LOW 5-STARS AT 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE FROM 7 TO 9

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Abstract

In 1940, Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $P_5$ of 3-polytopes with minimum degree 5.

Given a 3-polytope $P$, by $h_5(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in $P$.

Recently, Borodin, Ivanova and Jensen showed that if a polytope $P$ in $P_5$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-vertex, then $h_5(P)$ can be arbitrarily large. Therefore, we consider the subclass $P_5^*$ of 3-polytopes in $P_5$ that avoid $(5, 5, 6, 6, \infty)$-vertices.

For each $P^*$ in $P_5^*$ without vertices of degree from 7 to 9, it follows from Lebesgue’s Theorem that $h_5(P^*) \leq 17$. Recently, this bound was lowered by Borodin, Ivanova, and Kazak to the sharp bound $h_5(P^*) \leq 15$ assuming the absence of vertices of degree from 7 to 11 in $P^*$.

In this note, we extend the bound $h_5(P^*) \leq 15$ to all $P^*$s without vertices of degree from 7 to 9.

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1. Introduction

The degree of a vertex or face $x$ in a convex finite 3-dimensional polytope (called a 3-polytope) is denoted by $d(x)$. As proved by Steinitz [14], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs. A $k$-vertex is a vertex $v$ with $d(v) = k$. A $k^+$-vertex ($k^-$-vertex) is one of degree at least $k$ (at most $k$). Similar notation is used for the faces. The set of 3-polytopes with minimum degree 5 is denoted by $P_5$, and its elements are $P_5$s. We will drop the argument whenever it is clear from context.

The height of a subgraph $S$ of a 3-polytope is the maximum degree of the vertices of $S$ in the 3-polytope. A $k$-star, a star with $k$ rays, is minor if its center $v$ has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $h_k(P_5)$ we denote the minimum height of minor $k$-stars in a given 3-polytope $P_5$.

In 1904, Wernicke [15] proved that every $P_5$ has a 5-vertex adjacent to a 6$^-$-vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 6$^-$-neighbors. So $h_1 \leq h_2 \leq 6$ in $P_5$, where both bounds are sharp.

In 1940, in attempts to solve the Four Color Problem, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in $P_5$s.

In particular, this description implies the results in [11, 15] and shows that there is a 5-vertex with three 7$^-$-neighbors. Thus $h_3 \leq 7$, which is sharp due to Borodin [1]. Jendrol’ and Madaras [12] gave a precise description of minor 3-stars in $P_5$s.

Lebesgue [13] also proved $h_4(P_5) \leq 11$, which was strengthened by Borodin and Woodall [10] to the tight bound $h_4(P_5) \leq 10$. Recently, Borodin and Ivanova [2] obtained a precise description of 4-stars in $P_5$s.

The more general problem of describing 5-stars at 5-vertices in $P_5$ remains widely open.

Recently, precise upper bounds have been obtained for the minimum height $h_5(P_5)$ of minor 5-stars in several natural subclasses of $P_5$.

Note that Borodin, Ivanova and Jensen [5] showed that if a polytope $P_5$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-vertex, then $h_5(P_5)$ can be arbitrarily large. (In fact, every 5-vertex in the construction in [5] has two 5-neighbors and two 6-neighbors.) Therefore, from now on we restrict ourselves to the subclass $P^*_5$ of the 3-polytopes in $P_5$ avoiding $(5, 5, 6, 6, \infty)$-vertices.

For each $P^*_5$ in $P^*_5$, it follows from Lebesgue’s Theorem that $h_5(P^*_5) \leq 41$. This bound was lowered to $h_5(P^*_5) \leq 28$ by Borodin, Ivanova, and Jensen [5] and then to $h_5(P^*_5) \leq 23$ in Borodin-Ivanova [4]. On the other hand, it was shown in [5] that the upper bound for $h_5(P^*_5)$ cannot go down below 20. We conjecture
that \( h_5(P^*) \leq 20 \) whenever \( P^*_5 \in \mathbf{P}^*_5 \).

Back in 1996, Jendrol' and Madaras [12] showed that if a polytope \( P^*_5 \) has a 5-vertex adjacent to four 5-vertices, then \( h_5(P^*_5) \) can be arbitrarily large. Therefore, considering subclasses of \( \mathbf{P}^*_5 \) without vertices of degree from 6 to a certain \( k_6 \) with \( k_6 > 6 \), we should deal only with 3-polytopes \( P^*_5 \)'s having no 5-vertices with four 5-neighbors.

For every \( P^*_5 \) with \( k_6 = 9 \), Lebesgue's bound \( h_5(P^*_5) \leq 14 \) was improved by Borodin and Ivanova [3] to the sharp bound \( h_5(P^*_5) \leq 12 \). Later on, Borodin, Ivanova and Nikiforov [9] proved the same bound assuming the absence only of vertices of degree from 6 to 8, improving Lebesgue's bound \( h_5(P^*_5) \leq 17 \).

For each \( P^*_5 \) with no vertices of degree 6 or 7, it follows from Lebesgue's Theorem that \( h_5(P^*_5) \leq 23 \), and Borodin, Ivanova, Kazak and Vasil'eva [7] have obtained the best possible bound \( h_5(P^*_5) \leq 14 \).

Another natural direction of research towards a tight version of Lebesgue's Theorem is considering subclasses of \( \mathbf{P}^*_5 \) with no vertices of degree from 7 to a certain integer \( k_7 \) with \( k_7 > 7 \).

For \( k_7 = 11 \), Lebesgue's bound \( h_5(P^*_5) \leq 17 \) was lowered by Borodin, Ivanova, and Kazak [6] to the sharp bound \( h_5(P^*_5) \leq 15 \). The purpose of this note is to extend this bound to all \( P^*_5 \)'s such that \( k_7 = 9 \).

**Theorem 1.** Every 3-polytope \( P^* \) with minimum degree 5 and neither \((5, 5, 6, 6, \infty)\)-vertices nor vertices of degree from 7 to 9 satisfies \( h_5(P^*) \leq 15 \), which bound is best possible.

**Problem 2.** Is it true that every 3-polytope \( P^* \) with minimum degree 5 and no \((5, 5, 6, 6, \infty)\)-vertices satisfies \( h_5(P^*) \leq 15 \) provided that

(a) \( P^* \) has no vertices of degree 7 and 8?

(b) only 7-vertices are forbidden in \( P^* \)?

2. Proof of Theorem 1

The sharpness of the bound 15 in Theorem 1 follows from a construction in [6].

Now suppose a 3-polytope \( P^*_5 \) is a counterexample to the main statement of Theorem 1. In particular, each minor 5-star in \( P^*_5 \) contains a 16+ vertex along with either another 10+ vertex or at least three 6-vertices.

Let \( P_3 \) be a counterexample on the same vertices as \( P^*_5 \) with the maximum possible number of edges. For brevity, a vertex \( v \) with \( d(v) \neq 6 \) is a **non-6-vertex**.
Remark 3. $P_5$ has no two non-6-vertices being nonconsecutive along the boundary of a $4^+$-face. Indeed, otherwise adding a diagonal between these vertices would result in a counterexample with greater edges than $P_5$.

Corollary 4. In $P_5$, each $4^+$-face has at most two non-6-vertices, and if it has two such vertices, then they are adjacent to each other.

Discharging.

Let $V$, $E$, and $F$ be the sets of vertices, edges, and faces of $P_5$. Euler’s formula $|V| - |E| + |F| = 2$ for $P_5$ implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12.$$  

We assign an initial charge $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The final charge $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R9 below (see Figure 1).

For a vertex $v$, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a fixed cyclic order. If $f$ is a face, then $v_1, \ldots, v_{d(f)}$ are the vertices incident with $f$ in the same cyclic order.

A vertex is simplicial if it is completely surrounded by 3-faces.

R1. Every $4^+$-face gives 1 to every incident non-6-vertex.

R2. Suppose $f =uvw$ is a 3-face with $d(u) = 5$ and $d(v) \geq 10$.

(a) If $d(w) \geq 6$, then $u$ receives from $v$ either $\frac{2}{3}$ if $d(v) \leq 15$ or $\frac{2}{3}$ otherwise.

(b) If $d(w) = 5$, then $u$ (as well as $w$) receives from $v$ either $\frac{1}{3}$ if $d(v) \leq 15$ or $\frac{1}{3}$ otherwise.

R3. A non-simplicial 5-vertex $v$ such that there are 3-faces $v_1v_2v$ and $v_2v_3v$ with $d(v_2) \geq 16$ gives $\frac{2}{3}$ to $v_2$.

R4. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$ and $d(v_1) \geq 10$ gives $\frac{1}{3}$ to $v_2$.

R5. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = 5$, and $d(v_4) \geq 10$ gives $\frac{2}{3}$ to $v_2$.

R6. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = d(v_4) = 5$ (hence $d(v_5) \geq 10$) gives $\frac{1}{2}$ to $v_2$.

R7. A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = d(v_4) = 5$ (hence $d(v_5) \geq 10$) gives $\frac{1}{2}$ to $v_2$. 


Low 5-stars at 5-vertices in 3-polytopes ...

**Remark 5.** Note that a simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = d(v_4) = 6$, and $d(v_3) = 5$ gives nothing to $v_2$.

**R8.** A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = d(v_3) = d(v_4) = 5$, and $d(v_5) \geq 10$ gives $\frac{1}{15}$ to $v_2$.

**R9.** A simplicial 5-vertex $v$ with $d(v_2) \geq 16$, $d(v_1) = d(v_3) = 5$, $d(v_4) \geq 6$, and $d(v_5) \geq 10$ gives $\frac{4}{15}$ to $v_2$.

![Figure 1. Rules of discharging.](image)

**Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$.**

First consider a face $f$ in $P_5$. If $d(f) = 3$, then $f$ does not participate in discharging, and so $\mu'(v) = \mu(f) = 2 \times 3 - 6 = 0$. Note that every 4+-face is incident with at most two non-6-vertices due to Corollary 4, which implies that $\mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0$ by R1.

Now suppose $v \in V$. 

Case 1. \(d(v) \geq 18\). Since \(v\) sends at most \(\frac{2}{3}\) to its 5-neighbors through each 3-face by R2, we have \(\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 18}{3} \geq 0\).

Case 2. \(16 \leq d(v) \leq 17\). If \(v\) is not simplicial, then it sends at most \(\frac{2}{3}\) through each of at most \(d(v) - 1\) faces, so \(\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{d(v) - 16}{3} \geq 0\), as desired. From now on, suppose \(v\) is simplicial.

If \(v\) has two consecutive 6-neighbor \(v_2\), then \(v\) receives \(\frac{2}{3}\) from \(v_2\) by R3, which implies \(\mu'(v) \geq d(v) - 6 + \frac{2}{3} - d(v) \times \frac{2}{3} = \frac{d(v) - 16}{3} \geq 0\). Thus suppose all 5-vertices adjacent to \(v\) are simplicial.

If \(v\) has a 10-neighbor \(v_2\), then \(v\) receives \(\frac{1}{3} + \frac{1}{3}\) from the 5-vertices \(v_1\) and \(v_3\) by R4, which again implies \(\mu'(v) \geq 0\).

Summarizing, from now on our \(v\) is simplicial, has no 10-neighbor, no two consecutive 6-neighbors, and no non-simplicial 5-neighbors.

Suppose \(S_k = v_0, \ldots, v_k\) is a sequence of neighbors of \(v\) with \(d(v_0) = 6\), \(d(v_k) = 6\), and \(d(v_i) = 5\) whenever \(1 \leq i \leq k - 1\) and \(k \geq 2\). (It is not excluded that \(S_k = S_d(v)\), which happens when \(v\) has precisely one 6-neighbor.) Let \(w_i, 1 \leq i \leq k - 1, k \geq 2\), be the common neighbor of \(v_{i-1}\) and \(v_i\) different from \(v\).

Since \(\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 16}{3}\), we can say that \(v\) has the deficiency equal to \(\frac{1}{3}\) if \(d(v) = 17\) or \(\frac{2}{3}\) if \(d(v) = 16\).

Our next goal is to estimate the total return to \(v\) from its 5-neighbors by R4–R9 and show that it is not less than the deficiency of \(v\).

Remark 6. As we remember, our \(v\) has no \(S_1\)S. Note that \(v_1\) in \(S_2\) returns \(\frac{1}{3}\) to \(v\) by R5. As for \(S_3\), it can happen that neither \(v_1\) nor \(v_2\) returns anything to \(v\), which is the case only when \(v_1\) and \(v_2\) have a common 6-neighbor (see Remark 5).

Lemma 7. The total return from (the three 5-vertices of) an \(S_k\) is at least \(\frac{2}{3}\).

Proof. If \(d(w_2) \geq 10\) or \(d(w_2) = 5\), then \(v\) receives at least \(\frac{2}{3}\) from its 5-neighbor \(v_1\) by R6 or R7, respectively. The same is true for \(v_3\). So, if \(d(w_2) \neq 6\) and \(d(w_3) \neq 6\), our \(v\) returns at least \(\frac{4}{3}\), which is more than enough. Thus we can assume by symmetry that \(d(w_2) = 6\). Note that in this case \(d(w_3) \geq 10\), for \(v_2\) is not a \((5, 5, 6, 6, \infty)\)-vertex. Since \(v_2\) gives \(\frac{4}{15}\) to \(v\) by R9, while \(v_3\) gives \(\frac{2}{5}\) by R6, we have the desired return of \(\frac{2}{3}\).

Lemma 8. The total return from the three extreme 5-vertices \(v_1, v_2,\) and \(v_3\) of an \(S_k\) with \(k \geq 5\) is at least \(\frac{1}{3}\).

Proof. We have nothing to prove unless \(d(w_2) = 6\), which implies that \(d(w_3) \geq 10\). Now \(v_2\) still gives \(\frac{1}{15}\) to \(v\) by R9, while \(v_3\) gives at least \(\frac{1}{15}\) by R8 or R9, which returns sum up to the desired \(\frac{1}{3}\).
By symmetry, we deduce the following fact from Lemma 8.

**Corollary 9.** The total return from an $S_k$ is at least $\frac{1}{3}$ if $5 \leq k \leq 6$ and at least $\frac{2}{3}$ if $k \geq 7$.

If $v$ is completely surrounded by 5-vertices (which means that no $S_k$ is defined), then the total return to $v$ is at least $16 \times \frac{1}{17} > \frac{2}{3}$, and hence we can assume from now on that the neighborhood of $v$ is partitioned into $S_k$s.

If $d(v) = 17$, then to pay off the deficiency of $\frac{1}{3}$ it suffices to note that every $S_k$ with $k \neq 3$ returns at least $\frac{1}{3}$ to $v$, while 3 does not divide 17 (which implies that $v$ cannot be surrounded only by $S_3$s).

Finally, suppose that $d(v) = 16$. As follows from Lemma 7 combined with Corollary 9, we are able to cover the deficiency of $\frac{2}{3}$ unless the neighborhood of $v$ consists of several $S_3$ and at most one $S_k$ such that $k \in \{2, 5, 6\}$. However, the residue of 16 modulo 3 is neither 0 nor 2, a contradiction.

**Case 3.** $10 \leq d(v) \leq 15$. Now R2 implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{3(d(v) - 10)}{3} \geq 0$ since $v$ sends either nothing or $\frac{2}{3}$ through each incident face.

**Case 4.** $d(v) = 6$. Since $v$ does not participate in discharging, we have $\mu'(v) = \mu(v) = 6 - 6 = 0$.

**Case 5.** $d(v) = 5$. If $v$ is incident with a $4^+$-face, then $\mu'(v) \geq 5 - 6 + 1 = 0$ due to R1 combined with the fact that each $16^+$-neighbor $v_2$ gives more to $v$ by R2 than $v$ returns to $v_2$ by R3. Therefore, in what follows we can assume that $v$ is simplicial.

**Remark 10.** Each $16^+$-neighbor $v_2$ gives $v$ through the faces $v_1v_2$, $v_2v_3$ by R2 and returns from $v$ along edge $vv_2$ by R4–R9:

(a) $\frac{4}{3}$ versus $\frac{1}{3}$ if $d(v_1) \geq 6$ and $d(v_3) \geq 6$,
(b) 1 versus at most $\frac{1}{2}$ if $d(v_1) = 5$ and $d(v_3) \geq 6$, or
(c) $\frac{2}{3}$ versus at most $\frac{4}{15}$ if $d(v_1) = 5$ and $d(v_3) = 5$.

Remark 10 combined with examining R4–R9 more carefully implies the following observation.

**Remark 11.** The donation of a $16^+$-neighbor $v_2$ to $v$ exceeds the return from $v$ to $v_2$ by less than $\frac{1}{2}$ only when $v$ obeys R9, in which case we have $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$.

**Subcase 5.1.** $v$ participates in R9. Thus suppose $d(v_1) = d(v_3) = 5$, $d(v_2) \geq 16$, $d(v_4) \geq 6$, and $d(v_5) \geq 10$. Note that $v$ acquires $\frac{2}{5} - \frac{4}{15} = \frac{2}{5}$ from $v_2$ by R2 combined with R9.

If $d(v_5) \geq 16$, then $v_5$ gives 1 to $v$ by R2, and $v$ returns to $v_5$ either $\frac{1}{2}$ by R4 if $d(v_4) \geq 10$ or $\frac{2}{5}$ by R6 if $d(v_4) = 6$. Thus the total acquisition of $v$ from $v_5$ is at least $\frac{2}{5}$, and we are done.
If \( d(v_5) \leq 15 \), then \( v_5 \) gives \( \frac{3}{5} \) to \( v \) by R2, and we are done again.

\textbf{Subcase 5.2.} \( v \) does not participates in R9. In view of Remark 11, we already have nothing to prove if \( v \) has at least two \( 16^+ \)-neighbors. So suppose \( v_2 \) is the only \( 16^+ \)-neighbor of \( v \).

If \( d(v_1) \geq 10 \), then \( v_1 \) gives \( v \) at least \( \frac{2}{5} \) by R2, while \( v_2 \)'s resulting donation to \( v \) is \( 1 - \frac{1}{5} \) by R2 and R4. This implies \( \mu'(v) > 0 \).

By symmetry, suppose \( d(v_1) \leq d(v_3) \leq 6 \). If \( d(v_1) = d(v_3) = 6 \), then \( v_1 \) gives \( \frac{4}{7} \) to \( v \) by R2 and takes back \( \frac{1}{7} \) from \( v \) by R5, which implies \( \mu'(v) \geq 0 \).

\textbf{Subcase 5.2.1.} \( d(v_1) = 5 \) and \( d(v_3) = 6 \). Now \( v_2 \) gives 1 to \( v \) by R2. If \( d(v_5) > 6 \), which means that in fact \( 10 \leq d(v_4) \leq 15 \), then we have \( \mu'(v) \geq -1 + 1 - \frac{2}{5} + \frac{2}{5} = 0 \) by R2 and R6.

If \( d(v_5) = 6 \), then we have \( d(v_4) = 6 \) or \( d(v_4) \geq 10 \) due to the absence of a \((5, 5, 6, 6, \infty)\)-vertex. In both cases, \( \mu'(v) \geq -1 + 1 = 0 \) by R2 since \( v \) returns nothing to \( v_2 \).

Finally, \( d(v_5) = 5 \). Now \( d(v_4) \geq 10 \) due to the absence of \((5, 5, 6, 6, \infty)\)-vertex, and we have \( \mu'(v) \geq -1 + 1 - \frac{2}{5} + \frac{2}{5} > 0 \) by R2 and R7.

\textbf{Subcase 5.2.2.} \( d(v_1) = d(v_3) = 5 \). Here \( v_2 \) gives \( \frac{2}{5} \) to \( v \) by R2. Since \( v \) is not a \((5, 5, 6, 6, \infty)\)-vertex, we can assume that \( 10 \leq d(v_4) \leq 15 \). Furthermore, R9 is not applicable to \( v \) by an above assumption, so \( d(v_5) = 5 \). This means that \( v \) obeys R8, and we have \( \mu'(v) = -1 + 1 + \frac{2}{5} - \frac{1}{5} + \frac{2}{5} = 0 \), as desired.

Thus we have proved \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \), which contradicts (1) and completes the proof of Theorem 1.

\section*{References}


Low 5-stars at 5-vertices in 3-polytopes ...


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