ON THE n-PARTITE TOURNAMENTS WITH EXACTLY
n − m + 1 CYCLES OF LENGTH m

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Abstract

Gutin and Rafiey [Multipartite tournaments with small number of cycles, Australas J. Combin. 34 (2006) 17–21] raised the following two problems: (1) Let m ∈ {3, 4, . . . , n}. Find a characterization of strong n-partite tournaments having exactly n − m + 1 cycles of length m; (2) Let 3 ≤ m ≤ n and n ≥ 4. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly n − m + 1 cycles of length m for two values of m? In this paper, we discuss the strong n-partite tournaments D containing exactly n − m + 1 cycles of length m for 4 ≤ m ≤ n − 1. We describe the substructure of such D satisfying a given condition and we also show that, under this condition, the second problem has a negative answer.

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1. Introduction

An n-partite or multipartite tournament is an orientation of a complete n-partite graph. A tournament is an n-partite tournament with exactly n vertices. A digraph D is transitive if, for every pair of arcs xy and yz in D such that x ≠ z, the arc xz is also in D. It is easy to show that a tournament is transitive if and only if it is acyclic.

A digraph D is said to be strong, if for every pair of vertices x and y, D contains a path from x to y and a path from y to x. A directed path from x to y in D is denoted by an (x, y)-path. An l-cycle is a cycle of length l. A cycle or path in a digraph D is Hamiltonian if it includes all the vertices of D.

In 1966, Moon discussed the number of m-cycle in a strong tournament.
Theorem 1 (Moon [9]). Let $T$ be a strong tournament of order $n$. Then $T$ contains at least $n - m + 1$ cycles of length $m$ for $3 \leq m \leq n$.

The tournaments which gain the lower bound in Theorem 1 were characterized by Burzio and Demaria [2] for $m = 3$, Douglas [3] for $m = n$ and Las Vergnas [8] for $4 \leq m \leq n - 1$. We list the result of Las Vergnas especially because we will use it to prove our main results.

Theorem 2 (Las Vergnas [8]). Every strong tournament of order $n$ having exactly $n - m + 1$ cycles of given length $m$ with $4 \leq m \leq n - 1$ is isomorphic to $Q_n$, where $Q_n$ is a tournament of order $n \geq 3$ obtained by reversing the arcs in the unique Hamiltonian path of a transitive tournament.

In 2002, Volkmann extended Theorem 1 from tournaments to multipartite tournaments.

Theorem 3 (Volkmann [10]). Let $D$ be a strong $n$-partite tournament. Then $D$ contains at least $n - m + 1$ cycles of length $m$ for $3 \leq m \leq n$.

It is notable that the bound in Theorem 3 is sharp which can be seen in [4] for $n = 3$ and in [6] for $4 \leq m \leq n$. In addition, Gutin and Rafiey [6] raised the following two interesting and natural problems.

Problem 4 (Gutin and Rafiey [6]). Given $m \in \{3, 4, \ldots, n\}$, find a characterization of strong $n$-partite tournaments having exactly $n - m + 1$ cycles of length $m$.

Problem 5 (Gutin and Rafiey [6]). Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong $n$-partite tournaments, which are not themselves tournaments, with exactly $n - m + 1$ cycles of length $m$ for two values of $m$?

Problem 4 seems to be especially interesting for the case $m = n$ which was already solved by Gutin et al. in [7]. In this paper, we investigate strong $n$-partite tournaments $D$, which are not themselves tournaments and contain exactly $n - m + 1$ cycles of length $m$ for any given $4 \leq m \leq n - 1$. We prove that if $D$ has an $(n - 1)$-cycle with no pair of vertices from the same partite set, then $D$ must contain some given multipartite tournament as its subdigraph.

As far as Problem 5, Gutin and Rafiey [6] gave a negative answer for two values of $n - 1$ and $m$. This also implies that Problem 5 has a negative answer for $n = 4$. In this paper, we give a necessary condition to Problem 5 and show that if a strong $n$-partite tournament $D$, which is not itself a tournament, contains exactly $n - m + 1$ cycles of length $m$ for two values of $m \in \{4, 5, \ldots, n - 1\}$, then there is no an $(n - 1)$-cycle with no pair of vertices from the same partite set in $D$. 

2. Terminology and Preliminaries

We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

Let $D$ be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. We call the number of vertices of $D$ the order of $D$. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D(A)$. We use $V(D) \setminus V(A)$ to stand for the set of vertices which are in $V(D)$ but not in $V(A)$.

If $xy$ is an arc in $D$, then we say that $x$ dominates $y$ and write $x \rightarrow y$. For two disjoint subsets $X$ and $Y$ of $V(D)$, if every vertex of $X$ dominates every vertex of $Y$, we say $X$ dominates $Y$ and write $X \rightarrow Y$. Furthermore, $X \Rightarrow Y$ denotes the property that there is no arc from $Y$ to $X$.

The out-neighborhood $N^+(x)$ of a vertex $x$ is the set of vertices dominated by $x$ and the in-neighborhood $N^-(x)$ of a vertex $x$ is the set of vertices dominating $x$. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the outdegree and indegree of $x$, respectively. The global irregularity of $D$ is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. We denote by $D^{-1}$ the inverse digraph of $D$.

In order to present our main results, we define a class of $n$-partite tournaments $D_n$ of order $n + 1$ as described in the following figure, where $3 \leq m \leq n$, \{v_2, \ldots, v_{m-2}, v_m, \ldots, v_n\} \rightarrow y \rightarrow v_1$, $y$ and $v_{m-1}$ belong to the same partite set and $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n$.

The following two theorems on cycles in strong $n$-partite tournaments are very useful to prove our main results.

**Theorem 6** (Guo and Volkmann [5]). Every partite set of a strong $n$-partite tournament, $n \geq 3$, contains a vertex which lies on an $m$-cycle for each $m \in \{3, 4, \ldots, n\}$.

**Theorem 7** (Gutin and Rafiey [6]). Let $D$ be a strong $n$-partite tournament containing exactly $n - m + 1$ cycles of length $m$ for some $m \in \{3, 4, \ldots, n\}$. Then every $m$-cycle of $D$ has no pair of vertices from the same partite set.
3. Main Results

Before presenting the main results, we first prove the following lemma.

**Lemma 8.** Let $D$ be a strong $n$-partite tournament, $n \geq 5$, containing exactly $n - m + 1$ cycles of length $m$ for some $3 \leq m \leq n - 1$. If $D$ has an $(n - 1)$-cycle $C$ with no pair of vertices from the same partite set, then the following statements hold.

(a) There are no two vertices $u, w$ in $V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$.

(b) There exists a vertex $v \notin V(C)$ such that $D(V(C) \cup \{v\})$ is strong.

**Proof.** Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$. Suppose, without loss of generality, that $C = v_1v_2 \cdots v_{n-1}v_1$ with $v_i \in V_i$, $i = 1, 2, \ldots, n - 1$. By Theorem 1, $D(V(C))$ contains at least $(n - 1) - m + 1 = n - m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$.

(a) Suppose to the contrary that there exist two vertices $u, w \in V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$. Obviously, $u \notin V_n$ or $w \notin V_n$. Assume, without loss of generality, that $w \in V_j$ for some $1 \leq j \leq n - 1$. Since $n \geq 5$, there exist at least two vertices $v_k$ and $v_l$, such that $u, w, v_k$ and $v_l$ are in different partite sets. If $m = 3$, then $uvw_ku$ and $uvv_lu$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$. This contradicts the fact that $D$ contains exactly $n - m + 1$ cycles of length $m$.

If $m \geq 4$, then $uvw_{j+1}v_j \cdots v_{j+m-4}u$ (if $u \notin V_{j+m-4}$) or $uvv_{j+1}v_{j+2} \cdots v_{j+m-2}u$ (if $u \in V_{j+m-4}$) is an $m$-cycle with $v_j \in V_j$ or $u, v_{j+m-4} \in V_{j+m-4}$ (where all indices are modulo $n - 1$). This is impossible by Theorem 7.

(b) Assume that there is no vertex $v \in V(D) \setminus V(C)$ such that $D(V(C) \cup \{v\})$ is strong. Let $S = \{x \in V(D) \setminus V(C) : C \Rightarrow u \} \rightarrow x \}$ and $T = \{z \in V(D) \setminus V(C) : z \Rightarrow C\}$. Since $D$ is strong, we have that $S$ and $T$ are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Thus, we have $C \Rightarrow u \rightarrow w \Rightarrow C$, which contradicts (a).

**Theorem 9.** Let $D$ be a strong $n$-partite tournament which is not itself a tournament and contains exactly $n - m + 1$ cycles of length $m$ for some $4 \leq m \leq n - 1$. If $D$ has an $(n - 1)$-cycle $C$ with no pair of vertices from the same partite set, then $D$ contains some $D_i$ or $D_i^{-1}$ as its subdigraph for $i \in \{n-1, n\}$, where $D_i$ is defined in Section 2.

**Proof.** Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$ and let $C = v_1v_2 \cdots v_{n-1}v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n - 1$. By Theorem 1, $D(V(C))$ contains at least $n - m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$. By Theorem 6, there exists a vertex in $V_n$, say $x$, which lies on an $m$-cycle $C_{n-m+1}$ different from $C_1, C_2, \ldots, C_{n-m}$. We consider the following two cases.

Case 1. $D(V(C) \cup \{x\})$ is not strong. Since $D$ contains exactly $n - m + 1$ cycles of length $m$, we have that $D(V(C))$ contains exactly $n - m$ cycles of length
Claim 10. There exists a vertex $y \in V(D) \setminus V(C')$ such that $D(V(C')) \cup \{y\}$ is strong.
Proof. Assume that there is no vertex \( y \in V(D) \setminus V(C') \) such that \( D(V(C') \cup \{y\}) \) is strong. Let \( S = \{x \in V(D) \setminus V(C') : C' \Rightarrow x\} \) and \( T = \{z \in V(D) \setminus V(C') : z \Rightarrow V(C')\} \). Since \( D \) is strong, we have that \( S \) and \( T \) are non-empty and there are vertices \( u \in S \) and \( w \in T \) such that \( u \rightarrow w \). Suppose that \( u \in V_i \) and \( w \in V_j \) for \( 1 \leq i \neq j \leq n \). Then \( uwv_{j+1}v_{j+2} \cdots v_{j+m-2}u \) (if \( i \neq j + m - 2 \)) or \( uwv_{j+2}v_{j+3} \cdots v_{j+m-1}u \) (if \( i = j + m - 2 \)) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \). Note that \( D \) contains exactly \( n - m + 1 \) cycles of length \( m \). This is a contradiction.

By Claim 10, there are two vertices \( v_a, v_b \) (\( 1 \leq a, b \leq n \)), such that \( v_a \rightarrow y \rightarrow v_b \). Assume that \( v_k \) is the first vertex from \( v_1 \) to \( v_n \) dominating \( y \).

Claim 11. \( v_1 \Rightarrow y \) for all \( k \leq i \leq n \).

Proof. Otherwise, there exists some index \( t \) such that either \( v_t \rightarrow y \rightarrow v_{t+1} \) (\( k \leq t \leq n - 1 \)) or \( y, v_{t+1} \in V_{t+1} \) but \( v_t \rightarrow y \rightarrow v_{t+2} \) (\( k \leq t \leq n - 2 \)). We still assume that \( t \) is such a minimum index.

If \( k = t \leq n - m + 1 \), then either \( v_tv_{t+1} \cdots v_{t+m-2}v_t \) or \( v_tv_{t+2} \cdots v_{t+m-1}v_t \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( n - m + 2 \leq k \leq n - 2 \), then either \( v_tv_{t+1} \cdots v_{n-m+2}v_{t+1} \) or \( v_tv_{t+2} \cdots v_nv_{n-m+1}v_{t+1} \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( t = n - 1 \), then \( y \rightarrow v_n \) and \( v_{n-1}yv_nv{n-m+2} \cdots v_{n-1} \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

Claim 12. \( y \rightarrow v_1 \).

Proof. If \( y \in V_1 \), then \( y \rightarrow v_2 \) (otherwise, \( k = 2 \) and \( \{v_2, v_3, \ldots, v_n\} \rightarrow y \) by Claim 11, which contradicts the assumption that \( D(V(C') \cup \{y\}) \) is strong). By Claim 11, we have \( v_n \rightarrow y \). Therefore, \( D(v_2, \ldots, v_n, y) \) is a strong tournament. Then \( y \) is in an \( m \)-cycle of \( D(v_2, \ldots, v_n, y) \), which is different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

Therefore, \( y \notin V_1 \). If \( v_1 \rightarrow y \), then \( \{v_2, v_3, \ldots, v_n\} \Rightarrow y \) by Claim 11, which contradicts the assumption that \( D(V(C') \cup \{y\}) \) is strong. So we have \( y \rightarrow v_1 \). □

By Claim 12, we have that \( 2 \leq k \leq n \) and \( y \Rightarrow v_{m-1} \). Otherwise, \( yv_1v_2 \cdots v_{m-1}y \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( k = 2 \), then by \( m \geq 4 \) and Claim 11, we have \( y \in V_{m-1} \), and hence, \( \{v_2, v_3, \ldots, v_n\} \Rightarrow y \rightarrow v_1 \). Now, \( D \) contains \( D_n \) as its subdigraph.

If \( 2 < k < m - 1 \), then \( y \in V_{m-1} \), \( y \rightarrow v_2 \) and \( v_m \rightarrow y \). Thus, \( yv_2 \cdots v_my \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( m - 1 \leq k \leq n - 1 \), then \( 1 \leq k - m + 2 \leq k - 2 \) and \( y \Rightarrow v_{k-m+2} \). Now \( v_kv_{k-m+2} \cdots v_k (y \rightarrow v_{k-m+2}) \) or \( v_{k+1}yv_{k-m+3} \cdots v_{k+1} (y \in V_{k-m+2}) \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.
If \( k = n \), then \( v_n \rightarrow y \Rightarrow \{ v_1, v_2 \cdots v_{n-1} \} \) by the choice of \( k \). It is easy to see that \( y \in V_{n-m+2} \), as otherwise \( yv_{n-m+2} \cdots v_n y \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction. Now \( D \) contains \( D_n^{-1} \) as its subdigraph. □

**Theorem 13.** Let \( D \) be a strong \( n \)-partite tournament, \( n \geq 5 \), which is not itself a tournament. If \( D \) contains an \((n-1)\)-cycle with no pair of vertices from the same partite set, then \( D \) does not contain exactly \( n-m+1 \) cycles of length \( m \) for two values of \( m \in \{4, 5, \ldots, n-1\} \).

**Proof.** Let \( m \) and \( m_1 \) be two distinct values from the set \( \{4, 5, \ldots, n-1\} \) and assume that \( D \) has exactly \( n-m+1 \) cycles of length \( m \). Let \( V_1, V_2, \ldots, V_n \) be the partite sets of \( D \). By Theorem 9, \( D \) contains some \( D_i \) or \( D_i^{-1} \) as its subdigraph for \( i \in \{n-1, n\} \).

If \( D \) contains \( D_{n-1} \left(D_{n-1}^{-1}\right) \) as its subdigraph, then let \( C = v_1 v_2 \cdots v_{n-1} v_1 \) be an \((n-1)\)-cycle of \( D_{n-1} \left(D_{n-1}^{-1}\right) \) with \( v_i \in V_i \) (\( i = 1, 2, \ldots, n-1 \)), \( v_i \rightarrow v_j \) for all \( 1 < j + 1 < i \leq n-1, y \in V_{m-1}, \{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow y \rightarrow v_1 \) (\( y \in V_{n-m+1} \) and \( v_{n-1} \rightarrow y \Rightarrow \{v_1, v_2, \ldots, v_{n-1}\} \)). By Theorem 1, \( D\langle V(C) \rangle \) contains at least \((n-1)-m_1+1 = n-m_1 \) cycles of length \( m_1 \). Note that \( yv_1 v_2 \cdots v_{m_1-1} y \) \( (v_{n-1} y v_{n-(m_1-1)} v_{n-(m_1-2)} \cdots v_{n-1}) \) is another \( m_1 \)-cycle of \( D_{n-1}(D_{n-1}^{-1}) \). In addition, there exists a vertex in \( V_n \), say \( x \), which is in an \( m_1 \)-cycle of \( D \) different from the above \( m_1 \)-cycles. Thus, \( D \) contains at least \( n-m_1+2 \) cycles of length \( m_1 \).

If \( D \) contains \( D_n \left(D_n^{-1}\right) \) as its subdigraph, then let \( C = v_1 v_2 \cdots v_n v_1 \) be an \( n \)-cycle of \( D_n \left(D_n^{-1}\right) \) with \( v_i \in V_i \) (\( i = 1, 2, \ldots, n \)), \( v_i \rightarrow v_j \) for all \( 1 < j + 1 < i \leq n, y \in V_{m-1}, \{v_2, v_3, \ldots, v_n\} \Rightarrow y \rightarrow v_1 \) (\( y \in V_{n-m+2}, v_n \rightarrow y \Rightarrow \{v_1, v_2, \ldots, v_n\} \)). By Theorem 1, \( D\langle V(C) \rangle \) contains at least \( n-m_1+1 \) cycles of length \( m_1 \). It is easy to see that \( yv_1 v_2 \cdots v_{m_1-1} y \) \( (v_n y v_{n-(m_1-2)} v_{n-(m_1-3)} \cdots v_n) \) is another \( m_1 \)-cycle of \( D_n \left(D_n^{-1}\right) \). Then \( D \) contains at least \( n-m_1+2 \) cycles of length \( m_1 \). The theorem is complete. □

In 2004, Winzen [11] showed that an \( n \)-partite tournament \( D \) with \( n \geq 4 \) and \( i_g(D) \leq 2 \) contains a strong subtournament of order \( p \) for every \( p \in \{3, 4, \ldots, n-1\} \). So \( D \) contains an \((n-1)\)-cycle with no pair of vertices from the same partite set, which yields the following result.

**Corollary 14.** If \( D \) is a strong \( n \)-partite tournament with \( n \geq 5 \) and \( i_g(D) \leq 2 \), which is not itself a tournament, then \( D \) does not contain exactly \( n-m+1 \) cycles of length \( m \) for two values of \( m \in \{4, 5, \ldots, n-1\} \).

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