LOW 5-STARS AT 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE FROM 7 TO 9

OLEG V. BORODIN, MIKHAIL A. BYKOV

AND

ANNA O. IVANOVA

Sobolev Institute of Mathematics
Novosibirsk, 630090, Russia

E-mail: brdnoleg@math.nsc.ru
131093@mail.ru
shmganna@mail.ru

Abstract

In 1940, Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5.

Given a 3-polytope $P$, by $h_5(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in $P$.

Recently, Borodin, Ivanova and Jensen showed that if a polytope $P$ in $P_5$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-vertex, then $h_5(P)$ can be arbitrarily large. Therefore, we consider the subclass $P_5^*$ of 3-polytopes in $P_5$ that avoid $(5, 5, 6, 6, \infty)$-vertices.

For each $P^*$ in $P_5^*$ without vertices of degree from 7 to 9, it follows from Lebesgue’s Theorem that $h_5(P^*) \leq 17$. Recently, this bound was lowered by Borodin, Ivanova, and Kazak to the sharp bound $h_5(P^*) \leq 15$ assuming the absence of vertices of degree from 7 to 11 in $P^*$.

In this note, we extend the bound $h_5(P^*) \leq 15$ to all $P^*$’s without vertices of degree from 7 to 9.

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1. Introduction

The degree of a vertex or face $x$ in a convex finite 3-dimensional polytope (called a 3-polytope) is denoted by $d(x)$. As proved by Steinitz [14], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs. A $k$-vertex is a vertex $v$ with $d(v) = k$. A $k^+$-vertex ($k^-$-vertex) is one of degree at least $k$ (at most $k$). Similar notation is used for the faces. The set of 3-polytopes with minimum degree 5 is denoted by $P_5$, and its elements are $P_5$s. We will drop the argument whenever it is clear from context.

The height of a subgraph $S$ of a 3-polytope is the maximum degree of the vertices of $S$ in the 3-polytope. A $k$-star, a star with $k$ rays, is minor if its center $v$ has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $h_k(P_5)$ we denote the minimum height of minor $k$-stars in a given 3-polytope $P_5$.

In 1904, Wernicke [15] proved that every $P_5$ has a 5-vertex adjacent to a 6-vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 6-neighbors. So $h_1 \leq h_2 \leq 6$ in $P_5$, where both bounds are sharp.

In 1940, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in $P_5$s.

In particular, this description implies the results in [11, 15] and shows that there is a 5-vertex with three 7-neighbors. Thus $h_3 \leq 7$, which is sharp due to Borodin [1]. Jendrol’ and Madaras [12] gave a precise description of minor 3-stars in $P_5$s.

Lebesgue [13] also proved $h_4(P_5) \leq 11$, which was strengthened by Borodin and Woodall [10] to the tight bound $h_4(P_5) \leq 10$. Recently, Borodin and Ivanova [2] obtained a precise description of 4-stars in $P_5$s.

The more general problem of describing 5-stars at 5-vertices in $P_5$ remains widely open.

Recently, precise upper bounds have been obtained for the minimum height $h_5(P_5)$ of minor 5-stars in several natural subclasses of $P_5$.

Note that Borodin, Ivanova and Jensen [5] showed that if a polytope $P_5$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-vertex, then $h_5(P_5)$ can be arbitrarily large. (In fact, every 5-vertex in the construction in [5] has two 5-neighbors and two 6-neighbors.) Therefore, from now on we restrict ourselves to the subclass $P_5^*$ of the 3-polytopes in $P_5$ avoiding $(5, 5, 6, 6, \infty)$-vertices.

For each $P_5^*$ in $P_5^*$, it follows from Lebesgue’s Theorem that $h_5(P_5^*) \leq 41$. This bound was lowered to $h_5(P_5^*) \leq 28$ by Borodin, Ivanova, and Jensen [5] and then to $h_5(P_5^*) \leq 23$ in Borodin-Ivanova [4]. On the other hand, it was shown in [5] that the upper bound for $h_5(P_5^*)$ cannot go down below 20. We conjecture...
that \( h_5(P_5^*) \leq 20 \) whenever \( P_5^* \in P_5^* \).

Back in 1996, Jendrol’ and Madaras [12] showed that if a polytope \( P_{5^*} \) has a 5-vertex adjacent to four 5-vertices, then \( h_5(P_{5^*}) \) can be arbitrarily large. Therefore, considering subclasses of \( P_5^* \) without vertices of degree from 6 to a certain \( k_6 \) with \( k_6 > 6 \), we should deal only with 3-polytopes \( P_{5^*} \)'s having no 5-vertices with four 5-neighbors.

For every \( P_{5^*} \) in \( P_5^* \) with \( k_6 = 9 \), Lebesgue’s bound \( h_5(P_{5^*}) \leq 14 \) was improved by Borodin and Ivanova [3] to the sharp bound \( h_5(P_{5^*}) \leq 12 \). Later on, Borodin, Ivanova and Nikiforov [9] proved the same bound assuming the absence only of vertices of degree from 6 to 8, improving Lebesgue’s bound \( h_5(P_{5^*}) \leq 17 \).

Another natural direction of research towards a tight version of Lebesgue’s Theorem is considering subclasses of \( P_5^* \) with no vertices of degree from 7 to a certain integer \( k_7 \) with \( k_7 > 7 \).

For \( k_7 = 11 \), Lebesgue’s bound \( h_5(P^*) \leq 17 \) was lowered by Borodin, Ivanova, and Kazak [6] to the sharp bound \( h_5(P^*) \leq 15 \). The purpose of this note is to extend this bound to all \( P^* \)’s such that \( k_7 = 9 \).

**Theorem 1.** Every 3-polytope \( P^* \) with minimum degree 5 and neither \((5, 5, 6, 6, \infty)\)-vertices nor vertices of degree from 7 to 9 satisfies \( h_5(P^*) \leq 15 \), which bound is best possible.

**Problem 2.** Is it true that every 3-polytope \( P^* \) with minimum degree 5 and no \((5, 5, 6, 6, \infty)\)-vertices satisfies \( h_5(P^*) \leq 15 \) provided that

(a) \( P^* \) has no vertices of degree 7 and 8?
(b) only 7-vertices are forbidden in \( P^* \)?

2. **Proof of Theorem 1**

The sharpness of the bound 15 in Theorem 1 follows from a construction in [6].

Now suppose a 3-polytope \( P_5^* \) is a counterexample to the main statement of Theorem 1. In particular, each minor 5-star in \( P_5^* \) contains a 16\(^{+}\)-vertex along with either another 10\(^{+}\)-vertex or at least three 6-vertices.

Let \( P_5 \) be a counterexample on the same vertices as \( P_5^* \) with the maximum possible number of edges. For brevity, a vertex \( v \) with \( d(v) \neq 6 \) is a non-6-vertex.
Remark 3. \( P_5 \) has no two non-6-vertices being nonconsecutive along the boundary of a 4\(^+\)-face. Indeed, otherwise adding a diagonal between these vertices would result in a counterexample with greater edges than \( P_5 \).

Corollary 4. In \( P_5 \), each 4\(^+\)-face has at most two non-6-vertices, and if it has two such vertices, then they are adjacent to each other.

Discharging.

Let \( V \), \( E \), and \( F \) be the sets of vertices, edges, and faces of \( P_5 \). Euler’s formula \(|V| - |E| + |F| = 2\) for \( P_5 \) implies

\[
\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12.
\]

We assign an initial charge \( \mu(v) = d(v) - 6 \) to each \( v \in V \) and \( \mu(f) = 2d(f) - 6 \) to each \( f \in F \), so that only 5-vertices have negative initial charge. Using the properties of \( P_5 \) as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge \( \mu(x) \) is non-negative for all \( x \in V \cup F \). This will contradict the fact that the sum of the final charges is, by (1), equal to \(-12\).

The final charge \( \mu'(x) \) whenever \( x \in V \cup F \) is defined by applying the rules R1–R9 below (see Figure 1).

For a vertex \( v \), let \( v_1, \ldots, v_{d(v)} \) be the vertices adjacent to \( v \) in a fixed cyclic order. If \( f \) is a face, then \( v_1, \ldots, v_{d(f)} \) are the vertices incident with \( f \) in the same cyclic order.

A vertex is simplicial if it is completely surrounded by 3-faces.

R1. Every 4\(^+\)-face gives 1 to every incident non-6-vertex.

R2. Suppose \( f = uvw \) is a 3-face with \( d(u) = 5 \) and \( d(v) \geq 10 \).

(a) If \( d(w) \geq 6 \), then \( u \) receives from \( v \) either \( \frac{2}{3} \) if \( d(v) \leq 15 \) or \( \frac{2}{3} \) otherwise.

(b) If \( d(w) = 5 \), then \( u \) (as well as \( w \)) receives from \( v \) either \( \frac{1}{3} \) if \( d(v) \leq 15 \) or \( \frac{1}{3} \) otherwise.

R3. A non-simplicial 5-vertex \( v \) such that there are 3-faces \( v_1v_2v \) and \( v_2v_3v \) with \( d(v_2) \geq 16 \) gives \( \frac{2}{3} \) to \( v_2 \).

R4. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \) and \( d(v_1) \geq 10 \) gives \( \frac{1}{3} \) to \( v_2 \).

R5. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \) and \( d(v_1) = d(v_3) = 6 \) gives \( \frac{1}{3} \) to \( v_2 \).

R6. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = 6 \), \( d(v_3) = 5 \), and \( d(v_4) \geq 10 \) gives \( \frac{2}{3} \) to \( v_2 \).

R7. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = 6 \), \( d(v_3) = d(v_4) = 5 \) (hence \( d(v_5) \geq 10 \)) gives \( \frac{1}{2} \) to \( v_2 \).
Remark 5. Note that a simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_4) = 6 \), and \( d(v_3) = 5 \) gives nothing to \( v_2 \).

R8. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_3) = d(v_4) = 5 \), and \( d(v_5) \geq 10 \) gives \( \frac{1}{15} \) to \( v_2 \).

R9. A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_3) = 5 \), \( d(v_4) \geq 6 \), and \( d(v_5) \geq 10 \) gives \( \frac{4}{15} \) to \( v_2 \).

Checking \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \).

First consider a face \( f \) in \( P_5 \). If \( d(f) = 3 \), then \( f \) does not participate in discharging, and so \( \mu'(v) = \mu(f) = 2 \times 3 - 6 = 0 \). Note that every \( 4^+ \)-face is incident with at most two non-6-vertices due to Corollary 4, which implies that \( \mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0 \) by R1.

Now suppose \( v \in V \).
Case 1. \( d(v) \geq 18 \). Since \( v \) sends at most \( \frac{2}{3} \) to its 5-neighbors through each 3-face by R2, we have \( \mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{(d(v) - 18)}{3} \geq 0 \).

Case 2. \( 16 \leq d(v) \leq 17 \). If \( v \) is not simplicial, then it sends at most \( \frac{2}{3} \) through each of at most \( d(v) - 1 \) faces, so \( \mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{(d(v) - 16)}{3} \geq 0 \), as desired. From now on, suppose \( v \) is simplicial.

If \( v \) has two consecutive \( 6^+ \)-neighbors, then again \( \mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} \geq 0 \). So we can assume from now on that each 3-face incident with \( v \) is incident with a 5-vertex.

If \( v \) has at least one non-simplicial 5-neighbor \( v_2 \), then \( v \) receives \( \frac{2}{3} \) from \( v_2 \) by R3, which implies \( \mu'(v) \geq d(v) - 6 + \frac{2}{3} - d(v) \times \frac{2}{3} = \frac{(d(v) - 16)}{3} \geq 0 \). Thus suppose all 5-vertices adjacent to \( v \) are simplicial.

If \( v \) has a \( 10^+ \)-neighbor \( v_2 \), then \( v \) receives \( \frac{1}{3} + \frac{1}{3} \) from the 5-vertices \( v_1 \) and \( v_3 \) by R4, which again implies \( \mu'(v) \geq 0 \).

Summarizing, from now on our \( v \) is simplicial, has no \( 10^+ \)-neighbors, no two consecutive 6-neighbors, and no non-simplicial 5-neighbors.

Suppose \( S_k = v_0, \ldots, v_k \) is a sequence of neighbors of \( v \) with \( d(v_0) = 6 \), \( d(v_k) = 6 \), while \( d(v_i) = 5 \) whenever \( 1 \leq i \leq k - 1 \) and \( k \geq 2 \). (It is not excluded that \( S_k = S_{d(v)} \), which happens when \( v \) has precisely one 6-neighbor.) Let \( w_i \), \( 1 \leq i \leq k - 1 \), \( k \geq 2 \), be the common neighbor of \( v_{i-1} \) and \( v_i \) different from \( v \).

Since \( \mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{(d(v) - 16)}{3} \), we can say that \( v \) has the deficiency equal to \( \frac{1}{3} \) if \( d(v) = 17 \) or \( \frac{2}{3} \) if \( d(v) = 16 \).

Our next goal is to estimate the total return to \( v \) from its 5-neighbors by R4–R9 and show that it is not less than the deficiency of \( v \).

Remark 6. As we remember, our \( v \) has no \( S_1 \). Note that \( v_1 \) in \( S_2 \) returns \( \frac{1}{3} \) to \( v \) by R5. As for \( S_3 \), it can happen that neither \( v_1 \) nor \( v_2 \) returns anything to \( v \), which is the case only when \( v_1 \) and \( v_2 \) have a common 6-neighbor (see Remark 5).

Lemma 7. The total return from (the three 5-vertices of) an \( S_4 \) is at least \( \frac{2}{3} \).

Proof. If \( d(w_2) \geq 10 \) or \( d(w_2) = 5 \), then \( v \) receives at least \( \frac{2}{3} \) from its 5-neighbor \( v_1 \) by R6 or R7, respectively. The same is true for \( v_3 \). So, if \( d(w_2) \neq 6 \) and \( d(w_3) \neq 6 \), our \( v \) returns at least \( \frac{4}{5} \), which is more than enough. Thus we can assume by symmetry that \( d(w_2) = 6 \). Note that in this case \( d(w_3) \geq 10 \), for \( v_2 \) is not a \((5, 5, 6, 6, \infty)\)-vertex. Since \( v_2 \) gives \( \frac{1}{15} \) to \( v \) by R9, while \( v_3 \) gives \( \frac{2}{3} \) by R6, we have the desired return of \( \frac{2}{3} \). ■

Lemma 8. The total return from the three extreme 5-vertices \( v_1, v_2, \) and \( v_3 \) of an \( S_k \) with \( k \geq 5 \) is at least \( \frac{1}{3} \).

Proof. We have nothing to prove unless \( d(w_2) = 6 \), which implies that \( d(w_3) \geq 10 \). Now \( v_2 \) still gives \( \frac{1}{15} \) to \( v \) by R9, while \( v_3 \) gives at least \( \frac{1}{15} \) by R8 or R9, which returns sum up to the desired \( \frac{1}{3} \). ■
By symmetry, we deduce the following fact from Lemma 8.

**Corollary 9.** The total return from an $S_k$ is at least $\frac{1}{\delta}$ if $5 \leq k \leq 6$ and at least $\frac{2}{\delta}$ if $k \geq 7$.

If $v$ is completely surrounded by 5-vertices (which means that no $S_k$ is defined), then the total return to $v$ is at least $16 \times \frac{1}{17} > \frac{2}{3}$, and hence we can assume from now on that the neighborhood of $v$ is partitioned into $S_k$s.

If $d(v) = 17$, then to pay off the deficiency of $\frac{1}{3}$ it suffices to note that every $S_k$ with $k \neq 3$ returns at least $\frac{1}{3}$ to $v$, while 3 does not divide 17 (which implies that $v$ cannot be surrounded only by $S_3$s).

Finally, suppose that $d(v) = 16$. As follows from Lemma 7 combined with Corollary 9, we are able to cover the deficiency of $16$, while 3 does not divide 17 (which implies that every $S_k$ with $k \neq 3$ returns at least $\frac{1}{3}$ to $v$, while 3 does not divide 17 (which implies that $v$ cannot be surrounded only by $S_3$s).

Remark 10. Each $16^+\text{-neighbor } v_2$ gives $v$ through the faces $v_1v_2, v_2v_3$ by $R2$ and returns from $v$ along edge $vv_2$ by $R4$–$R9$:

(a) $\frac{4}{3}$ versus $\frac{1}{3}$ if $d(v_1) \geq 6$ and $d(v_3) \geq 6$,
(b) 1 versus at most $\frac{1}{2}$ if $d(v_1) = 5$ and $d(v_3) \geq 6$, or
(c) $\frac{2}{3}$ versus at most $\frac{4}{15}$ if $d(v_1) = 5$ and $d(v_3) = 5$.

Remark 10 combined with examining $R4$–$R9$ more carefully implies the following observation.

**Remark 11.** The donation of a $16^+\text{-neighbor } v_2$ to $v$ exceeds the return from $v$ to $v_2$ by less than $\frac{1}{2}$ only when $v$ obeys $R9$, in which case we have $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$.

Subcase 5.1. $v$ participates in $R9$. Thus suppose $d(v_1) = d(v_3) = 5$, $d(v_2) \geq 16$, $d(v_4) \geq 6$, and $d(v_5) \geq 10$. Note that $v$ acquires $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$ from $v_2$ by $R2$ combined with $R9$.

If $d(v_5) \geq 16$, then $v_5$ gives 1 to $v$ by $R2$, and $v$ returns to $v_5$ either $\frac{1}{2}$ by $R4$ if $d(v_4) \geq 10$ or $\frac{2}{5}$ by $R6$ if $d(v_4) = 6$. Thus the total acquisition of $v$ from $v_5$ is at least $\frac{2}{5}$, and we are done.
If \(d(v_5) \leq 15\), then \(v_5\) gives \(\frac{3}{5}\) to \(v\) by R2, and we are done again.

**Subcase 5.2.** \(v\) does not participate in R9. In view of Remark 11, we already have nothing to prove if \(v\) has at least two \(16^+\)-neighbors. So suppose \(v_2\) is the only \(16^+\)-neighbor of \(v\).

If \(d(v_1) \geq 10\), then \(v_1\) gives \(v\) at least \(\frac{2}{5}\) by R2, while \(v_2\)’s resulting donation to \(v\) is \(1 - \frac{1}{5}\) by R2 and R4. This implies \(\mu'(v) > 0\).

By symmetry, suppose \(d(v_1) \leq d(v_3) \leq 6\). If \(d(v_1) = d(v_3) = 6\), then \(v_1\) gives \(\frac{4}{5}\) to \(v\) by R2 and takes back \(\frac{1}{5}\) from \(v\) by R5, which implies \(\mu'(v) \geq 0\).

**Subcase 5.2.1.** \(d(v_1) = 5\) and \(d(v_3) = 6\). Now \(v_2\) gives 1 to \(v\) by R2. If \(d(v_5) > 6\), which means that in fact \(10 \leq d(v_4) \leq 15\), then we have \(\mu'(v) \geq -1 + 1 - \frac{2}{5} + \frac{2}{5} = 0\) by R2 and R6.

If \(d(v_5) = 6\), then we have \(d(v_4) = 6\) or \(d(v_4) \geq 10\) due to the absence of a \((5, 5, 6, 6, \infty)\)-vertex. In both cases, \(\mu'(v) \geq -1 + 1 = 0\) by R2 since \(v\) returns nothing to \(v_2\).

Finally, \(d(v_5) = 5\). Now \(d(v_4) \geq 10\) due to the absence of \((5, 5, 6, 6, \infty)\)-vertex, and we have \(\mu'(v) \geq -1 + 1 - \frac{1}{5} + \frac{1}{5} = 0\) by R2 and R7.

**Subcase 5.2.2.** \(d(v_1) = d(v_3) = 5\). Here \(v_2\) gives \(\frac{2}{5}\) to \(v\) by R2. Since \(v\) is not a \((5, 5, 6, 6, \infty)\)-vertex, we can assume that \(10 \leq d(v_4) \leq 15\). Furthermore, R9 is not applicable to \(v\) by an above assumption, so \(d(v_5) = 5\). This means that \(v\) obeys R8, and we have \(\mu'(v) = -1 + \frac{2}{5} - \frac{1}{15} + \frac{1}{5} = 0\), as desired.

Thus we have proved \(\mu'(x) \geq 0\) whenever \(x \in V \cup F\), which contradicts (1) and completes the proof of Theorem 1.

**References**


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