T-COLORINGS, DIVISIBILITY AND THE CIRCULAR CHROMATIC NUMBER

ROBERT JANCZEWSKI
Department of Algorithms and Systems Modelling
Gdańsk University of Technology
Narutowicza 11/12, Gdańsk, Poland
e-mail: skalar@eti.pg.gda.pl

ANNA MARIA TRZASKOWSKA
Department of Applied Informatics in Management
Gdańsk University of Technology
Narutowicza 11/12, Gdańsk, Poland
e-mail: anna.trzaskowska@pg.edu.pl

AND

KRZYSZTOF TUROWSKI
Center for Science of Information
Purdue University West Lafayette, Indiana, USA
e-mail: krzysztof.szymon.turowski@gmail.com

Abstract

Let $T$ be a $T$-set, i.e., a finite set of nonnegative integers satisfying $0 \in T$, and $G$ be a graph. In the paper we study relations between the $T$-edge spans $\text{esp}_T(G)$ and $\text{esp}_{d \odot T}(G)$, where $d$ is a positive integer and

$$d \odot T = \{0 \leq t \leq d (\max T + 1) : d \mid t \Rightarrow t/d \in T\}.$$

We show that $\text{esp}_{d \odot T}(G) = d \text{esp}_T(G) - r$, where $r$, $0 \leq r \leq d - 1$, is an integer that depends on $T$ and $G$. Next we focus on the case $T = \{0\}$ and show that

$$\text{esp}_{d \odot \{0\}}(G) = \lceil d(\chi_c(G) - 1) \rceil,$$

where $\chi_c(G)$ is the circular chromatic number of $G$. This result allows us to formulate several interesting conclusions that include a new formula for the circular chromatic number

$$\chi_c(G) = 1 + \inf \{ \text{esp}_{d \odot \{0\}}(G)/d : d \geq 1 \}.$$
and a proof that the formula for the $T$-edge span of powers of cycles, stated as conjecture in [Y. Zhao, W. He and R. Cao, The edge span of $T$-coloring on graph $C_d^n$, Appl. Math. Lett. 19 (2006) 647–651], is true.

**Keywords:** $T$-coloring, circular chromatic number.

**2010 Mathematics Subject Classification:** 05C15.

1. Introduction

In the paper we study relations between two different generalizations of ordinary vertex colorings: $T$-colorings and $(k, d)$-colorings. Let $G$ be a graph with $n$-vertex set $V$ and edge set $E$. Given integers $1 \leq d \leq k$, by a $(k, d)$-coloring of $G$ we mean any function $c : V \to [0, k - 1]$ ($[a, b] := \{a, a + 1, \ldots, b\}$ for any integers $a \leq b$) such that

$$d \leq |c(u) - c(v)| \leq k - d$$

whenever $uv \in E$. This notion may be viewed as a generalization of a $k$-coloring since $(k, d)$-colorings of $G$ are $k$-colorings of $G$ and $(k, 1)$-colorings are the same as $k$-colorings that use colors from the interval $[0, k - 1]$. The **circular chromatic number**, introduced by Vince [12] as a generalization of the chromatic number, is defined by the formula

$$\chi_c(G) = \inf \{k/d : G \text{ has a } (k, d)\text{-coloring}\}.$$

The circular chromatic number was studied by many authors, see [14, 15] for a survey of results. It was shown for example [12] that the distance between the circular and ordinary chromatic number does not exceed 1, i.e.

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

In the same paper Vince proved two useful facts: (1) $G$ has a $(k, d)$-coloring if and only if $\chi_c(G) \leq k/d$; (2) $\chi_c(G)$ is a rational number which has a form $k/d$, where $k \leq n$. We will use these observations to show that there is a relation between $\chi_c(G)$ and $\text{es}_{T}(G)$ the $T$-edge span defined below. Given a $T$-set $T$, i.e., a finite set that consists of nonnegative integers and satisfies $0 \in T$, by a $T$-**coloring** of $G$ we mean any function $c : V \to \mathbb{Z}$ such that

$$|c(u) - c(v)| \notin T$$

whenever $uv \in E$. $T$-colorings were introduced as a model for the frequency assignment problem in [5]. This notion also may be viewed as a generalization of ordinary vertex colorings since $T$-colorings are vertex colorings and vertex
colorings are \{0\}-colorings. The $T$-edge span, introduced by Cozzens and Roberts [1], is defined as
\[
\text{esp}_T(G) = \min\{\text{esp}(c) : c \text{ is a } T\text{-coloring of } G\},
\]
where $\text{esp}(c) = \max\{|c(u) - c(v)| : uv \in E\}$ is the edge span of $c$ (if $G$ is an empty graph then $\text{esp}(c) = 0$). If we replace $\text{esp}(c)$ by $\text{sp}(c)$ (the span of $c$, i.e., $\max\{|c(u) - c(v)| : u, v \in V\}$) we will receive the $T$-span of $G$. Both parameters were studied by many authors, there are results concerning computational complexity of the problem of computing $\text{sp}_T(G)$ [2, 3], the behaviour of the greedy algorithm [7] and formulas describing $\text{sp}_T(G)$ and $\text{esp}_T(G)$ for some $T$-sets $T$ and some graphs $G$ [8, 9, 13].

The remainder of the paper is organized as follows. In Section 2 we study relations between $\text{esp}_T(G)$ and $\text{sp}_{d \odot T}(G)$, where $d$ is a positive integer and $d \odot T = \{0 \leq t \leq d (\max T + 1) : d | t \Rightarrow t/d \in T\}$. We show that $\text{esp}_{d \odot T}(G) = d \text{esp}_T(G) - r$, where $r, 0 \leq r \leq d - 1$, is an integer that depends on $T$ and $G$. In Section 3 we study the distance between the $T$-span and $T$-edge span and show that it cannot exceed $\max T$. We also give examples that prove that this bound is tight. Section 4 contains our main results. We show that if $T$ is an interval, i.e., $T = [0, d - 1]$ (or equivalently $T = d \odot \{0\}$), then $(k, d)$-colorings ($k \geq d$) are nonnegative $T$-colorings with span bounded by $k - 1$ and edge span bounded by $k - d$. We use this relation to show that
\[
\text{esp}_{d \odot \{0\}}(G) = \lceil d(\chi_c(G) - 1) \rceil.
\]
We also discuss whether it is possible to extend this relation to all $T$-sets. Using the above formula we show that
\[
\chi_c(G) = 1 + \inf \{ \text{esp}_{d \odot \{0\}}(G)/d : d \geq 1 \}
\]
and discuss how these formulas allow us to move known results from the world of the $T$-edge span to the world of the circular chromatic number and vice versa. The last section is devoted to the powers of cycles investigated in [13]. The authors conjectured and partially proved that
\[
\text{esp}_{d \odot \{0\}}(C^p_n) = pd + \lceil rd/q \rceil,
\]
where $q \geq 2$ and $r$ are the quotient and the remainder of the division of $n$ by $p + 1$, respectively. We show that it is true in general.

2. $T$-Edge Span and $d \odot T$-Edge Span

The operation $\odot$ was introduced in [6], where it was shown that $\text{sp}_{d \odot T}(G) = d \text{sp}_T(G)$. Below we prove a similar formula for the $T$-edge span, but before we proceed we need to recall the following result.
Lemma 1 (Lemma 2.2(i) of [6]). If $a$ and $b$ are real numbers, then $||a - b|| \leq ||a|| - ||b|| \leq ||a - b||$.

Lemma 2. Let $G$ be a graph, $T$ be a $T$-set and $d$ be a positive integer.

1. If $c$ is a $T$-coloring of $G$, then $dc$ is a $d \circ T$-coloring of $G$.
2. If $c$ is a $d \circ T$-coloring of $G$, then $\lceil c/d \rceil$ is a $T$-coloring of $G$.

Proof. Let $uv$ be an edge of $G$ (if $G$ is empty, then our claim is obvious).

1. If $|c(u) - c(v)| \geq max T + 1$, then $|dc(u) - dc(v)| \geq d (max T + 1) = max d \circ T + 1$. If $|c(u) - c(v)| < max T + 1$ and $|dc(u) - dc(v)| \in d \circ T$, then the definition of $d \circ T$ gives $|c(u) - c(v)| \in T$, a contradiction. Hence $|dc(u) - dc(v)| \notin d \circ T$ in both cases.

2. If $|c(u) - c(v)| \geq max d \circ T + 1 = d (max T + 1)$, then $|c(u)/d - c(v)/d| \geq |c(u)/d - c(v)/d| \geq max T + 1$ by Lemma 1. If $|c(u) - c(v)| < max d \circ T + 1$, then the definition of $d \circ T$ gives $d|c(u) - c(v)|$ and, by Lemma 1, $|c(u)/d - c(v)/d| = |c(u) - c(v)|/d \notin T$. Hence $|c(u)/d - c(v)/d| \notin T$ in both cases. \qed

Theorem 3. Let $G$ be a graph, $T$ be a $T$-set and $d$ be a positive integer. There is an integer $0 \leq r \leq d - 1$ such that $sp_{d \circ T}(G) = dsp_T(G) - r$.

Proof. Let $c$ be a $T$-coloring of $G$ such that $sp(c) = sp_T(G)$. By Lemma 2, $dc$ is a $d \circ T$-coloring of $G$. Hence

$$sp_{d \circ T}(G) \leq sp(dc) = dsp(c) = dsp_T(G).$$

Let $c'$ be a $d \circ T$-coloring of $G$ such that $sp(c') = sp_{d \circ T}(G)$. By Lemma 2, $|c'/d|$ is a $T$-coloring of $G$. Let $uv$ be an edge of $G$ such that $sp(|c'/d|) = |c'(u)/d| - |c'(v)/d|$ (if $G$ is empty our claim is obvious). Then

$$dsp_T(G) - d \leq dsp(|c'/d|) - d = d|c'(u)/d| - |c'(v)/d| - d$$

$$\leq d |c'(u) - c'(v)/d| - d \leq d |sp(c')/d| - d$$

$$= d [esp_{d \circ T}(G)/d] - d < esp_{d \circ T}(G).$$

To complete the proof it suffices to combine (1) with (2). \qed

The open problem is a formula for $r$. Later we will show how to compute $r$ provided that $T = \{0\}$ and that $r$ can be any integer from $[0, d - 1]$.

Corollary 4. Let $G$ be a graph, $T$ be a $T$-set and $d$ be a positive integer. Then $sp_T(G) = [esp_{d \circ T}(G)/d]$. 

3. The Distance Between the T-Span and T-Edge Span

It is known [1] that \( \text{esp}_T(G) \leq \text{sp}_T(G) \). We are going to show that \( \text{sp}_T(G) \leq \text{esp}_T(G) + \max T \) and give examples in which the difference \( \text{sp}_T(G) - \text{esp}_T(G) \) equals \( \max T \).

**Lemma 5.** Let \( G \) be a graph and \( T \) be a \( T \)-set. If \( c : V \rightarrow \mathbb{Z} \) is a \( T \)-coloring of \( G \) and \( c : V \rightarrow \mathbb{Z} \) is the remainder of the division of \( c' \) by \( \text{esp}(c') + \max T + 1 \), i.e., \( c(v) = c'(v) \mod (\text{esp}(c') + \max T + 1) \) for \( v \in V \), then

1. \( c \) is a \( T \)-coloring of \( G \);
2. \( \text{sp}(c) \leq \text{esp}(c') + \max T \);
3. \( \text{esp}(c) \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T) \).

**Proof.** Observe that (2) follows immediately from the definition of \( c \). To prove (1) and (3), take an edge \( uv \) of \( G \) (if \( G \) is empty, our claim is obvious). Let \( q \) be the quotient of the division of \( c' \) by \( \text{esp}(c') + \max T + 1 \). Without loss of generality we may assume that \( q(u) \geq q(v) \). It is easy to see that \( q(u) \leq q(v) + 1 \) since otherwise

\[
\text{esp}(c') \geq |c'(u) - c'(v)| = |(\text{esp}(c') + \max T + 1)(q(u) - q(v)) + c(u) - c(v)| \\
\geq (\text{esp}(c') + \max T + 1)|q(u) - q(v)| - |c(u) - c(v)| \\
\geq 2(\text{esp}(c') + \max T + 1) - \text{esp}(c') - \max T \\
= \text{esp}(c') + \max T + 2 > \text{esp}(c')
\]

Hence there are two cases to consider.

(a) \( q(u) = q(v) + 1 \). Then \( |c'(u) - c'(v)| = |(\text{esp}(c') + \max T + 1)(q(u) - q(v)) + (c(u) - c(v))| = |(\text{esp}(c') + \max T + 1 + (c(u) - c(v))| \). Since \( \text{esp}(c') + \max T + 1 > \text{esp}(c') \geq |c'(u) - c'(v)| \) and \( |c(u) - c(v)| \leq \text{esp}(c') + \max T \), we have \( |c(u) - c(v)| = \text{esp}(c') + \max T + 1 - |c'(u) - c'(v)| \). This gives \( |c(u) - c(v)| \geq \max T + 1 \) and \( |c(u) - c(v)| \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T) \) since \( |c'(u) - c'(v)| \notin T \) implies \( |c'(u) - c'(v)| \geq \min(\mathbb{N} \setminus T) \).

(b) \( q(u) = q(v) \). Then \( |c'(u) - c'(v)| = |c(u) - c(v)| \), which gives \( |c(u) - c(v)| \notin T \) and \( |c(u) - c(v)| \leq \text{esp}(c') \leq \text{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T) \).

**Corollary 6.** Let \( G \) be a graph and \( T \) be a \( T \)-set. Then

1. There is a \( T \)-coloring \( c \) of \( G \) such that \( \text{sp}(c) \leq \text{esp}_T(G) + \max T \) and \( \text{esp}(c) \leq \text{esp}_T(G) + \max T + 1 - \min(\mathbb{N} \setminus T) \).
2. If \( T \) is an interval, then there is a \( T \)-coloring \( c \) of \( G \) such that \( \text{esp}(c) = \text{esp}_T(G) \) and \( \text{sp}(c) \leq \text{esp}_T(G) + \max T \).
3. \( \text{esp}_T(G) \leq \text{sp}_T(G) \leq \text{esp}_T(G) + \max T \).
Proof. (1) Let $c'$ be a $T$-coloring of $G$ satisfying $\esp(c') = \esp_T(G)$ and $c$ be the remainder of the division of $c'$ by $\esp_T(G) + \max T + 1$. The claim follows from Lemma 5.

(2) Follows from (1) since $\min(\mathbb{N} \setminus T) = \max T + 1$ if $T$ is an interval.

(3) Follows from (1) and the definition of the $T$-span.

The above inequalities are tight. It is known [1] that $\esp_T(G) = \sp_T(G)$ for all weakly perfect graphs and all $T$-sets $T$. It is also easy to see that if $T$ is an interval, then $\sp_T(C_{2n+1}) = 2\max T + 2$ ($\sp_T(G) = (\max T + 1)(\chi(G) - 1)$ if $T$ is an interval, see [1]) and $\esp_T(C_{2n+1}) = \lceil(\max T + 1)(1 + 1/n)\rceil$ (see Theorem 8) which gives $\sp_T(C_{2n+1}) = \esp_T(C_{2n+1}) + \max T$ provided that $n \geq \max T + 1$.

4. The Relation Between $(k,d)$-Colorings and $T$-Colorings

Now we are ready to prove that there is a relation between $(k,d)$-colorings and $T$-colorings provided that $T$ is an interval.

Lemma 7. Let $G$ be a graph and $d$ be a positive integer. If $T = [0,d-1]$, then for every function $c : V \to Z$ and every integer $k \geq d$ the following conditions are equivalent:

1. $c$ is a $T$-coloring of $G$ such that $\sp(c) \leq k - 1$ and $\esp(c) \leq k - d$;
2. $c - \min c(V)$ is a $(k,d)$-coloring of $G$.

Proof. Let $uv$ be an edge of $G$ (our claim is obvious if $G$ is empty) and $c' = c - \min c(V)$.

$(\Rightarrow)$ $c$ is a $T$-coloring of $G$ and $T$ is an interval, so $|c'(u) - c'(v)| = |c(u) - c(v)| \geq d$. Moreover, $|c'(u) - c'(v)| = |c(u) - c(v)| \leq \esp(c) \leq k - d$ and $c'(V) \subseteq \lceil 0, \esp(c) \rceil \subseteq [0,k-1]$.

$(\Leftarrow)$ $c'$ is a $(k,d)$-coloring of $G$, so $|c(u) - c(v)| = |c'(u) - c'(v)| \geq d$ and $|c(u) - c(v)| = |c'(u) - c'(v)| \leq k - d$. This proves that $c$ is a $T$-coloring and gives $\esp(c) \leq k - d$. To complete the proof it suffices to observe that $c'(V) \subseteq [0,k-1]$ implies $\sp(c) = \sp(c') \leq k - 1$.

Theorem 8. Let $G$ be a graph and $d$ be a positive integer. If $T = [0,d-1]$, then

$$\esp_T(G) = \lceil d(\chi_e(G) - 1) \rceil.$$ 

Proof. Without loss of generality we assume that $G$ is not empty. Then $k = \lceil d(\chi_e(G)) \rceil - 1 \geq d$. If $\esp_T(G) \leq k - d$, then, by Corollary 6, there is a $T$-coloring $c$ of $G$ such that $\esp(c) = \esp_T(G) \leq k - d$ and $\sp(c) \leq \esp_T(G) + d - 1 \leq k - 1$. 

Lemma 7 implies now that \( c - \min c(V) \) is a \((k, d)\)-coloring, which finally gives 
\[ d\chi_c(G) \leq k \], a contradiction. Hence 
\[ \text{esp}_T(G) \geq k - d + 1. \]

On the other hand, \((k + 1)/d \geq \chi_c(G)\) so there exists a \((k + 1, d)\)-coloring \( c \) of \( G \).
Without loss of generality we assume that \( \min c(V) = 0 \). By Lemma 7, \( c \) has to be a \( T \)-coloring of \( G \) with \( \text{esp}(c) \leq k - d + 1 \). This gives
\[ \text{esp}_T(G) \leq k - d + 1. \]

Combining these inequalities together, we get \( \text{esp}_T(G) = k - d + 1 = \lceil d\chi_c(G) \rceil - d = \lceil d(\chi_c(G) - 1) \rceil \).

Since \( T \) is an interval, we know that \( |T| = \max T + 1 \) and the above formula may be expressed as 
\[ \text{esp}_T(G) = |T|(\chi_c(G) - 1). \]

This resembles Tesman’s inequality \( \text{sp}_T(G) \leq |T|(\chi(G) - 1) \) which holds for all \( T \)-sets \( T \) and all graphs \( G \) [11], so it is interesting to ask the following question.

Does \( \text{esp}_T(G) \leq |T|(\chi_c(G) - 1) \) for all \( T \)-sets \( T \) and all graphs \( G \)?

Unfortunately, the answer is negative even for odd cycles. To show this, let us consider integers \( 1 \leq k \leq n - 1 \) and set \( T = \{0, 2, \ldots, 2k\} \) and \( G = C_{2n+1}. \)
Then \( |T|(\chi_c(G) - 1) = \lceil (k + 1)(1 + 1/n) \rceil = k + 2 \) and \( \text{esp}_T(C_{2n+1}) \geq 2k + 2 \) since otherwise the differences of colors assigned to adjacent vertices of \( G \) in any \( T \)-coloring of \( G \) with minimal edge span would be odd and their sum would not be 0, a contradiction.

Theorem 8 shows also that the value of integer \( r \) of Theorem 3 can be arbitrary. Indeed, if we take \( 0 \leq r \leq d - 1 \) and a planar graph \( G \) such that \( \chi_c(G) = 3 - r/d \) (which exists by [10]), then \( \chi(G) = 3 \) and \( \text{esp}_{d\in\{0\}}(G) = [d(\chi_c(G) - 1)] = [d(2 - r/d)] = 2d - r = d(\chi(G) - 1) - r = d\chi_{\{0\}}(G) - r \). The open question is if this is true for all \( T \)-sets \( T \).

**Theorem 9.** Let \( G \) be a graph. Then
\[ \chi_c(G) = 1 + \inf \left\{ \text{esp}_{d\in\{0\}}(G)/d : d \geq 1 \right\}. \]

Moreover, if \( \chi_c(G) = k/d \) (\( 1 \leq d \leq k \)), then \( \chi_c(G) = 1 + \text{esp}_{d\in\{0\}}(G)/d \).

**Proof.** \( \chi_c(G) - 1 \leq \text{esp}_{d\in\{0\}}(G)/d \) by Theorem 8. To complete the proof it suffices to observe that if \( \chi_c(G) = k/d \), then the same theorem gives \( \chi_c(G) - 1 = \text{esp}_{d\in\{0\}}(G)/d \). □
Theorems 8 and 9 have two important consequences. Firstly, if we know a formula for $\chi_c(G)$, then we can easily obtain a formula for $\esp_T(G)$ for all $T$-sets $T$ that are intervals. For example, Fan [4] proved that $\chi_c(G) = \chi(G)$ if the complement of $G$ is non-Hamiltonian, which gives

**Corollary 10.** If $G$ is a graph whose complement is non-Hamiltonian, then

$$\esp_{d\oplus\{0\}}(G) = d(\chi(G) - 1) = \esp_{d\oplus\{0\}}(G)$$

for every $d \geq 1$.

Secondly, if the problem of computing $\chi_c(G)$ for graphs $G$ from a certain class $\mathcal{G}$ is polynomially solvable, then we can compute $\esp_T(G)$ for $G \in \mathcal{G}$ and any interval $T$ in a polynomial time, too.

5. **Powers of Cycles**

Let $p \geq 1$ and $n \geq 2p + 2$ be integers. Let $q$ and $r$ are the quotient and the remainder of the division of $n$ by $p + 1$, respectively.

Zhao et al. in [13] proved the following theorem.

**Theorem 11.** If $q = pl + t$ for $l \geq 0$, $0 \leq t \leq p - 1$ such that $p \geq td$, then

$$\esp_{d\oplus\{0\}}(C_{pn}) = pd + \lceil rd/q \rceil.$$  

Moreover, they conjectured that this equality holds for any $n \geq 2p + 2$, not only when $p \geq td$. We will show that it is true. Recall that it is known that if $G$ is a $n$-vertex graph, then $\chi_c(G) \geq n/\alpha(G)$, where $\alpha(G)$ is the independence number of $G$.

**Theorem 12.** $\chi_c(C_{pn}) = n/q$.

**Proof.** Let $v_0, v_1, \ldots, v_{n-1}$ be a cyclic ordering of vertices of $C_{pn}$. We claim that a function given by

$$c(v_i) = (iq) \mod n$$

is a $(n,q)$-coloring of $C_{pn}$. Indeed, the definition of $c$ gives $0 \leq c \leq n - 1$ and, if $v_iv_j$ ($i > j$) is an edge of $C_{pn}$, then either $1 \leq i - j \leq p$ and $|c(v_i) - c(v_j)| = (i-j)q$ or $1 \leq n + j - i \leq p$ and $|c(v_i) - c(v_j)| = (n - i + j)q$. In both cases it is easy to verify that $q \leq |c(v_i) - c(v_j)| \leq qp \leq n - q$.

To complete the proof it suffices to observe that $\alpha(C_{pn}) \leq q$ and use inequality $\chi_c(G) \geq n/\alpha(G)$.

**Theorem 13.** $\esp_{d\oplus\{0\}}(C_{pn}) = pd + \lceil rd/q \rceil$.

**Proof.** Follows immediately from Theorems 8 and 12.
6. Conclusion

We proved the general relation between the circular chromatic number and $T$-edge span for $T = d \odot \{0\}$. Moreover, we applied it to solve an open conjecture concerning the $T$-edge span for powers of cycles $C_n^p$.

Possible further fields of research include for example finding the necessary conditions for $\esp_T(G) \leq \lceil |T| (\chi_c(G) - 1) \rceil$, or analyzing dependence between $\esp_T(G)$ and $\chi_c(G)$ on the structure of a set $T$.

References


Received 19 March 2018
Revised 16 October 2018
Accepted 7 December 2018