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T-COLORINGS, DIVISIBILITY AND THE CIRCULAR CHROMATIC NUMBER

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Abstract

Let T be a T-set, i.e., a finite set of nonnegative integers satisfying $0 \in T$, and G be a graph. In the paper we study relations between the T-edge spans $\mathrm{esp}_T(G)$ and $\mathrm{esp}_{d\odot T}(G)$, where d is a positive integer and

$$d \odot T = \{0 \leq t \leq d\,(\max T + 1)\colon d \mid t \Rightarrow t/d \in T\}\,.$$

We show that $\mathrm{esp}_{d\odot T}(G) = d\,\mathrm{esp}_T(G) - r$, where r, $0 \leq r \leq d-1$, is an integer that depends on T and G. Next we focus on the case $T = \{0\}$ and show that

$$\mathrm{esp}_{d\odot\{0\}}(G) = \lceil d(\chi_c(G) - 1)\rceil,$$

where $\chi_c(G)$ is the circular chromatic number of G. This result allows us to formulate several interesting conclusions that include a new formula for the circular chromatic number

$$\chi_c(G) = 1 + \inf\left\{ \mathrm{esp}_{d\odot\{0\}}(G)/d\colon d \geq 1\right\}$$

and a proof that the formula for the T-edge span of powers of cycles, stated as conjecture in [Y. Zhao, W. He and R. Cao, *The edge span of T-coloring on graph C_n^d*, Appl. Math. Lett. 19 (2006) 647–651], is true.

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1. Introduction

In the paper we study relations between two different generalizations of ordinary vertex colorings: T-colorings and (k, d)-colorings. Let G be a graph with n-vertex set V and edge set E. Given integers $1 \leq d \leq k$, by a *(k, d)-coloring* of G we mean any function $c\colon V \to [0, k-1]$ ($[a, b] := \{a, a+1, \ldots, b\}$ for any integers $a \leq b$) such that

$$d \leq |c(u) - c(v)| \leq k - d$$

whenever $uv \in E$. This notion may be viewed as a generalization of a k-coloring since (k, d)-colorings of G are k-colorings of G and $(k, 1)$-colorings are the same as k-colorings that use colors from the interval $[0, k-1]$. The *circular chromatic number*, introduced by Vince [12] as a generalization of the chromatic number, is defined by the formula

$$\chi_c(G) = \inf \{k/d\colon G \text{ has a } (k, d)\text{-coloring}\}.$$

The circular chromatic number was studied by many authors, see [14, 15] for a survey of results. It was shown for example [12] that the distance between the circular and ordinary chromatic number does not exceed 1, i.e.

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

In the same paper Vince proved two useful facts: (1) G has a (k, d)-coloring if and only if $\chi_c(G) \leq k/d$; (2) $\chi_c(G)$ is a rational number which has a form k/d, where $k \leq n$. We will use these observations to show that there is a relation between $\chi_c(G)$ and $\mathrm{esp}_T(G)$ the T-edge span defined below. Given a T-set T, i.e., a finite set that consists of nonnegative integers and satisfies $0 \in T$, by a *T-coloring* of G we mean any function $c\colon V \to \mathbb{Z}$ such that

$$|c(u) - c(v)| \notin T$$

whenever $uv \in E$. T-colorings were introduced as a model for the frequency assignment problem in [5]. This notion also may be viewed as a generalization of ordinary vertex colorings since T-colorings are vertex colorings and vertex colorings are $\{0\}$-colorings. The T*-edge span*, introduced by Cozzens and Roberts [1], is defined as

$$\operatorname{esp}_T(G) = \min\{\operatorname{esp}(c)\colon c \text{ is a } T\text{-coloring of } G\},$$

where $\operatorname{esp}(c) = \max\{|c(u) - c(v)|\colon uv \in E\}$ is the *edge span* of c (if G is an empty graph then $\operatorname{esp}(c) = 0$). If we replace $\operatorname{esp}(c)$ by $\operatorname{sp}(c)$ (the *span* of c, i.e., $\max\{|c(u) - c(v)|\colon u, v \in V\}$) we will receive the T*-span* of G. Both parameters were studied by many authors, there are results concerning computational complexity of the problem of computing $\operatorname{sp}_T(G)$ [2, 3], the behaviour of the greedy algorithm [7] and formulas describing $\operatorname{sp}_T(G)$ and $\operatorname{esp}_T(G)$ for some T-sets T and some graphs G [8, 9, 13].

The remainder of the paper is organized as follows. In Section 2 we study relations between $\operatorname{esp}_T(G)$ and $\operatorname{esp}_{d\odot T}(G)$, where d is a positive integer and $d \odot T = \{0 \le t \le d(\max T + 1)\colon d\,|\,t \Rightarrow t/d \in T\}$. We show that $\operatorname{esp}_{d\odot T}(G) = d\operatorname{esp}_T(G) - r$, where r, $0 \le r \le d-1$, is an integer that depends on T and G. In Section 3 we study the distance between the T-span and T-edge span and show that it cannot exceed $\max T$. We also give examples that prove that this bound is tight. Section 4 contains our main results. We show that if T is an interval, i.e., $T = [0, d-1]$ (or equivalently $T = d \odot \{0\}$), then (k, d)-colorings ($k \ge d$) are nonnegative T-colorings with span bounded by $k - 1$ and edge span bounded by $k - d$. We use this relation to show that

$$\operatorname{esp}_{d\odot\{0\}}(G) = \lceil d(\chi_c(G) - 1)\rceil .$$

We also discuss whether it is possible to extend this relation to all T-sets. Using the above formula we show that

$$\chi_c(G) = 1 + \inf\left\{ \operatorname{esp}_{d\odot\{0\}}(G)/d\colon d \ge 1\right\}$$

and discuss how these formulas allow us to move known results from the world of the T-edge span to the world of the circular chromatic number and vice versa. The last section is devoted to the powers of cycles investigated in [13]. The authors conjectured and partially proved that

$$\operatorname{esp}_{d\odot\{0\}}(C_n^p) = pd + \lceil rd/q\rceil ,$$

where $q \ge 2$ and r are the quotient and the remainder of the division of n by $p + 1$, respectively. We show that it is true in general.

2. T-EDGE SPAN AND $d \odot T$-EDGE SPAN

The operation $\odot$ was introduced in [6], where it was shown that $\operatorname{sp}_{d\odot T}(G) = d\operatorname{sp}_T(G)$. Below we prove a similar formula for the T-edge span, but before we proceed we need to recall the following result.

Lemma 1 (Lemma 2.2(i) of [6])**.** *If a and b are real numbers, then* $\lfloor |a-b| \rfloor \leq |\lfloor a \rfloor - \lfloor b \rfloor| \leq \lceil |a-b| \rceil$.

Lemma 2. *Let G be a graph, T be a T-set and d be a positive integer.*

(1) *If c is a T-coloring of G, then dc is a $d \odot T$-coloring of G.*

(2) *If c is a $d \odot T$-coloring of G, then $\lfloor c/d \rfloor$ is a T-coloring of G.*

Proof. Let uv be an edge of G (if G is empty, then our claim is obvious).

(1) If $|c(u)-c(v)| \geq \max T+1$, then $|dc(u)-dc(v)| \geq d\,(\max T+1) = \max d \odot T+1$. If $|c(u)-c(v)| < \max T+1$ and $|dc(u)-dc(v)| \in d \odot T$, then the definition of $d \odot T$ gives $|c(u)-c(v)| \in T$, a contradiction. Hence $|dc(u)-dc(v)| \notin d \odot T$ in both cases.

(2) If $|c(u)-c(v)| \geq \max d \odot T+1 = d\,(\max T+1)$, then $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| \geq \lfloor |c(u)-c(v)|/d \rfloor \geq \max T+1$ by Lemma 1. If $|c(u)-c(v)| < \max d \odot T+1$, then the definition of $d \odot T$ gives $d \mid |c(u)-c(v)|$ and, by Lemma 1, $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| = |c(u)-c(v)|/d \notin T$. Hence $|\lfloor c(u)/d \rfloor - \lfloor c(v)/d \rfloor| \notin T$ in both cases. ■

Theorem 3. *Let G be a graph, T be a T-set and d be a positive integer. There is an integer $0 \leq r \leq d-1$ such that $\operatorname{esp}_{d \odot T}(G) = d\operatorname{esp}_T(G) - r$.*

Proof. Let c be a T-coloring of G such that $\operatorname{esp}(c) = \operatorname{esp}_T(G)$. By Lemma 2, dc is a $d \odot T$-coloring of G. Hence

$$\operatorname{esp}_{d \odot T}(G) \leq \operatorname{esp}(dc) = d\operatorname{esp}(c) = d\operatorname{esp}_T(G). \tag{1}$$

Let c' be a $d \odot T$-coloring of G such that $\operatorname{esp}(c') = \operatorname{esp}_{d \odot T}(G)$. By Lemma 2, $\lfloor c'/d \rfloor$ is a T-coloring of G. Let uv be an edge of G such that $\operatorname{esp}(\lfloor c'/d \rfloor) = |\lfloor c'(u)/d \rfloor - \lfloor c'(v)/d \rfloor|$ (if G is empty our claim is obvious). Then

$$\begin{aligned} d\operatorname{esp}_T(G) - d &\leq d\operatorname{esp}(\lfloor c'/d \rfloor) - d = d|\lfloor c'(u)/d \rfloor - \lfloor c'(v)/d \rfloor| - d \\ &\leq d\left\lceil |c'(u)-c'(v)|/d \right\rceil - d \leq d\left\lceil \operatorname{esp}(c')/d \right\rceil - d \\ &= d\left\lceil \operatorname{esp}_{d \odot T}(G)/d \right\rceil - d < \operatorname{esp}_{d \odot T}(G). \end{aligned} \tag{2}$$

To complete the proof it suffices to combine (1) with (2). ■

The open problem is a formula for r. Later we will show how to compute r provided that $T = \{0\}$ and that r can be any integer from $[0, d-1]$.

Corollary 4. *Let G be a graph, T be a T-set and d be a positive integer. Then* $\operatorname{esp}_T(G) = \left\lceil \operatorname{esp}_{d \odot T}(G)/d \right\rceil$.

3. The Distance Between the T-Span and T-Edge Span

It is known [1] that $\mathrm{esp}_T(G) \le \mathrm{sp}_T(G)$. We are going to show that $\mathrm{sp}_T(G) \le \mathrm{esp}_T(G) + \max T$ and give examples in which the difference $\mathrm{sp}_T(G) - \mathrm{esp}_T(G)$ equals $\max T$.

Lemma 5. *Let G be a graph and T be a T-set. If $c' \colon V \to \mathbb{Z}$ is a T-coloring of G and $c \colon V \to \mathbb{Z}$ is the remainder of the division of c' by $\mathrm{esp}(c') + \max T + 1$, i.e., $c(v) = c'(v) \bmod (\mathrm{esp}(c') + \max T + 1)$ for $v \in V$, then*

(1) *c is a T-coloring of G;*

(2) *$\mathrm{sp}(c) \le \mathrm{esp}(c') + \max T$;*

(3) *$\mathrm{esp}(c) \le \mathrm{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$.*

Proof. Observe that (2) follows immediately from the definition of c. To prove (1) and (3), take an edge uv of G (if G is empty, our claim is obvious). Let q be the quotient of the division of c' by $\mathrm{esp}(c') + \max T + 1$. Without loss of generality we may assume that $q(u) \ge q(v)$. It is easy to see that $q(u) \le q(v) + 1$ since otherwise

$$\begin{aligned}
\mathrm{esp}(c') &\ge |c'(u) - c'(v)| \\
&= |(\mathrm{esp}(c') + \max T + 1)(q(u) - q(v)) + c(u) - c(v)| \\
&\ge (\mathrm{esp}(c') + \max T + 1)|q(u) - q(v)| - |c(u) - c(v)| \\
&\ge 2(\mathrm{esp}(c') + \max T + 1) - \mathrm{esp}(c') - \max T \\
&= \mathrm{esp}(c') + \max T + 2 > \mathrm{esp}(c').
\end{aligned}$$

Hence there are two cases to consider.

(a) $q(u) = q(v) + 1$. Then $|c'(u) - c'(v)| = |(\mathrm{esp}(c') + \max T + 1)(q(u) - q(v)) + (c(u) - c(v))| = |\,\mathrm{esp}(c') + \max T + 1 + (c(u) - c(v))|$. Since $\mathrm{esp}(c') + \max T + 1 > \mathrm{esp}(c') \ge |c'(u) - c'(v)|$ and $|c(u) - c(v)| \le \mathrm{esp}(c') + \max T$, we have $|c(u) - c(v)| = \mathrm{esp}(c') + \max T + 1 - |c'(u) - c'(v)|$. This gives $|c(u) - c(v)| \ge \max T + 1$ and $|c(u) - c(v)| \le \mathrm{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$ since $|c'(u) - c'(v)| \notin T$ implies $|c'(u) - c'(v)| \ge \min(\mathbb{N} \setminus T)$.

(b) $q(u) = q(v)$. Then $|c'(u) - c'(v)| = |c(u) - c(v)|$, which gives $|c(u) - c(v)| \notin T$ and $|c(u) - c(v)| \le \mathrm{esp}(c') \le \mathrm{esp}(c') + \max T + 1 - \min(\mathbb{N} \setminus T)$. ■

Corollary 6. *Let G be a graph and T be a T-set. Then*

(1) *There is a T-coloring c of G such that $\mathrm{sp}(c) \le \mathrm{esp}_T(G) + \max T$ and $\mathrm{esp}(c) \le \mathrm{esp}_T(G) + \max T + 1 - \min(\mathbb{N} \setminus T)$.*

(2) *If T is an interval, then there is a T-coloring c of G such that $\mathrm{esp}(c) = \mathrm{esp}_T(G)$ and $\mathrm{sp}(c) \le \mathrm{esp}_T(G) + \max T$.*

(3) *$\mathrm{esp}_T(G) \le \mathrm{sp}_T(G) \le \mathrm{esp}_T(G) + \max T$.*

Proof. (1) Let c' be a T-coloring of G satisfying $\text{esp}(c') = \text{esp}_T(G)$ and c be the remainder of the division of c' by $\text{esp}_T(G) + \max T + 1$. The claim follows from Lemma 5.

(2) Follows from (1) since $\min(\mathbb{N} \setminus T) = \max T + 1$ if T is an interval.

(3) Follows from (1) and the definition of the T-span. ■

The above inequalities are tight. It is known [1] that $\text{esp}_T(G) = \text{sp}_T(G)$ for all weakly perfect graphs and all T-sets T. It is also easy to see that if T is an interval, then $\text{sp}_T(C_{2n+1}) = 2\max T + 2$ ($\text{sp}_T(G) = (\max T + 1)(\chi(G) - 1)$ if T is an interval, see [1]) and $\text{esp}_T(C_{2n+1}) = \lceil(\max T + 1)(1 + 1/n)\rceil$ (see Theorem 8) which gives $\text{sp}_T(C_{2n+1}) = \text{esp}_T(C_{2n+1}) + \max T$ provided that $n \geq \max T + 1$.

4. The Relation Between (k, d)-Colorings and T-Colorings

Now we are ready to prove that there is a relation between (k, d)-colorings and T-colorings provided that T is an interval.

Lemma 7. *Let G be a graph and d be a positive integer. If $T = [0, d - 1]$, then for every function $c \colon V \to \mathbb{Z}$ and every integer $k \geq d$ the following conditions are equivalent:*

(1) *c is a T-coloring of G such that* $\text{sp}(c) \leq k - 1$ *and* $\text{esp}(c) \leq k - d$*;*

(2) *$c - \min c(V)$ is a (k, d)-coloring of G.*

Proof. Let uv be an edge of G (our claim is obvious if G is empty) and $c' = c - \min c(V)$.

($\Rightarrow$) c is a T-coloring of G and T is an interval, so $|c'(u) - c'(v)| = |c(u) - c(v)| \geq d$. Moreover, $|c'(u) - c'(v)| = |c(u) - c(v)| \leq \text{esp}(c) \leq k - d$ and $c'(V) \subseteq [0, \text{sp}(c)] \subseteq [0, k - 1]$.

($\Leftarrow$) c' is a (k, d)-coloring of G, so $|c(u) - c(v)| = |c'(u) - c'(v)| \geq d$ and $|c(u) - c(v)| = |c'(u) - c'(v)| \leq k - d$. This proves that c is a T-coloring and gives $\text{esp}(c) \leq k - d$. To complete the proof it suffices to observe that $c'(V) \subseteq [0, k - 1]$ implies $\text{sp}(c) = \text{sp}(c') \leq k - 1$. ■

Theorem 8. *Let G be a graph and d be a positive integer. If $T = [0, d - 1]$, then*

$$\text{esp}_T(G) = \lceil d(\chi_c(G) - 1)\rceil.$$

Proof. Without loss of generality we assume that G is not empty. Then $k = \lceil d\chi_c(G)\rceil - 1 \geq d$. If $\text{esp}_T(G) \leq k - d$, then, by Corollary 6, there is a T-coloring c of G such that $\text{esp}(c) = \text{esp}_T(G) \leq k - d$ and $\text{sp}(c) \leq \text{esp}_T(G) + d - 1 \leq k - 1$.

Lemma 7 implies now that $c - \min c(V)$ is a (k, d)-coloring, which finally gives $d\chi_c(G) \le k$, a contradiction. Hence

$$\mathrm{esp}_T(G) \ge k - d + 1.$$

On the other hand, $(k+1)/d \ge \chi_c(G)$ so there exists a $(k+1, d)$-coloring c of G. Without loss of generality we assume that $\min c(V) = 0$. By Lemma 7, c has to be a T-coloring of G with $\mathrm{esp}(c) \le k - d + 1$. This gives

$$\mathrm{esp}_T(G) \le k - d + 1.$$

Combining these inequalities together, we get $\mathrm{esp}_T(G) = k - d + 1 = \lceil d\chi_c(G)\rceil - d = \lceil d(\chi_c(G) - 1)\rceil$. ■

Since T is an interval, we know that $|T| = \max T + 1$ and the above formula may be expressed as

$$\mathrm{esp}_T(G) = \lceil |T|(\chi_c(G) - 1)\rceil.$$

This resembles Tesman's inequality $\mathrm{sp}_T(G) \le |T|(\chi(G) - 1)$ which holds for all T-sets T and all graphs G [11], so it is interesting to ask the following question.

Does $\mathrm{esp}_T(G) \le \lceil |T|(\chi_c(G) - 1)\rceil$ for all T-sets T and all graphs G?

Unfortunately, the answer is negative even for odd cycles. To show this, let us consider integers $1 \le k \le n - 1$ and set $T = \{0, 2, \ldots, 2k\}$ and $G = C_{2n+1}$. Then $\lceil |T|(\chi_c(G) - 1)\rceil = \lceil (k+1)(1 + 1/n)\rceil = k + 2$ and $\mathrm{esp}_T(C_{2n+1}) \ge 2k + 2$ since otherwise the differences of colors assigned to adjacent vertices of G in any T-coloring of G with minimal edge span would be odd and their sum would not be 0, a contradiction.

Theorem 8 shows also that the value of integer r of Theorem 3 can be arbitrary. Indeed, if we take $0 \le r \le d - 1$ and a planar graph G such that $\chi_c(G) = 3 - r/d$ (which exists by [10]), then $\chi(G) = 3$ and $\mathrm{esp}_{d\odot\{0\}}(G) = \lceil d(\chi_c(G) - 1)\rceil = \lceil d(2 - r/d)\rceil = 2d - r = d(\chi(G) - 1) - r = d\,\mathrm{esp}_{\{0\}}(G) - r$. The open question is if this is true for all T-sets T.

Theorem 9. *Let G be a graph. Then*

$$\chi_c(G) = 1 + \inf\bigl\{\mathrm{esp}_{d\odot\{0\}}(G)/d \colon d \ge 1\bigr\}.$$

Moreover, if $\chi_c(G) = k/d$ $(1 \le d \le k)$, then $\chi_c(G) = 1 + \mathrm{esp}_{d\odot\{0\}}(G)/d$.

Proof. $\chi_c(G) - 1 \le \mathrm{esp}_{d\odot\{0\}}(G)/d$ by Theorem 8. To complete the proof it suffices to observe that if $\chi_c(G) = k/d$, then the same theorem gives $\chi_c(G) - 1 = \mathrm{esp}_{d\odot\{0\}}(G)/d$. ■

Theorems 8 and 9 have two important consequences. Firstly, if we know a formula for $\chi_c(G)$, then we can easily obtain a formula for $\mathrm{esp}_T(G)$ for all T-sets T that are intervals. For example, Fan [4] proved that $\chi_c(G) = \chi(G)$ if the complement of G is non-Hamiltonian, which gives

Corollary 10. *If G is a graph whose complement is non-Hamiltonian, then*

$$\mathrm{esp}_{d\odot\{0\}}(G) = d(\chi(G) - 1) = \mathrm{sp}_{d\odot\{0\}}(G)$$

for every $d \geq 1$.

Secondly, if the problem of computing $\chi_c(G)$ for graphs G from a certain class $\mathcal{G}$ is polynomially solvable, then we can compute $\mathrm{esp}_T(G)$ for $G \in \mathcal{G}$ and any interval T in a polynomial time, too.

5. Powers of Cycles

Let $p \geq 1$ and $n \geq 2p + 2$ be integers. Let q and r are the quotient and the remainder of the division of n by $p + 1$, respectively.

Zhao *et al.* in [13] proved the following theorem.

Theorem 11. *If $q = pl + t$ for $l \geq 0$, $0 \leq t \leq p - 1$ such that $p \geq td$, then*

$$\mathrm{esp}_{d\odot\{0\}}(C_n^p) = pd + \lceil rd/q \rceil .$$

Moreover, they conjectured that this equality holds for any $n \geq 2p + 2$, not only when $p \geq td$. We will show that it is true. Recall that it is known that if G is a n-vertex graph, then $\chi_c(G) \geq n/\alpha(G)$, where $\alpha(G)$ is the independence number of G.

Theorem 12. $\chi_c(C_n^p) = n/q$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be a cyclic ordering of vertices of C_n^p. We claim that a function given by

$$c(v_i) = (iq) \bmod n$$

is a (n, q)-coloring of C_n^p. Indeed, the definition of c gives $0 \leq c \leq n - 1$ and, if v_iv_j $(i > j)$ is an edge of C_n^p, then either $1 \leq i-j \leq p$ and $|c(v_i)-c(v_j)| = (i-j)q$ or $1 \leq n + j - i \leq p$ and $|c(v_i) - c(v_j)| = (n - i + j)q$. In both cases it is easy to verify that $q \leq |c(v_i) - c(v_j)| \leq qp \leq n - q$.

To complete the proof it suffices to observe that $\alpha(C_n^p) \leq q$ and use inequality $\chi_c(G) \geq n/\alpha(G)$. ■

Theorem 13. $\mathrm{esp}_{d\odot\{0\}}(C_n^p) = pd + \lceil rd/q \rceil$.

Proof. Follows immediately from Theorems 8 and 12. ■

6. Conclusion

We proved the general relation between the circular chromatic number and T-edge span for $T = d \odot \{0\}$. Moreover, we applied it to solve an open conjecture concerning the T-edge span for powers of cycles C_n^p.

Possible further fields of research include for example finding the necessary conditions for $\mathrm{esp}_T(G) \leq \lceil |T|(\chi_c(G) - 1)\rceil$, or analyzing dependence between $\mathrm{esp}_T(G)$ and $\chi_c(G)$ on the structure of a set T.

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