LOW 5-STARS AT 5-VERTECES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE FROM 7 TO 9

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Abstract
In 1940, Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5.

Given a 3-polytope $P$, by $h_5(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in $P$.

Recently, Borodin, Ivanova and Jensen showed that if a polytope $P$ in $P_5$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5,5,6,6,\infty)$-vertex, then $h_5(P)$ can be arbitrarily large. Therefore, we consider the subclass $P_5^*$ of 3-polytopes in $P_5$ that avoid $(5,5,6,6,\infty)$-vertices.

For each $P^*$ in $P_5^*$ without vertices of degree from 7 to 9, it follows from Lebesgue’s Theorem that $h_5(P^*) \leq 17$. Recently, this bound was lowered by Borodin, Ivanova, and Kazak to the sharp bound $h_5(P^*) \leq 15$ assuming the absence of vertices of degree from 7 to 11 in $P^*$.

In this note, we extend the bound $h_5(P^*) \leq 15$ to all $P^*$s without vertices of degree from 7 to 9.

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1. Introduction

The degree of a vertex or face \( x \) in a convex finite 3-dimensional polytope (called a 3-polytope) is denoted by \( d(x) \). As proved by Steinitz [14], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs. A \( k \)-vertex is a vertex \( v \) with \( d(v) = k \). A \( k^+ \)-vertex (\( k^- \)-vertex) is one of degree at least \( k \) (at most \( k \)). Similar notation is used for the faces. The set of 3-polytopes with minimum degree 5 is denoted by \( P_5 \), and its elements are \( P_5 \)-s. We will drop the argument whenever it is clear from context.

The height of a subgraph \( S \) of a 3-polytope is the maximum degree of the vertices of \( S \) in the 3-polytope. A \( k \)-star, a star with \( k \) rays, is minor if its center \( v \) has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By \( h_k(P_5) \) we denote the minimum height of minor \( k \)-stars in a given 3-polytope \( P_5 \).

In 1904, Wernicke [15] proved that every \( P_5 \) has a 5-vertex adjacent to a 6-vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 6-neighbors. So \( h_1 \leq h_2 \leq 6 \) in \( P_5 \), where both bounds are sharp.

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In 1940, in attempts to solve the Four Color Problem, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in \( P_5 \)-s.

In particular, this description implies the results in [11, 15] and shows that there is a 5-vertex with three 7-neighbors. Thus \( h_3 \leq 7 \), which is sharp due to Borodin [1]. Jendrol’ and Madaras [12] gave a precise description of minor 3-stars in \( P_5 \)-s.

Lebesgue [13] also proved \( h_4(P_5) \leq 11 \), which was strengthened by Borodin and Woodall [10] to the tight bound \( h_4(P_5) \leq 10 \). Recently, Borodin and Ivanova [2] obtained a precise description of 4-stars in \( P_5 \)-s.

The more general problem of describing 5-stars at 5-vertices in \( P_5 \) remains widely open.

Recently, precise upper bounds have been obtained for the minimum height \( h_5(P_5) \) of minor 5-stars in several natural subclasses of \( P_5 \).

Note that Borodin, Ivanova and Jensen [5] showed that if a polytope \( P_5 \) is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a \((5,5,6,6,\infty)\)-vertex, then \( h_5(P_5) \) can be arbitrarily large. (In fact, every 5-vertex in the construction in [5] has two 5-neighbors and two 6-neighbors.) Therefore, from now on we restrict ourselves to the subclass \( P_5^* \) of the 3-polytopes in \( P_5 \) avoiding \((5,5,6,6,\infty)\)-vertices.

For each \( P_5^* \) in \( P_5^* \), it follows from Lebesgue’s Theorem that \( h_5(P_5^*) \leq 41 \). This bound was lowered to \( h_5(P_5^*) \leq 28 \) by Borodin, Ivanova, and Jensen [5] and then to \( h_5(P_5^*) \leq 23 \) in Borodin-Ivanova [4]. On the other hand, it was shown in [5] that the upper bound for \( h_5(P_5^*) \) cannot go down below 20. We conjecture...
that $h_5(P_5^*) \leq 20$ whenever $P_5^* \in P_5^*$.

Back in 1996, Jendrol’ and Madaras [12] showed that if a polytope $P_5^{***}$ has a 5-vertex adjacent to four 5-vertices, then $h_5(P_5^{***})$ can be arbitrarily large. Therefore, considering subclasses of $P_5^n$ without vertices of degree from 6 to a certain $k_6$ with $k_6 > 6$, we should deal only with 3-polytopes $P_5^{***}$’s having no 5-vertices with four 5-neighbors.

For every $P_5^{**}$ in $P_5^*$ with $k_6 = 9$, Lebesgue’s bound $h_5(P_5^{**}) \leq 14$ was improved by Borodin and Ivanova [3] to the sharp bound $h_5(P_5^{**}) \leq 12$. Later on, Borodin, Ivanova and Nikiforov [9] proved the same bound assuming the absence only of vertices of degree from 6 to 8, improving Lebesgue’s bound $h_5(P_5^{**}) \leq 17$.

For each $P_5^{**}$ with no vertices of degree 6 or 7, it follows from Lebesgue’s Theorem that $h_5(P_5) \leq 23$, and Borodin, Ivanova, Kazak and Vasil’eva [7] have obtained the best possible bound $h_5(P_5^{**}) \leq 14$.

For each $P_5^{**}$ with no 6-vertices, Lebesgue’s bound $h_5(P_5^{**}) \leq 41$ was improved by Borodin, Ivanova, and Nikiforov [8] to the sharp bound $h_5(P_5^{**}) \leq 17$. We note that the sharpness was confirmed in [8] by a construction on almost 3000 vertices.

Another natural direction of research towards a tight version of Lebesgue’s Theorem is considering subclasses of $P_5^*$ with no vertices of degree from 7 to a certain integer $k_7$ with $k_7 > 7$.

For $k_7 = 11$, Lebesgue’s bound $h_5(P_5) \leq 17$ was lowered by Borodin, Ivanova, and Kazak [6] to the sharp bound $h_5(P_5) \leq 15$. The purpose of this note is to extend this bound to all $P_5^*$’s such that $k_7 = 9$.

**Theorem 1.** Every 3-polytope $P_5^*$ with minimum degree 5 and neither $(5, 5, 6, 6, \infty)$-vertices nor vertices of degree from 7 to 9 satisfies $h_5(P_5^*) \leq 15$, which bound is best possible.

**Problem 2.** Is it true that every 3-polytope $P_5^*$ with minimum degree 5 and no $(5, 5, 6, 6, \infty)$-vertices satisfies $h_5(P_5^*) \leq 15$ provided that

(a) $P_5^*$ has no vertices of degree 7 and 8?
(b) only 7-vertices are forbidden in $P_5^*$?

2. **Proof of Theorem 1**

The sharpness of the bound 15 in Theorem 1 follows from a construction in [6].

Now suppose a 3-polytope $P_5^*$ is a counterexample to the main statement of Theorem 1. In particular, each minor 5-star in $P_5$ contains a $16^+$-vertex along with either another $10^+$-vertex or at least three 6-vertices.

Let $P_5$ be a counterexample on the same vertices as $P_5^*$ with the maximum possible number of edges. For brevity, a vertex $v$ with $d(v) \neq 6$ is a non-6-vertex.
Remark 3. $P_5$ has no two non-6-vertices being nonconsecutive along the boundary of a $4^+$-face. Indeed, otherwise adding a diagonal between these vertices would result in a counterexample with greater edges than $P_5$.

Corollary 4. In $P_5$, each $4^+$-face has at most two non-6-vertices, and if it has two such vertices, then they are adjacent to each other.

Discharging.

Let $V$, $E$, and $F$ be the sets of vertices, edges, and faces of $P_5$. Euler’s formula $|V| - |E| + |F| = 2$ for $P_5$ implies

\[
\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12.
\]

We assign an initial charge $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The final charge $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R9 below (see Figure 1).

For a vertex $v$, let $v_1, \ldots, v_d(v)$ be the vertices adjacent to $v$ in a fixed cyclic order. If $f$ is a face, then $v_1, \ldots, v_d(f)$ are the vertices incident with $f$ in the same cyclic order.

A vertex is simplicial if it is completely surrounded by 3-faces.

R1. Every $4^+$-face gives 1 to every incident non-6-vertex.

R2. Suppose $f =uvw$ is a 3-face with $d(u) = 5$ and $d(v) \geq 10$.

(a) If $d(w) \geq 6$, then $u$ receives from $v$ either $\frac{2}{3}$ if $d(v) \leq 15$ or $\frac{2}{5}$ otherwise.

(b) If $d(w) = 5$, then $u$ (as well as $w$) receives from $v$ either $\frac{1}{3}$ if $d(v) \leq 15$ or $\frac{1}{5}$ otherwise.

R3. A non-simplicial 5-vertex $v$ such that there are 3-faces $v_1v_2$ and $v_2v_3$ with $d(v_2) \geq 16$ gives $\frac{2}{3}$ to $v_2$.

R4. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$ and $d(v_1) \geq 10$ gives $\frac{1}{3}$ to $v_2$.

R5. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$ and $d(v_1) = d(v_3) = 6$ gives $\frac{1}{3}$ to $v_2$.

R6. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = 5$, and $d(v_4) \geq 10$ gives $\frac{2}{5}$ to $v_2$.

R7. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = d(v_4) = 5$ (hence $d(v_5) \geq 10$) gives $\frac{1}{2}$ to $v_2$. 

Remark 5. Note that a simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_3) = 6 \), and \( d(v_5) = 5 \) gives nothing to \( v_2 \).

**R8.** A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_3) = 5 \), and \( d(v_4) \geq 6 \) gives \( \frac{1}{15} \) to \( v_2 \).

**R9.** A simplicial 5-vertex \( v \) with \( d(v_2) \geq 16 \), \( d(v_1) = d(v_3) = 5 \), and \( d(v_5) \geq 10 \) gives \( \frac{4}{15} \) to \( v_2 \).

Figure 1. Rules of discharging.

Checking \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \).

First consider a face \( f \) in \( P_5 \). If \( d(f) = 3 \), then \( f \) does not participate in discharging, and so \( \mu'(v) = \mu(f) = 2 \times 3 - 6 = 0 \). Note that every 4+-face is incident with at most two non-6-vertices due to Corollary 4, which implies that \( \mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0 \) by R1.

Now suppose \( v \in V \).
Case 1. $d(v) \geq 18$. Since $v$ sends at most $\frac{2}{3}$ to its 5-neighbors through each 3-face by R2, we have $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 18}{3} \geq 0$.

Case 2. $16 \leq d(v) \leq 17$. If $v$ is not simplicial, then it sends at most $\frac{2}{3}$ through each of at most $d(v) - 1$ faces, so $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{d(v) - 16}{3} \geq 0$, as desired. From now on, suppose $v$ is simplicial.

If $v$ has two consecutive $6^+\text{-}neighbors$, then again $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} \geq 0$. So we can assume from now on that each 3-face incident with $v$ is incident with a 5-vertex.

If $v$ has at least one non-simplicial 5-neighbor $v_2$, then $v$ receives $\frac{2}{3}$ from $v_2$ by R3, which implies $\mu'(v) \geq d(v) - 6 + \frac{2}{3} - d(v) \times \frac{2}{3} = \frac{d(v) - 16}{3} \geq 0$. Thus suppose all 5-vertices adjacent to $v$ are simplicial.

If $v$ has a $10^+\text{-}neighbor v_2$, then $v$ receives $\frac{1}{3} + \frac{1}{3}$ from the 5-vertices $v_1$ and $v_3$ by R4, which again implies $\mu'(v) \geq 0$.

Summarizing, from now on our $v$ is simplicial, has no $10^+\text{-}neighbors$, no two consecutive 6-neighbors, and no non-simplicial 5-neighbors.

Suppose $S_k = v_0, \ldots, v_k$ is a sequence of neighbors of $v$ with $d(v_0) = 6$, $d(v_k) = 6$, while $d(v_i) = 5$ whenever $1 \leq i \leq k - 1$ and $k \geq 2$. (It is not excluded that $S_k = S_{d(v)}$, which happens when $v$ has precisely one 6-neighbor.) Let $w_i$, $1 \leq i \leq k - 1$, $k \geq 2$, be the common neighbor of $v_{i-1}$ and $v_1$ different from $v$.

Since $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 16}{3}$, we can say that $v$ has the deficiency equal to $\frac{1}{3}$ if $d(v) = 17$ or $\frac{2}{3}$ if $d(v) = 16$.

Our next goal is to estimate the total return to $v$ from its 5-neighbors by R4–R9 and show that it is not less than the deficiency of $v$.

Remark 6. As we remember, our $v$ has no $S_{18}$. Note that $v_1$ in $S_2$ returns $\frac{1}{3}$ to $v$ by R5. As for $S_3$, it can happen that neither $v_1$ nor $v_2$ returns anything to $v$, which is the case only when $v_1$ and $v_2$ have a common 6-neighbor (see Remark 5).

Lemma 7. The total return from the three 5-vertices of an $S_k$ is at least $\frac{2}{3}$.

Proof. If $d(w_2) \geq 10$ or $d(w_2) = 5$, then $v$ receives at least $\frac{2}{3}$ from its 5-neighbor $v_1$ by R6 or R7, respectively. The same is true for $v_2$. So, if $d(w_2) \neq 6$ and $d(w_3) \neq 6$, our $v$ returns at least $\frac{4}{5}$, which is more than enough. Thus we can assume by symmetry that $d(w_2) = 6$. Note that in this case $d(w_3) \geq 10$, for $v_2$ is not a $(5, 5, 6, 6, \infty)$-vertex. Since $v_2$ gives $\frac{1}{10}$ to $v$ by R9, while $v_3$ gives $\frac{2}{5}$ by R6, we have the desired return of $\frac{2}{3}$.

Lemma 8. The total return from the three extreme 5-vertices $v_1$, $v_2$, and $v_3$ of an $S_k$ with $k \geq 5$ is at least $\frac{1}{3}$.

Proof. We have nothing to prove unless $d(w_2) = 6$, which implies that $d(w_3) \geq 10$. Now $v_2$ still gives $\frac{1}{10}$ to $v$ by R9, while $v_3$ gives at least $\frac{1}{10}$ by R8 or R9, which returns sum up to the desired $\frac{1}{3}$. ■
By symmetry, we deduce the following fact from Lemma 8.

**Corollary 9.** The total return from an $S_k$ is at least $\frac{1}{3}$ if $5 \leq k \leq 6$ and at least $\frac{2}{3}$ if $k \geq 7$.

If $v$ is completely surrounded by 5-vertices (which means that no $S_k$ is defined), then the total return to $v$ is at least $16 \times \frac{1}{15} > \frac{2}{3}$, and hence we can assume from now on that the neighborhood of $v$ is partitioned into $S_k$s.

If $d(v) = 17$, then to pay off the deficiency of $\frac{1}{3}$ it suffices to note that every $S_k$ with $k \neq 3$ returns at least $\frac{1}{3}$ to $v$, while 3 does not divide 17 (which implies that $v$ cannot be surrounded only by $S_3$s).

Finally, suppose that $d(v) = 16$. As follows from Lemma 7 combined with Corollary 9, we are able to cover the deficiency of $\frac{2}{3}$ unless the neighborhood of $v$ consists of several $S_3$ and at most one $S_k$ such that $k \in \{2, 5, 6\}$. However, the residue of 16 modulo 3 is neither 0 nor 2, a contradiction.

- **Case 3.** $10 \leq d(v) \leq 15$. Now R2 implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{3(d(v)-10)}{5} \geq 0$ since $v$ sends either nothing or $\frac{2}{5}$ through each incident face.

- **Case 4.** $d(v) = 6$. Since $v$ does not participate in discharging, we have $\mu'(v) = \mu(v) = 6 - 6 = 0$.

- **Case 5.** $d(v) = 5$. If $v$ is incident with a 4$^+$-face, then $\mu'(v) \geq 5 - 6 + 1 = 0$ due to R1 combined with the fact that each 16$^+$-neighbor $v_2$ gives more to $v$ by R2 than $v$ returns to $v_2$ by R3. Therefore, in what follows we can assume that $v$ is simplicial.

**Remark 10.** Each 16$^+$-neighbor $v_2$ gives $v$ through the faces $v_1v_2$, $v_2v_3$ by R2 and returns from $v$ along edge $vv_2$ by R4–R9:

- (a) $\frac{4}{3}$ versus $\frac{1}{3}$ if $d(v_1) \geq 6$ and $d(v_3) \geq 6$,
- (b) 1 versus at most $\frac{1}{2}$ if $d(v_1) = 5$ and $d(v_3) \geq 6$, or
- (c) $\frac{2}{3}$ versus at most $\frac{4}{15}$ if $d(v_1) = 5$ and $d(v_3) = 5$.

Remark 10 combined with examining R4–R9 more carefully implies the following observation.

**Remark 11.** The donation of a 16$^+$-neighbor $v_2$ to $v$ exceeds the return from $v$ to $v_2$ by less than $\frac{1}{2}$ only when $v$ obeys R9, in which case we have $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$.

**Subcase 5.1.** $v$ participates in R9. Thus suppose $d(v_1) = d(v_3) = 5$, $d(v_2) \geq 16$, $d(v_4) \geq 6$, and $d(v_5) \geq 10$. Note that $v$ acquires $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$ from $v_2$ by R2 combined with R9.

If $d(v_5) \geq 16$, then $v_5$ gives 1 to $v$ by R2, and $v$ returns to $v_5$ either $\frac{1}{7}$ by R4 if $d(v_4) \geq 10$ or $\frac{2}{5}$ by R6 if $d(v_4) = 6$. Thus the total acquisition of $v$ from $v_5$ is at least $\frac{2}{5}$, and we are done.
If \( d(v_5) \leq 15 \), then \( v_5 \) gives \( \frac{3}{5} \) to \( v \) by R2, and we are done again.

Subcase 5.2. \( v \) does not participates in R9. In view of Remark 11, we already have nothing to prove if \( v \) has at least two 16+-neighbors. So suppose \( v_2 \) is the only 16+-neighbor of \( v \).

If \( d(v_1) \geq 10 \), then \( v_1 \) gives \( v \) at least \( \frac{2}{5} \) by R2, while \( v_2 \)'s resulting donation to \( v \) is \( 1 - \frac{1}{5} \) by R2 and R4. This implies \( \mu'(v) > 0 \).

By symmetry, suppose \( d(v_1) \leq d(v_3) \leq 6 \). If \( d(v_1) = d(v_3) = 6 \), then \( v_1 \) gives \( \frac{1}{5} \) to \( v \) by R2 and takes back \( \frac{1}{5} \) from \( v \) by R5, which implies \( \mu'(v) \geq 0 \).

Subcase 5.2.1. \( d(v_1) = 5 \) and \( d(v_3) = 6 \). Now \( v_2 \) gives \( 1 \) to \( v \) by R2. If \( d(v_5) > 6 \), which means that in fact \( 10 \leq d(v_4) \leq 15 \), then we have \( \mu'(v) \geq 1 + 1 - \frac{2}{5} + \frac{2}{5} = 0 \) by R2 and R6.

If \( d(v_5) = 6 \), then we have \( d(v_4) = 6 \) or \( d(v_4) \geq 10 \) due to the absence of a \((5,5,6,6,\infty)\)-vertex. In both cases, \( \mu'(v) \geq 1 + 1 = 0 \) by R2 since \( v \) returns nothing to \( v_2 \).

Finally, \( d(v_5) = 5 \). Now \( d(v_4) \geq 10 \) due to the absence of \((5,5,6,6,\infty)\)-vertex, and we have \( \mu'(v) \geq -1 + 1 - \frac{2}{5} + \frac{2}{5} > 0 \) by R2 and R7.

Subcase 5.2.2. \( d(v_1) = d(v_3) = 5 \). Here \( v_2 \) gives \( \frac{2}{5} \) to \( v \) by R2. Since \( v \) is not a \((5,5,6,6,\infty)\)-vertex, we can assume that \( 10 \leq d(v_4) \leq 15 \). Furthermore, R9 is not applicable to \( v \) by an above assumption, so \( d(v_5) = 5 \). This means that \( v \) obeys R8, and we have \( \mu'(v) = -1 + \frac{2}{5} - \frac{1}{15} + \frac{2}{5} = 0 \), as desired.

Thus we have proved \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \), which contradicts (1) and completes the proof of Theorem 1.

References


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