ON THE \( n \)-PARTITE TOURNAMENTS WITH EXACTLY \( n - m + 1 \) CYCLES OF LENGTH \( m \)

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Abstract

Gutin and Rafiey [Multipartite tournaments with small number of cycles, Australas J. Combin. 34 (2006) 17–21] raised the following two problems: (1) Let \( m \in \{3, 4, \ldots, n\} \). Find a characterization of strong \( n \)-partite tournaments having exactly \( n - m + 1 \) cycles of length \( m \); (2) Let \( 3 \leq m \leq n \) and \( n \geq 4 \). Are there strong \( n \)-partite tournaments, which are not themselves tournaments, with exactly \( n - m + 1 \) cycles of length \( m \) for two values of \( m \)? In this paper, we discuss the strong \( n \)-partite tournaments \( D \) containing exactly \( n - m + 1 \) cycles of length \( m \) for \( 4 \leq m \leq n - 1 \). We describe the substructure of such \( D \) satisfying a given condition and we also show that, under this condition, the second problem has a negative answer.

Keywords: multipartite tournaments, tournaments, cycles.

2010 Mathematics Subject Classification: 05C20, 05C38.

1. Introduction

An \( n \)-partite or multipartite tournament is an orientation of a complete \( n \)-partite graph. A tournament is an \( n \)-partite tournament with exactly \( n \) vertices. A digraph \( D \) is transitive if, for every pair of arcs \( xy \) and \( yz \) in \( D \) such that \( x \neq z \), the arc \( xz \) is also in \( D \). It is easy to show that a tournament is transitive if and only if it is acyclic.

A digraph \( D \) is said to be strong, if for every pair of vertices \( x \) and \( y \), \( D \) contains a path from \( x \) to \( y \) and a path from \( y \) to \( x \). A directed path from \( x \) to \( y \) in \( D \) is denoted by an \((x, y)\)-path. An \( l \)-cycle is a cycle of length \( l \). A cycle or path in a digraph \( D \) is Hamiltonian if it includes all the vertices of \( D \).

In 1966, Moon discussed the number of \( m \)-cycle in a strong tournament.
Theorem 1 (Moon [9]). Let $T$ be a strong tournament of order $n$. Then $T$ contains at least $n - m + 1$ cycles of length $m$ for $3 \leq m \leq n$.

The tournaments which gain the lower bound in Theorem 1 were characterized by Burzio and Demaria [2] for $m = 3$, Douglas [3] for $m = n$ and Las Vergnas [8] for $4 \leq m \leq n - 1$. We list the result of Las Vergnas especially because we will use it to prove our main results.

Theorem 2 (Las Vergnas [8]). Every strong tournament of order $n$ having exactly $n - m + 1$ cycles of given length $m$ with $4 \leq m \leq n - 1$ is isomorphic to $Q_n$, where $Q_n$ is a tournament of order $n \geq 3$ obtained by reversing the arcs in the unique Hamiltonian path of a transitive tournament.

In 2002, Volkmann extended Theorem 1 from tournaments to multipartite tournaments.

Theorem 3 (Volkmann [10]). Let $D$ be a strong $n$-partite tournament. Then $D$ contains at least $n - m + 1$ cycles of length $m$ for $3 \leq m \leq n$.

It is notable that the bound in Theorem 3 is sharp which can be seen in [4] for $n = 3$ and in [6] for $4 \leq m \leq n$. In addition, Gutin and Rafiey [6] raised the following two interesting and natural problems.

Problem 4 (Gutin and Rafiey [6]). Given $m \in \{3, 4, \ldots, n\}$, find a characterization of strong $n$-partite tournaments having exactly $n - m + 1$ cycles of length $m$.

Problem 5 (Gutin and Rafiey [6]). Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong $n$-partite tournaments, which are not themselves tournaments, with exactly $n - m + 1$ cycles of length $m$ for two values of $m$?

Problem 4 seems to be especially interesting for the case $m = n$ which was already solved by Gutin et al. in [7]. In this paper, we investigate strong $n$-partite tournaments $D$, which are not themselves tournaments and contain exactly $n - m + 1$ cycles of length $m$ for any given $4 \leq m \leq n - 1$. We prove that if $D$ has an $(n - 1)$-cycle with no pair of vertices from the same partite set, then $D$ must contain some given multipartite tournament as its subdigraph.

As far as Problem 5, Gutin and Rafiey [6] gave a negative answer for two values $n - 1$ and $n$ of $m$. This also implies that Problem 5 has a negative answer for $n = 4$. In this paper, we give a necessary condition to Problem 5 and show that if a strong $n$-partite tournament $D$, which is not itself a tournament, contains exactly $n - m + 1$ cycles of length $m$ for two values of $m \in \{4, 5, \ldots, n - 1\}$, then there is no an $(n - 1)$-cycle with no pair of vertices from the same partite set in $D$. 


2. Terminology and Preliminaries

We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

Let \( D \) be a digraph with the vertex set \( V(D) \) and the arc set \( A(D) \). We call the number of vertices of \( D \) the order of \( D \). A subdigraph induced by a subset \( A \subseteq V(D) \) is denoted by \( D(A) \). We use \( V(D) \setminus V(A) \) to stand for the set of vertices which are in \( V(D) \) but not in \( V(A) \).

If \( xy \) is an arc in \( D \), then we say that \( x \) dominates \( y \) and write \( x \rightarrow y \). For two disjoint subsets \( X \) and \( Y \) of \( V(D) \), if every vertex of \( X \) dominates every vertex of \( Y \), we say \( X \) dominates \( Y \) and write \( X \rightarrow Y \). Furthermore, \( X \Rightarrow Y \) denotes the property that there is no arc from \( Y \) to \( X \).

The out-neighborhood \( N^+(x) \) of a vertex \( x \) is the set of vertices dominated by \( x \) and the in-neighborhood \( N^-(x) \) of a vertex \( x \) is the set of vertices dominating \( x \). The numbers \( d^+(x) = |N^+(x)| \) and \( d^-(x) = |N^-(x)| \) are the outdegree and indegree of \( x \), respectively. The global irregularity of \( D \) is defined as \( ig(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\} \). We denote by \( D^{-1} \) the inverse digraph of \( D \).

In order to present our main results, we define a class of \( n \)-partite tournaments \( D_n \) of order \( n + 1 \) as described in the following figure, where \( 3 \leq m \leq n \), \( \{v_2, \ldots, v_{m-2}, v_m, \ldots, v_n\} \rightarrow y \rightarrow v_1 \), and \( v_{m-1} \) belong to the same partite set and \( v_i \rightarrow v_j \) for all \( 1 < j + 1 < i \leq n \).

The following two theorems on cycles in strong \( n \)-partite tournaments are very useful to prove our main results.

**Theorem 6** (Guo and Volkmann [5]). Every partite set of a strong \( n \)-partite tournament, \( n \geq 3 \), contains a vertex which lies on an \( m \)-cycle for each \( m \in \{3, 4, \ldots, n\} \).

**Theorem 7** (Gutin and Rafiey [6]). Let \( D \) be a strong \( n \)-partite tournament containing exactly \( n - m + 1 \) cycles of length \( m \) for some \( m \in \{3, 4, \ldots, n\} \). Then every \( m \)-cycle of \( D \) has no pair of vertices from the same partite set.
3. Main Results

Before presenting the main results, we first prove the following lemma.

Lemma 8. Let $D$ be a strong $n$-partite tournament, $n \geq 5$, containing exactly $n - m + 1$ cycles of length $m$ for some $3 \leq m \leq n - 1$. If $D$ has an $(n - 1)$-cycle $C$ with no pair of vertices from the same partite set, then the following statements hold.

(a) There are no two vertices $u, w$ in $V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$.
(b) There exists a vertex $v \notin V(C)$ such that $D(V(C) \cup \{v\})$ is strong.

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$. Suppose, without loss of generality, that $C = v_1v_2 \cdots v_{n-1}v_1$ with $v_i \in V_i$, $i = 1, 2, \ldots, n - 1$. By Theorem 1, $D(V(C))$ contains at least $(n - 1) - m + 1 = n - m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$.

(a) Suppose to the contrary that there exist two vertices $u, w \in V(D) \setminus V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$. Obviously, $u \notin V_n$ or $w \notin V_n$. Assume, without loss of generality, that $w \in V_j$ for some $1 \leq j \leq n - 1$. Since $n \geq 5$, there exist at least two vertices $v_k$ and $v_t$, such that $u, w, v_k$ and $v_t$ are in different partite sets. If $m = 3$, then $uwv_ku$ and $uwv_tu$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$. This contradicts the fact that $D$ contains exactly $n - m + 1$ cycles of length $m$. If $m \geq 4$, then $uwv_{j-1}v_j \cdots v_{j+m-4}u$ (if $u \notin V_{j+m-4}$) or $uwv_{j+1}v_{j+2} \cdots v_{j+m-2}u$ (if $u \in V_{j+m-4}$) is an $m$-cycle with $w, v_j \in V_j$ or $u, v_{j+m-4} \in V_{j+m-4}$ (where all indices are modulo $n - 1$). This is impossible by Theorem 7.

(b) Assume that there is no vertex $v \in V(D) \setminus V(C)$ such that $D(V(C) \cup \{v\})$ is strong. Let $S = \{x \in V(D) \setminus V(C) : C \Rightarrow x\}$ and $T = \{z \in V(D) \setminus V(C) : z \Rightarrow C\}$. Since $D$ is strong, we have that $S$ and $T$ are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Thus, we have $C \Rightarrow u \rightarrow w \Rightarrow C$, which contradicts (a).

Theorem 9. Let $D$ be a strong $n$-partite tournament which is not itself a tournament and contains exactly $n - m + 1$ cycles of length $m$ for some $4 \leq m \leq n - 1$. If $D$ has an $(n - 1)$-cycle $C$ with no pair of vertices from the same partite set, then $D$ contains some $D_i$ or $D_i^{-1}$ as its subdigraph for $i \in \{n - 1, n\}$, where $D_i$ is defined in Section 2.

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$ and let $C = v_1v_2 \cdots v_{n-1}v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n - 1$. By Theorem 1, $D(V(C))$ contains at least $n - m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$. By Theorem 6, there exists a vertex in $V_n$, say $x$, which lies on an $m$-cycle $C_{n-m+1}$ different from $C_1, C_2, \ldots, C_{n-m}$. We consider the following two cases.

Case 1. $D(V(C) \cup \{x\})$ is not strong. Since $D$ contains exactly $n - m + 1$ cycles of length $m$, we have that $D(V(C))$ contains exactly $n - m$ cycles of length
m. By Theorem 2, $D(V(C))$ is isomorphic to $Q_{n-1}$. So we may assume that $v_i \to v_j$ for all $1 < j + 1 < i \leq n - 1$. Since $D(V(C) \cup \{x\})$ is not strong, we have that $C \to x$ or $x \to C$.

First we consider the case $C \to x$. Let $S = \{u \in V(D) \setminus (V(C) \cup \{x\}) : D(V(C) \cup \{u\})$ is strong$\}$. By Lemma 8(b), $S$ is not empty. Since $D$ is strong, there is a path from $x$ to $S$. Let $P = x_1 x_2 \cdots x_t$ ($x_1 = x$) be such a path and assume that the $P$ is of minimum length. That is, $x_t \in S$ and $D(V(C) \cup \{x_i\})$ is not strong for each $i \in \{1, 2, \ldots, t - 1\}$. Since $C \to x_1$ and $x_1 \to x_2$, we have $x_2 \notin V(C)$. If $t > 2$, then by Lemma 8(a), we have $C \to x_2$. Successively, we can get that $x_i \notin V(C)$ and $C \Rightarrow x_i$ for all $i \in \{2, 3, \ldots, t - 1\}$ when $t > 2$.

If there exist two vertices $v_i, v_j$ on $C$ such that $x_t \to \{v_i, v_j\}$, then, when $t > m - 1$, we have that $x_t v_i x_t-(m-2)x_t-(m-3) \cdots x_t$ (if $x_t-(m-2) \notin V_i$) or $x_t v_j x_t-(m-2)x_t-(m-3) \cdots x_t$ (if $x_t-(m-2) \notin V_i$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{m-1}$, a contradiction; when $t = m - 1$, it is clear that $x_t v_i x_t \cdots x_t$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction; when $t \leq m - 2$, it is easy to see that $x_t v_i v_{i+1} \cdots v_{i+(m-1-t)} x_1 \cdots x_t$ and $x_t v_j v_{j+1} \cdots v_{j+(m-1-t)} x_1 \cdots x_t$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction.

Therefore, $x_t$ has only one out-neighbor on $C$. We will show that $x_t \to v_i$. In fact, if $x_t \to v_i$ and $i \geq 2$, then we have that $x_t v_i \cdots v_{i+m-2} x_t$ (when $i + m - 2 \leq n - 1$ and $x_t \notin V_{i+m-2}$) or $x_t v_i \cdots v_{i+m-3} v_{i+1} x_t$ (when $i + m - 2 \leq n - 1$ and $x_t \notin V_{i+m-2}$) or $x_t v_i \cdots v_{n-1} v_1 \cdots v_{n-i+1} x_t$ (when $i + m - 2 \geq n$ and $x_t \notin V_{n-n+i-1}$) or $x_t v_i \cdots v_{n-1} v_2 \cdots v_{n-i+1} x_t$ (when $i + m - 2 \geq n$ and $x_t \notin V_{n-n+i-1}$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So we have $\{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow x_t$. Furthermore, if $v_{n-1} \to x_t$, then $x_t v_1 v_2 \cdots v_{n-1} x_t$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So $x_t \in V_{n-1}$. Let $x_t = y$. Then $D$ contains $D_{n-1}$ as its subdigraph.

For the case $x \to C$, by considering the inverse of $D$, it is easy to see that $D$ contains $D_{n-1}$ as its subdigraph.

Case 2. $D(V(C) \cup \{x\})$ is strong. In this case, $D(V(C) \cup \{x\})$ is a strong tournament of order $n$. By Theorem 1, $D(V(C) \cup \{x\})$ contains at least $n - m + 1$ cycles of length $m$. Note that $D$ contains exactly $n - m + 1$ cycles of length $m$. We have that $D(V(C) \cup \{x\})$ contains exactly $n - m + 1$ cycles of length $m$. By Theorem 2, $D(V(C) \cup \{x\})$ is isomorphic to $Q_n$. So we may assume that $C' = v_1 v_2 \cdots v_n v_1, C_1 = v_2 v_3 \cdots v_{m+1} v_2, \ldots, C_{n-m+1} = v_{n-m+1} v_{n-m+2} \cdots v_n v_{n-m+1}$ are $n - m + 1$ cycles of length $m$ of $D$.

Claim 10. There exists a vertex $y \in V(D) \setminus V(C')$ such that $D(V(C') \cup \{y\})$ is strong.
Proof. Assume that there is no vertex \( y \in V(D) \backslash V(C') \) such that \( D(V(C') \cup \{y\}) \) is strong. Let \( S = \{x \in V(D) \backslash V(C') : x \Rightarrow y\} \) and \( T = \{z \in V(D) \backslash V(C') : z \Rightarrow V(C')\} \). Since \( D \) is strong, we have that \( S \) and \( T \) are non-empty and there are vertices \( u \in S \) and \( w \in T \) such that \( u \rightarrow w \). Suppose that \( u \in V_i \) and \( w \in V_j \) for \( 1 \leq i \neq j \leq n \). Then \( uwv_{j+1}v_{j+2} \cdots v_{j+m-2}u \) (if \( i \neq j + m - 2 \)) or \( uv_{j+2}v_{j+3} \cdots v_{j+m-1}u \) (if \( i = j + m - 2 \)) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \). Note that \( D \) contains exactly \( n-m+1 \) cycles of length \( m \). This is a contradiction.

By Claim 10, there are two vertices \( v_a, v_b \) (\( 1 \leq a, b \leq n \)), such that \( v_a \rightarrow y \rightarrow v_b \). Assume that \( yv \) is the first vertex from \( v_1 \) to \( v_n \) dominating \( y \).

Claim 11. \( v_i \Rightarrow y \) for all \( k \leq i \leq n \).

Proof. Otherwise, there exists some index \( t \) such that either \( v_t \rightarrow y \rightarrow v_{t+1} \) (\( k \leq t \leq n-1 \)) or \( v_{t+1} \in V_{t+1} \), but \( v_t \rightarrow y \rightarrow v_{t+2} \) (\( k \leq t \leq n-2 \)). We still assume that \( t \) is such a minimum index.

If \( t \leq n-m+1 \), then either \( v_tv_{t+1}v_{t+2} \cdots v_{t+m-2}v_t \) or \( v_tv_{t+2}v_{t+3} \cdots v_{t+m-1}v_t \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( n-m+2 \leq t \leq n-2 \), then either \( v_tv_{t+1}v_{t+2} \cdots v_{n-m-2}v_t \) or \( v_tv_{t+2} \cdots v_{n-m+1}v_t \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( t = n-1 \), then \( y \rightarrow v_n \) and \( v_{n-1}v_nv_{n-m+2} \cdots v_{n-1} \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction. \( \square \)

Claim 12. \( y \rightarrow v_1 \).

Proof. If \( y \in V_1 \), then \( y \rightarrow v_2 \) (otherwise, \( k = 2 \) and \( \{v_2, v_3, \ldots, v_n\} \rightarrow y \) by Claim 11, which contradicts the assumption that \( D(V(C') \cup \{y\}) \) is strong). By Claim 11, we have \( v_n \rightarrow y \). Therefore, \( D(v_2, \ldots, v_n, y) \) is a strong tournament. Then \( y \) is in an \( m \)-cycle of \( D(v_2, \ldots, v_n, y) \), which is different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

Therefore, \( y \notin V_1 \). If \( v_1 \rightarrow y \), then \( \{v_2, v_3, \ldots, v_n\} \rightarrow y \) by Claim 11, which contradicts the assumption that \( D(V(C') \cup \{y\}) \) is strong. So we have \( y \rightarrow v_1 \). \( \square \)

By Claim 12, we have that \( 2 \leq k \leq n \) and \( y \Rightarrow v_{m-1} \). Otherwise, \( yv_1v_2 \cdots v_{m-1}y \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( k = 2 \), then by \( m \geq 4 \) and Claim 11, we have \( y \in V_{m-1} \), and hence, \( \{v_2, v_3, \ldots, v_n\} \Rightarrow y \rightarrow v_1 \). Now, \( D \) contains \( D_n \) as its subdigraph.

If \( 2 < k < m-1 \), then \( y \in V_{m-1} \), \( y \rightarrow v_2 \) and \( v_m \rightarrow y \). Thus, \( yv_2 \cdots v_{m-1}y \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.

If \( m-1 \leq k \leq n-1 \), then \( 1 \leq k - m + 2 \leq k-2 \) and \( y \Rightarrow v_{k-m+2} \). Now \( v_kv_{k-m+2} \cdots v_k \) (if \( y \rightarrow v_{k-m+2} \)) or \( v_{k+1}v_{k-m+2} \cdots v_{k+1} \) (if \( y \in V_{k-m+2} \)) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction.
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If \( k = n \), then \( v_n \rightarrow y \Rightarrow \{ v_1, v_2, \ldots, v_{n-1} \} \) by the choice of \( k \). It is easy to see that \( y \in V_{n-m+2} \), as otherwise \( yv_{n-m+2} \cdots v_n y \) is an \( m \)-cycle different from \( C_1, C_2, \ldots, C_{n-m+1} \), a contradiction. Now \( D \) contains \( D_n^{-1} \) as its subdigraph.

**Theorem 13.** Let \( D \) be a strong \( n \)-partite tournament, \( n \geq 5 \), which is not itself a tournament. If \( D \) contains an \((n-1)\)-cycle with no pair of vertices from the same partite set, then \( D \) does not contain exactly \( n-m+1 \) cycles of length \( m \) for two values of \( m \in \{4, 5, \ldots, n-1\} \).

**Proof.** Let \( m \) and \( m_1 \) be two distinct values from the set \( \{4, 5, \ldots, n-1\} \) and assume that \( D \) has exactly \( n-m+1 \) cycles of length \( m \). Let \( V_1, V_2, \ldots, V_n \) be the partite sets of \( D \). By Theorem 9, \( D \) contains some \( D_i \) or \( D^{-1}_i \) as its subdigraph for \( i \in \{n-1, n\} \).

If \( D \) contains \( D_{n-1} (D_n^{-1}) \) as its subdigraph, then let \( C = v_1 v_2 \cdots v_{n-1} v_1 \) be an \((n-1)\)-cycle of \( D_{n-1} (D_n^{-1}) \) with \( v_i \in V_i (i = 1, 2, \ldots, n-1) \), \( v_i \rightarrow v_j \) for all \( 1 < j + 1 < i \leq n-1 \), \( y \in V_{n-1} \), \( \{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow y \rightarrow v_1 \) (\( y \in V_{n-m+1} \) and \( v_{n-1} \rightarrow y \Rightarrow \{v_1, v_2, \ldots, v_{n-2}\} \)). By Theorem 1, \( D(V(C)) \) contains at least \((n-1)-m_1+1 = n-m_1 \) cycles of length \( m_1 \). Note that \( yv_1 v_2 \cdots v_{m_1-1} y \) \( (v_{n-1} y v_{n-(m_1-1)} v_{n-(m_1-2)} \cdots v_{n-1}) \) is another \( m_1 \)-cycle of \( D_{n-1}(D_n^{-1}) \). In addition, there exists a vertex in \( V_n \), say \( x \), which is in an \( m_1 \)-cycle of \( D \) different from the above \( m_1 \)-cycles. Thus, \( D \) contains at least \( n-m_1+2 \) cycles of length \( m_1 \).

If \( D \) contains \( D_n (D_n^{-1}) \) as its subdigraph, then let \( C = v_1 v_2 \cdots v_n v_1 \) be an \( n \)-cycle of \( D_n (D_n^{-1}) \) with \( v_i \in V_i (i = 1, 2, \ldots, n) \), \( v_i \rightarrow v_j \) for all \( 1 < j + 1 < i \leq n \), \( y \in V_{n-1} \), \( \{v_2, v_3, \ldots, v_n\} \Rightarrow y \rightarrow v_1 \) (\( y \in V_{n-m+2} \), \( v_n \rightarrow y \Rightarrow \{v_1, v_2, \ldots, v_{n-1}\} \)). By Theorem 1, \( D(V(C)) \) contains at least \( n-m_1+1 \) cycles of length \( m_1 \). It is easy to see that \( yv_1 v_2 \cdots v_{m_1-1} y \) \( (v_{n} y v_{n-(m_1-2)} v_{n-(m_1-3)} \cdots v_{n}) \) is another \( m_1 \)-cycle of \( D_n (D_n^{-1}) \). Then \( D \) contains at least \( n-m_1+2 \) cycles of length \( m_1 \). The theorem is complete.

In 2004, Winzen [11] showed that an \( n \)-partite tournament \( D \) with \( n \geq 4 \) and \( i_g(D) \leq 2 \) contains a strong subtournament of order \( p \) for every \( p \in \{3, 4, \ldots, n-1\} \). So \( D \) contains an \((n-1)\)-cycle with no pair of vertices from the same partite set, which yields the following result.

**Corollary 14.** If \( D \) is a strong \( n \)-partite tournament with \( n \geq 5 \) and \( i_g(D) \leq 2 \), which is not itself a tournament, then \( D \) does not contain exactly \( n-m+1 \) cycles of length \( m \) for two values of \( m \in \{4, 5, \ldots, n-1\} \).

**Acknowledgments**

We would like to express our sincere thanks to the anonymous referees for their useful suggestions and detailed comments which lead to numerous improvements.
of this article. This work is supported by the Natural Science Young Foundation of China (No. 11701349) and by the Natural Science Foundation of Shanxi Province, China (No. 201601D011005) and by Shanxi Scholarship Council of China (2017–018).

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Received 18 December 2017
Revised 19 June 2018
Accepted 9 August 2018