receives at most $1-2\times\frac{1}{4} < \frac{3}{4} - \frac{1}{8} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$, while each 7^{+}-neighbor receives at most $4 \times \frac{1}{8} = \frac{1}{4} + 2 \times \frac{1}{8} = 2 \times \frac{1}{4} < \frac{5}{8}$ from v, as desired.

Case 3. $8 \leq d(v) \leq 15$. To satisfy R8, we first send $\xi_{d(v)}$ to each neighbor v_k, and then each 7^{+}-neighbor v_k transfers $\frac{\xi_{d(v)}}{2}$ to each 5-vertex in $\{v_{k-1}, v_{k+1}\}$. This shows that $\mu'(v) \geq d(v) - 6 - d(v) \times \xi_{d(v)} = 0$.

Case 4. $d(v) = 7$. If v has at most five simplicial 5-neighbors, then $\mu'(v) \geq 7 - 6 - 5 \times \frac{1}{5} = 0$ by R9(n). If v has precisely six simplicial 5-neighbors, then $\mu'(v) \geq 1 - 6 \times \frac{1}{6} = 0$ by R9(e).

Finally, suppose v is completely surrounded by simplicial 5-vertices. This implies that there is a 7-cycle $C_7 = w_1 \cdots w_7$ avoiding v, where each v_k lies in a 3-face $w_k v_k w_{k+1}$ (addition modulo 7). Note that $d(w_k)+d(w_{k+1}) \geq 45-3\times5-7 = 23$ whenever $1 \leq k \leq 7$. By the oddness of 7, v has a helpful neighbor, which gives $\frac{1}{6}$ to v by R10. As a result, we have $\mu'(v) \geq 1 + \frac{1}{6} - 7 \times \frac{1}{6} = 0$ in view of R9(e), as required.

Case 5. $d(v) = 5$. If v is incident with at least two 4^{+}-faces, then $\mu'(v) \geq 5 - 6 + 2 \times \frac{1}{2} = 0$ by R1.

If v is incident with precisely one 4^{+}-face, then we are done when v has a 12^{+}-neighbor since v receives $\frac{1}{2}$ by R1 and at least $\frac{1}{2}$ by R2 or R8.

So suppose otherwise. Note that v then has two 8^{+}-neighbors, for otherwise v would have an 11^{-}-neighbor and four 7^{-}-neighbors, which implies $w(S_5) \leq 5 + 4 \times 7 + 11 < 45$, a contradiction. Thus $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by R1 and R8.

From now on we can assume that v is simplicial.

Subcase 5.1. v is helpful, with $d(v_1) \geq 12$, $d(v_2) = d(v_4) = 5$, $d(v_3) = 7$, and $d(v_5) \geq 12$. Now v receives $\frac{1}{2}$ from each of v_1, v_5 by R3 and/or R8. Also v receives at least $\frac{1}{6}$ from v_3 by R9 and returns $\frac{1}{6}$ to v_3 by R10. This implies $\mu'(v) \geq -1 + 2 \times \frac{1}{2} + \frac{1}{6} - \frac{1}{6} = 0$, as desired.

Subcase 5.2. v is strong, with $d(v_1) = d(v_2) = 5$, $7 \le d(v_3) \le d(v_5) \le 11$, and $d(v_4) \ge 45 - 3 \times 5 - 2 \times 11 = 8$. Now v must collect the total of at least $\frac{4}{3}$ from v_3, v_4, v_5 in order to be able to give $2 \times \frac{1}{6}$ to v_1, v_2 according to R11 (and leave 1 for itself).

We are easily done if $d(v_4) \ge 12$, for then v_4 gives v at least 1 by R4(a) or R8(c) while each of v_3, v_5 gives at least $\frac{1}{6}$ by R8 and R9.

So suppose $d(v_4) \le 11$. Since $d(v_3) + d(v_4) + d(v_5) \ge w(S_5) - 3 \times 5 = 30$, this implies that v has no neighbors of degree less than $30 - 2 \times 11 = 8$. If $d(v_4) = 8$, then $d(v_3) = d(v_5) = 11$, which implies that v receives $\frac{1}{2}$ from v_4 by R8(c) and $2 \times \frac{15}{22}$ from v_3, v_5 by R8(b), as desired. If $d(v_4) \ge 9$, then v receives at least $\frac{2}{3}$ from v_4 by R8(c) and at least $2 \times \frac{3}{8}$ from v_3, v_5 by R8(b), and we are done.

Subcase 5.3. v does not give charge away by R10 and R11. So we must check that v collects the total of at least 1 from its neighbors by R3–R9. If v has at least two 12^+-neighbors, then $\mu'(v) \ge -1 + 2 \times \frac{1}{2} = 0$ by R3 or R8(a),(b). So in what follows we assume that v has at most one 12^+-neighbor, which means that R3 is not applied to v.

Subcase 5.3.1. v has at most one 5-neighbor. Here, v receives at least $\frac{3}{8}$ from an 8-neighbor and at least $\frac{1}{2}$ from a 9^+-neighbor by R4–R8. This implies, in view of R9, that $\mu'(v) \ge -1 + 3 \times \frac{1}{6} + \frac{1}{2} = 0$ in the presence of a 9^+-neighbor or $\mu'(v) \ge -1 + 2 \times \frac{1}{6} + 2 \times \frac{3}{8} > 0$ when v has at least two 8-neighbors. However, one of this situations is inevitable, since otherwise we would have $w(S_5) \le 5 + 4 \times 7 + 8 < 45$, which is impossible.

Subcase 5.3.2. v has precisely two 5-neighbors. Note that the total degree of the three 7^+-neighbors of v is at least $45 - 3 \times 5 = 30$.

Suppose v has no 7-neighbor. Each 8^+-neighbor v_2 gives v by R4–R8 at least $\frac{1}{4}$ if $d(v_1) = d(v_3) = 5$ and at least $\frac{3}{8}$ if $d(v_1) \ne 5$, so $\mu'(v) \ge -1 + \frac{1}{4} + 2 \times \frac{3}{8} = 0$, and we are done.

Next suppose v has at least one 7-neighbor. Now the other two 7^+-neighbors have the total degree at least $30 - 7 = 23$, so there is a 12^+-neighbor, say v_2, among them.

If v_2 gives v at least $\frac{3}{4}$ to v by R4 or R8, then $\mu'(v) > 0$, since the other two 7^+-neighbors give at least $2 \times \frac{1}{6}$ by R4–R9.

So suppose $d(v_1) = d(v_3) = 5$ and $7 = d(v_4) \le d(v_5)$. Now if $d(v_2) \ge 18$, then we have $\mu'(v) \ge -1 + \frac{2}{3} + 2 \times \frac{1}{6} = 0$ by R4–R9.

For $16 \le d(v_2) \le 17$ we are similarly done if v_2 gives $\frac{2}{3}$ by R6(e), so suppose R6(n) is applied to v_2 rather than R6(e). If $d(v_5) \ge 8$, then $\mu'(v) \ge -1 + \frac{5}{8} + \frac{1}{6} + \frac{3}{8} > 0$. It remains to assume that $d(v_4) = d(v_5) = 7$ and neither of v_4, v_5 has six simplicial 5-neighbors (as if we apply R9(e) to v_4 or v_5 it would mean we should apply R6(e) to v, and then $\mu'(v) \ge -1 + \frac{2}{3} + 2 \times \frac{1}{6} = 0$). This means that $\mu'(v) \ge -1 + \frac{5}{8} + 2 \times \frac{1}{5} = \frac{1}{40}$ by R6(n) and R9(n).

Finally, suppose $12 \leq d(v_2) \leq 15$. Now $d(v_5) \geq w(S_5) - 3 \times 5 - 7 - 15 = 8$, and it suffices to observe that v receives at least $\frac{1}{2}, \frac{1}{6}, \frac{3}{8}$ from v_2, v_4, v_5, respectively, which makes $\mu'(v) > 0$, as desired.

Subcase 5.3.3. v has precisely three 5-neighbors. Note that the total degree of the two 7^+-neighbors of v is at least $45 - 4 \times 5 = 25$.

First suppose $7 \leq d(v_1) \leq d(v_2)$. By the above assumption that R3 is not applied, we have $d(v_1) \leq 11$, which implies that v has a 14^+-neighbor. Note that v_2 gives v at least $\frac{3}{4}$ by R4(b) or R8, while v_1 gives v at least $\frac{3}{8}$ by R8 if $d(v_1) \geq 8$, and then we have $\mu'(v) \geq 0$. But if $d(v_1) = 7$, then $d(v_2) \geq 25 - 7 = 18$, and $\mu'(v) \geq -1 + \frac{5}{6} + \frac{1}{6} = 0$ by R4(b) combined with R9.

Thus from now on we can assume that $7 \leq d(v_1) \leq 11$ and $d(v_3) \geq 14$. If $d(v_3) \leq 15$, then v receives from v_1 and v_3 at least $1 = \frac{2}{5} + \frac{3}{5} = \xi_{10} + \xi_{15} < \xi_{11} + \xi_{14}$ by R8(a), as desired.

Next suppose $16 \leq d(v_3) \leq 17$, which implies that $d(v_1) \geq 8$. Since v_1 gives v at least $\frac{1}{4}$ by R8(a) while v_3 gives either $\frac{3}{4}$ or $\frac{5}{8}$ by R7, we are done unless v_3 gives $\frac{5}{8}$ by R7(e). The latter happens when v_5 is strong, in which case v receives $\frac{1}{6}$ from v_5 by R11, which yields $\mu'(v) \geq -1 + \frac{1}{4} + \frac{1}{6} + \frac{5}{8} > 0$.

Finally, suppose $d(v_3) \geq 18$. Now v_1 gives v at least $\frac{1}{6}$ by R9 while v_3 gives either $\frac{5}{6}$ or $\frac{2}{3}$ by R7. Since the donation of $\frac{2}{3}$ by R7(e) to v is accompanied by receiving $\frac{1}{6}$ by R11 from a strong vertex v_5, we have $\mu'(v) \geq -1 + 2 \times \frac{1}{6} + \frac{2}{3} = -1 + \frac{1}{6} + \frac{5}{6} = 0$.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

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