CONNECTED DOMINATION CRITICAL GRAPHS
WITH CUT VERTICES

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Abstract

A graph $G$ is said to be $k$-$\gamma_c$-critical if the connected domination number of $G$, $\gamma_c(G)$, is $k$ and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices $u$ and $v$ of $G$. Let $G$ be a $k$-$\gamma_c$-critical graph and $\zeta(G)$ the number of cut vertices of $G$. It was proved, in [1, 6], that, for $3 \leq k \leq 4$, every $k$-$\gamma_c$-critical graph satisfies $\zeta(G) \leq k - 2$. In this paper, we generalize that every $k$-$\gamma_c$-critical graph satisfies $\zeta(G) \leq k - 2$ for all $k \geq 5$. We also characterize all $k$-$\gamma_c$-critical graphs when $\zeta(G)$ is achieving the upper bound.

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1. Introduction

All graphs in this paper are finite, undirected and simple (no loops or multiple edges). For a graph $G$, let $V(G)$ denote the set of all vertices of $G$ and let $E(G)$ denote the set of all edges of $G$. The complement $\overline{G}$ of $G$ is the graph
having the same set of vertices as \( G \) but the edge \( e \) is in \( E(G) \) if and only if \( e \notin E(G) \). For \( S \subseteq V(G) \), \( G[S] \) denotes the subgraph of \( G \) induced by \( S \). The open neighborhood \( N_G(v) \) of a vertex \( v \) in \( G \) is \( \{ u \in V(G) : uv \in E(G) \} \). Further, the closed neighborhood \( N_G[v] \) of a vertex \( v \) in \( G \) is \( N_G(v) \cup \{ v \} \). For subsets \( X \) and \( Y \) of \( V(G) \), \( N_Y(X) \) is the set \( \{ y \in Y : yx \in E(G) \) for some \( x \in X \) \}. For a subgraph \( H \) of \( G \), we use \( N_Y(H) \) instead of \( N_Y(V(H)) \) and we use \( N_H(X) \) instead of \( N_Y(V(H)) \). If \( X = \{ x \} \), we use \( N_Y(x) \) instead of \( N_Y(\{ x \}) \). The degree \( \text{deg}(x) \) of a vertex \( x \) in \( G \) is \( |N_G(x)| \). When no ambiguity occur, we write \( N(x) \) and \( N(X) \) instead of \( N_G(x) \) and \( N_G(X) \), respectively. An end vertex is a vertex of degree one and a support vertex is the vertex which is adjacent to an end vertex. A star \( K_{1,n} \) is a graph of order \( n + 1 \) containing one support vertex and \( n \) end vertices. The support vertex of a star is called the center. For a connected graph \( G \), a vertex \( v \) of \( G \) is called a cut vertex if \( G - v \) is not connected. The number of cut vertices of \( G \) is denoted by \( \zeta(G) \). A block \( B \) of a graph \( G \) is a maximal connected subgraph such that \( B \) has no cut vertex. An end block of \( G \) is a block containing exactly one cut vertex of \( G \). The distance \( d(u,v) \) between vertices \( u \) and \( v \) of \( G \) is the length of a shortest \((u,v)\)-path in \( G \). The diameter of \( G \) \( \text{diam}(G) \) is the maximum distance of any two vertices of \( G \). For a connected graph \( G \), a bridge \( xy \) of \( G \) is an edge such that \( G - xy \) is not connected.

For a finite sequence of graphs \( G_1, \ldots, G_l \) for \( l \geq 2 \), the joins \( G_1 \vee \cdots \vee G_l \) is the graph consisting of the disjoint union of \( G_1, \ldots, G_l \) and each vertex in \( G_i \) is joined to all vertices in \( G_{i+1} \) for \( 1 \leq i \leq l-1 \) by edges. If \( V(G_i) = \{ x \} \), then we simply write \( G_1 \vee \cdots \vee G_{i-1} \vee x \vee G_{i+1} \vee \cdots \vee G_l \). Moreover, for a subgraph \( H \) of \( G_2 \), the join \( G_1 \vee_H G_2 \) is the graph consisting of the disjoint union of \( G_1 \) and \( G_2 \) and edges that join each vertex in \( G_1 \) to each vertex in \( H \).

For subsets \( D \) and \( X \) of \( V(G) \), \( D \) dominates \( X \) if every vertex in \( X \) is either in \( D \) or adjacent to a vertex in \( D \). If \( D \) dominates \( X \), then we write \( D \searrow X \). We also write \( a \searrow X \) when \( D = \{ a \} \) and \( D \searrow x \) when \( X = \{ x \} \). Moreover, if \( X = V(G) \), then \( D \) is a dominating set of \( G \) and we write \( D \searrow G \). A connected dominating set of a graph \( G \) is a dominating set \( D \) of \( G \) such that \( G[D] \) is connected. If \( D \) is a connected dominating set of \( G \), we then write \( D \searrow_c G \). A smallest connected dominating set is called a \( \gamma_c \)-set. The cardinality of a \( \gamma_c \)-set is called the connected domination number of \( G \) and is denoted by \( \gamma_c(G) \). A graph \( G \) is said to be \( k-\gamma_c \)-critical if \( \gamma_c(G) = k \) and \( \gamma_c(G+uv) < k \) for any pair of non-adjacent vertices \( u \) and \( v \) of \( G \).

For related results on \( k-\gamma_c \)-critical graphs, Chen et al. [3] completely characterized these graphs when \( 1 \leq k \leq 2 \).

**Theorem 1** [3]. A graph \( G \) is \( 1-\gamma_c \)-critical if and only if \( G \) is a complete graph. Moreover, a graph \( G \) is \( 2-\gamma_c \)-critical if and only if \( \overline{G} = \bigcup_{i=1}^{l} K_{1,n_i} \), where \( l \geq 2 \) and \( n_i \geq 1 \) for all \( 1 \leq i \leq l \).
By Theorem 1, we observe that a $k$-$\gamma_c$-critical graph does not contain a cut vertex when $1 \leq k \leq 2$.

**Observation 2.** Let $G$ be a $k$-$\gamma_c$-critical graph with $1 \leq k \leq 2$. Then $G$ has no cut vertex.

For $k \geq 3$, there is no complete characterization of these graphs so far. However, there are some structural characterizations of $k$-$\gamma_c$-critical graphs when $3 \leq k \leq 4$ by focusing on the maximum number of cut vertices of the graphs. Ananchuen [1] proved that the number of cut vertices of a $3$-$\gamma_c$-critical graph does not exceed one.

**Theorem 3** [1]. Let $G$ be a $3$-$\gamma_c$-critical graph. Then $G$ contains at most one cut vertex.

In our previous work in [6], we established the maximum number of cut vertices that $4$-$\gamma_c$-critical graphs can have.

**Theorem 4** [6]. Let $G$ be a $4$-$\gamma_c$-critical graph. Then $G$ contains at most two cut vertices.

By these results, we naturally, ask for $k \geq 5$, whether every $k$-$\gamma_c$-critical graph contains at most $k - 2$ cut vertices. It turns out affirmatively as we shall see in the following theorem.

**Theorem 5.** For $k \geq 5$, let $G$ be a $k$-$\gamma_c$-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G) \leq k - 2$.

The proof of this theorem is presented in Section 4. In this paper, we also characterize all $k$-$\gamma_c$-critical graphs when the number of cut vertices is achieving the upper bound.

For the outline of this paper, we provide related results and prove that there exists a forbidden subgraph of $k$-$\gamma_c$-critical graphs in Section 2. In Section 3, we characterize some end blocks of $G$. We then use the results from Sections 2 and 3 to establish the upper bound of the number of cut vertices of $k$-$\gamma_c$-critical graphs in Section 4. We also characterize all $k$-$\gamma_c$-critical graphs when $\zeta(G) = k - 2$ in Section 5. Finally, we discuss our result with some related result in another type of domination critical graphs in Section 6.

### 2. Related Results

In this section, we state a number of results that we make use of in establishing our theorems. At the end of this section, we also prove some crucial results which will be used to settle the maximum number of cut vertices of $k$-$\gamma_c$-critical graphs.
in Section 4. We begin with a result of Chartrand and Oellermann [2] which gives the relationship between the numbers of end blocks and cut vertices.

**Lemma 6** (see [2], page 24). Let $G$ be a connected graph with at least one cut vertex. Then $G$ has at least two end blocks.

In [3], Chen et al. established fundamental properties of $k$-$\gamma_c$-critical graphs.

**Lemma 7** [3]. Let $G$ be a $k$-$\gamma_c$-critical graph and let $x$ and $y$ be a pair of non-adjacent vertices of $G$. Further, let $D_{xy}$ be a $\gamma_c$-set of $G + xy$. Then

1. $k - 2 \leq |D_{xy}| \leq k - 1$,
2. $D_{xy} \cap \{x, y\} \neq \emptyset$, and
3. if $\{x\} = \{x, y\} \cap D_{xy}$, then $N_G(y) \cap D_{xy} = \emptyset$.

In [5], we further observed some structure of the subgraph of $G$ (not $G + xy$) induced by $D_{xy}$. For completeness, we provide the proof.

**Observation 8.** If $\{x, y\} \subseteq D_{xy}$, then $G[D_{xy}]$ consists of 2 components and each of which contains exactly one vertex of $\{x, y\}$.

**Proof.** If $G[D_{xy}]$ is connected, then $D_{xy}$ is a connected dominating set of $G$. It follows by Lemma 7(1) that $\gamma_c(G) \leq k - 1$, contradiction. Thus $G[D_{xy}]$ is not connected. As $(G + xy)[D_{xy}]$ is connected and $xy$ is the only one edge which is added to $G$, it follows that $xy$ is a bridge of $(G + xy)[D_{xy}]$. Therefore, $G[D_{xy}]$ has exactly 2 components and each of which contains exactly one vertex of $\{x, y\}$. This completes the proof.

When $k \geq 3$, Ananchuen [1] established structures of $k$-$\gamma_c$-critical graphs with a cut vertex.

**Lemma 9** [1]. For $k \geq 3$, let $G$ be a $k$-$\gamma_c$-critical graph with a cut vertex $c$ and let $D$ be a connected dominating set. Then

1. $G - c$ contains exactly two components,
2. if $C_1$ and $C_2$ are the components of $G - c$, then $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete and
3. $c \in D$.

In our previous work in [6], we established the diameter of $k$-$\gamma_c$-critical graphs.

**Lemma 10** [6]. Let $G$ be a $k$-$\gamma_c$-critical graph. Then $\text{diam}(G) \leq k$.

We conclude this section by establishing a forbidden subgraph of $k$-$\gamma_c$-critical graphs when $k \geq 3$ in Lemma 12. We also need to prove the following lemma.
Lemma 11. Let $G$ be a $k$-$\gamma_c$-critical graph and let $x$ and $y$ be a pair of non-adjacent vertices of $G$ such that $|D_{xy} \cap \{x, y\}| = 1$. Then, for a pair of vertices $a$ and $b$ in $D_{xy}$, we have that $N(a) \not\subseteq N[b]$. 

Proof. Suppose to the contrary that $N(a) \subseteq N[b]$ for some $a, b \in D_{xy}$.

Claim. $D_{xy} \setminus \{a\} \succ c a$.

Proof. As $|D_{xy} \cap \{x, y\}| = 1$, we must have $G[D_{xy}]$ is connected. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $G[D_{xy} \setminus \{a\}]$ is connected. We next show that $D_{xy} \setminus \{a\} \succ a$. As $G[D_{xy}]$ is connected, $a$ must be adjacent to a vertex in $D_{xy}$. That is $D_{xy} \setminus \{a\} \succ a$. Therefore $D_{xy} \setminus \{a\} \succ c a$. This settles the claim.

Since $|D_{xy} \cap \{x, y\}| = 1$, we may assume without loss of generality that $\{x\} = D_{xy} \cap \{x, y\}$. We distinguish two cases.

Case 1. $a \neq x$. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $D_{xy} \setminus \{a\} \succ V(G + xy) \setminus \{y, a\}$. Thus, by the claim, we have $D_{xy} \setminus \{a\} \succ c G + xy$. This contradicts the minimality of $D_{xy}$. So Case 1 cannot occur.

Case 2. $a = x$. As $N(a) \subseteq N[b]$, we must have $D_{xy} \setminus \{a\} \succ V(G + xy) \setminus \{y, a\}$. By the claim, $D_{xy} \setminus \{a\} \succ V(G) \setminus \{y\}$. Because $G$ is connected, it follows that $N(y) \neq \emptyset$. Let $z \in N(y)$. By Lemma 7(3), $z \notin D_{xy}$. As $D_{xy} \succ G + xy$, we must have that $z$ is adjacent to a vertex in $D_{xy}$. If $za \notin E(G)$, then $(D_{xy} \setminus \{a\}) \cup \{z\} \succ c G$. Lemma 7(1) implies that $|(D_{xy} \setminus \{a\}) \cup \{z\}| \leq k - 1$ contradicting the minimality of $\gamma_c(G)$. Therefore, $za \in E(G)$. As $N(a) \subseteq N[b]$, we must have $zb \in E(G)$. Since $b \in D_{xy}$, it follows that $(D_{xy} \setminus \{a\}) \cup \{z\} \succ c G$. Similarly, $|(D_{xy} \setminus \{a\}) \cup \{z\}| \leq k - 1$, a contradiction. So Case 2 cannot occur and this completes the proof.

We are ready to provide the construction of a forbidden subgraph of $k$-$\gamma_c$-critical graphs. For a connected graph $G$, let $X, Y, X_1$ and $Y_1$ be disjoint vertex subsets of $V(G)$. We, further, let $Z = X \cup X_1 \cup Y \cup Y_1$ and $\overline{Z} = V(G) \setminus Z$. The induced subgraph $G[Z]$ is called a bad subgraph if

(i) $x_1 \succ X \cup X_1$ for any vertex $x_1 \in X_1$,
(ii) $N[x] \subseteq X \cup X_1$ for any vertex $x \in X$,
(iii) $y_1 \succ Y \cup Y_1$ for any vertex $y_1 \in Y_1$, and
(iv) $N[y] \subseteq Y \cup Y_1$ for any vertex $y \in Y$.

Figure 1 illustrates our set up.

Observe that $G[X_1]$ and $G[Y_1]$ are complete subgraphs. Further, if $\overline{Z} = \emptyset$, then there exists an edge $x_1y_1$ where $x_1 \in X_1$ and $y_1 \in Y_1$ because $G$ is connected. Thus $\{x_1, y_1\} \succ c G$. This implies that $\gamma_c(G) \leq 2$. Therefore, if $\gamma_c(G) \geq 3$, then
$Z \neq \emptyset$. The next lemma gives that every $k$-$\gamma_c$-critical graph has no bad subgraph as an induced subgraph.

![Figure 1. The induced subgraph $G[Z]$.](image)

**Lemma 12.** For $k \geq 3$, let $G$ be a $k$-$\gamma_c$-critical graph. Then $G$ does not contain a bad subgraph as an induced subgraph.

**Proof.** Suppose to the contrary that $G$ contains $G[Z]$ as a bad subgraph. Let $x \in X$ and $y \in Y$. Consider $G + xy$. Lemma 7(2) implies that $D_{xy} \cap \{x, y\} \neq \emptyset$.

We first show that $\{x, y\} \subseteq D_{xy}$. Suppose to the contrary that $|D_{xy} \cap \{x, y\}| = 1$. Without loss of generality let $\{x\} = D_{xy} \cap \{x, y\}$. Since $x$ is not adjacent to any vertex in $Y_1$, in order to dominate $Y_1$, $D_{xy} \cap (V(G) - X) \neq \emptyset$. Because $N[x] \subseteq X \cup X_1$, by the connectedness of $(G + xy)[D_{xy}]$, $D_{xy} \cap X_1 \neq \emptyset$.

Let $x_1 \in D_{xy} \cap X_1$. Thus $N(x) \subseteq N[x_1]$ contradicting Lemma 11. Hence $\{x, y\} \subseteq D_{xy}$.

By Observation 8, $G[D_{xy}]$ has exactly two components $H_1$ and $H_2$ containing $x$ and $y$, respectively. Let

$$U_1 = N(H_1) - V(H_1) \text{ and } U_2 = N(H_2) - V(H_2).$$

Thus $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$ because $D_{xy} = V(H_1) \cup V(H_2)$ and $D_{xy} \supseteq G + xy$. We next establish the following claim.

**Claim.** For a vertex $u \in V(H_1) \cup U_1$, if $(V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset$, then $V(H_1 - x) \cup \{u\} \supseteq U_1 \cup \{x\}$. Similarly, for a vertex $v \in V(H_2) \cup U_2$, if $(V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset$, then $V(H_2 - y) \cup \{v\} \supseteq U_2 \cup \{y\}$.

**Proof.** Suppose that there exists $x_1 \in (V(H_1) \cup \{u\}) \cap X_1$. By Property (i) of bad subgraph, $x_1 \supset X \cup X_1$. Hence, $N[x] \subseteq N[x_1]$. Clearly, $G[V(H_1) \cup \{u\}]$ is connected. Since $x_1 \in V(H_1 - x) \cup \{u\}$, it follows that $G[V(H_1 - x) \cup \{u\}]$ is connected. As $N[x] \subseteq N[x_1]$, we must have $V(H_1 - x) \cup \{u\} \supseteq x$. Thus, it remains to show that $V(H_1 - x) \cup \{u\} \supseteq U_1$. Let $w \in U_1$. So, $w$ is adjacent to a vertex of $H_1$. If $wx \notin E(G)$, then $w$ is adjacent to a vertex of $H_1 - x$. But, if $wx \in E(G)$, then $wx_1 \in E(G)$. These imply that $w$ is adjacent to a vertex in


\( V(H_1 - x) \cup \{u\} \). So \( V(H_1 - x) \cup \{u\} \succ_U U_1 \). Therefore, \( V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\} \). We can show that if \( (V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset \), then \( V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\} \) by the similar arguments. This settles the claim. \( \Box \)

We note by the claim that \( u \) can be a vertex in \( H_1 \). Thus if \( V(H_1) \cap X_1 \neq \emptyset \), then \( V(H_1 - x) \succ_c U_1 \cup \{x\} \). Clearly \( Z \neq \emptyset \) because \( k \geq 3 \). To dominate \( Z \), we have \( D_{xy} \cap (Z \cup X_1 \cup Y_1) \neq \emptyset \) because \( N[x] \subseteq X \cup X_1 \) and \( N[y] \subseteq Y \cup Y_1 \). Thus, by the connectedness of \( H_1 \) and \( H_2 \), \( V(H_1) \cap X_1 \neq \emptyset \) or \( V(H_2) \cap Y_1 \neq \emptyset \). Suppose without loss of generality that \( V(H_1) \cap X_1 \neq \emptyset \). By applying the claim, we have that

\[
(1) \quad V(H_1 - x) \succ_c U_1 \cup \{x\}.
\]

**Case 1.** \( U_1 \cap U_2 \neq \emptyset \). Thus there is a vertex \( v \in V(G) - (V(H_1) \cup V(H_2)) \) such that \( v \) is adjacent to a vertex of \( H_1 \) and a vertex of \( H_2 \). That is \( G[V(H_1) \cup \{v\} \cup V(H_2)] \) is connected.

We next show that \( (V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset \). Suppose that \( (V(H_2) \cup \{v\}) \cap Y_1 = \emptyset \). By the connectedness of \( H_2 \), \( V(H_2) \subseteq Y \) because \( y \in Y \). Moreover, \( v \in Y \) because \( v \) is adjacent to a vertex of \( H_2 \). So, Property (iv) implies that \( N[v] \subseteq Y \cup Y_1 \). As \( v \) is adjacent to a vertex of \( H_1 \), we must have that \( V(H_1) \cap (Y \cup Y_1) \neq \emptyset \). By the connectedness of \( H_1 \), \( V(H_1) \cap Y_1 \neq \emptyset \). Property (iii) yields that there exists a vertex of \( H_1 \) adjacent to a vertex of \( H_2 \). So \( H_1 \) and \( H_2 \) are the same component, a contradiction. Hence \( (V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset \). By the claim, we have that

\[
(2) \quad V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}.
\]

Since \( V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2) \), by (1) and (2), \( V(H_1 - x) \cup V(H_2 - y) \cup \{v\} \succ_c G \). Hence

\[
(D_{xy} \setminus \{x, y\}) \cup \{v\} = V(H_1 - x) \cup V(H_2 - y) \cup \{v\} \succ_c G.
\]

Lemma 7(1) yields that \(|(D_{xy} \setminus \{x, y\}) \cup \{v\}| \leq k - 1 \) contradicting \( \gamma_c(G) = k \). So Case 1 cannot occur.

**Case 2.** \( U_1 \cap U_2 = \emptyset \). Since \( G \) is connected, there exist vertices \( u \) and \( v \) in \( V(G) - (V(H_1) \cup V(H_2)) \) such that \( u \in U_1 \), \( v \in U_2 \) and \( uv \in E(G) \). Therefore \( G[V(H_1) \cup \{u\} \cup V(H_2)] \) is connected.

We will show that \( (V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset \) and \( (V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset \). Suppose to the contrary that \( (V(H_1) \cup \{u\}) \cap X_1 = \emptyset \). So \( V(H_1) \cup \{u\} \subseteq X \) by the connectedness of \( G[V(H_1) \cup \{u\}] \). Since \( H_1 \) and \( H_2 \) are different components, by Property (i), \( V(H_2) \cap X_1 = \emptyset \). Thus \( v \in X_1 \) because \( uv \in E(G) \) and \( N[u] \subseteq X \cup X_1 \). This implies by Property (i) that \( v \succ H_1 \), in particular \( v \in U_1 \). Thus \( v \in U_1 \cap U_2 \). This contradicts \( U_1 \cap U_2 = \emptyset \). Hence, \( (V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset \). By the same arguments, we have \( (V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset \).
Hence, by the claim, we have that $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$ and $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$. As $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$, we must have that $(D_{xy} - \{x, y\}) \cup \{u, v\} \succ_c G$. Lemma 7(1) gives that $|(D_{xy} - \{x, y\}) \cup \{u, v\}| \leq k - 1$ contradicting $\gamma_c(G) = k$. So Case 2 cannot occur. Therefore $G$ does not contain a bad subgraph as an induced subgraph. This completes the proof. 

By applying Lemma 12, we easily establish the maximum number of end vertices of $k$-$\gamma_c$-critical graphs.

**Corollary 13** [8]. For $k \geq 3$, every $k$-$\gamma_c$-critical graph has at most one end vertex.

**Proof.** Suppose to the contrary that $G$ has $x$ and $y$ as two end vertices. Let $x_1$ and $y_1$ be the support vertices adjacent to $x$ and $y$, respectively. Thus $x_1$ and $y_1$ are cut vertices. Since $\gamma_c(G) \geq 3$, $V(G) - \{x_1, x, y, y_1\} \neq \emptyset$. Thus, Lemma 9(1) implies that $x_1 \neq y_1$. Choose $X_1 = \{x_1\}, Y_1 = \{y_1\}, X = \{x\}$ and $Y = \{y\}$. Clearly $G[X_1 \cup Y_1 \cup X \cup Y]$ is a bad subgraph contradicting Lemma 12. Hence, $G$ has at most one end vertex and this completes the proof. 

It is worth noting that very recently Taylor and van der Merwe [8] proved Corollary 13 as well. They proved the corollary with contrapositive but did not apply the concept of a bad subgraph in their proof.

3. The Characterizations of Some end Blocks

In this section, we provide characterizations of some blocks of $k$-$\gamma_c$-critical graphs. For a connected graph $G$, we let $\mathcal{A}(G)$ be the set of all cut vertices of $G$.

We first show that for a connected graph $G$ and a pair of non-adjacent vertices $x$ and $y$ of $G$, $\mathcal{A}(G) = \mathcal{A}(G + xy)$ if $x$ and $y$ are in the same block of $G$.

**Lemma 14.** For a connected graph $G$, let $B$ be a block of $G$ and $x, y \in V(B)$ such that $xy \notin E(G)$. Then $\mathcal{A}(G) = \mathcal{A}(G + xy)$.

**Proof.** Since $G$ is a subgraph of $G + xy$, $\mathcal{A}(G + xy) \subseteq \mathcal{A}(G)$. Suppose there exists $c$ such that $c \in \mathcal{A}(G)$ but $c \notin \mathcal{A}(G + xy)$. Thus $(G + xy) - c$ is connected. Let $C$ be the component of $G - c$ containing vertices of $V(B) - \{c\}$ and $C'$ be a component of $G - c$ which is not $C$. Further, let $a \in N_{C'}(c)$ and $b \in N_{C}(c)$. Since $c$ is a cut vertex of $G$, there is only one path $a, c, b$ from $a$ to $b$. But $c$ is not a cut vertex in $G + xy$. This implies that $G - c$ has a path $P = p_1, p_2, \ldots, x, y, \ldots, p_r$ from $b$ to $a$ where $b = p_1, a = p_r, x = p_i$ and $y = p_{i+1}$ for some $1 \leq i \leq r - 1$ and $r \geq 2$. We see that $P$ must contain an edge $xy$ and $c \notin \{p_1, p_2, \ldots, p_r\}$. Since $C$ and $C'$ are the two different components of $G - c$, by the connectedness of the path $P$, $\{p_1, p_2, \ldots, p_i\} \subseteq V(C)$ and $\{p_{i+1}, \ldots, p_r\} \subseteq V(C')$. So $x \in V(C)$ and $y \in V(C')$.
contradicting \( x \) and \( y \) are in the same block. Therefore \( A(G) \subseteq A(G + xy) \) and thus, \( A(G) = A(G + xy) \). This completes the proof. \( \blacksquare \)

For a \( k \)-\( \gamma_c \)-critical graph \( G \) with a cut vertex, let \( B \) be an end block of \( G \) containing non-adjacent vertices \( x \) and \( y \). Clearly, \( V(B) = V(B + xy) \).

**Lemma 15.** For an integer \( k \geq 3 \), let \( G \) be a \( k \)-\( \gamma_c \)-critical graph with a \( \gamma_c \)-set \( D \) and let \( B \) be an end block of \( G \). For all \( x, y \in V(B) \) such that \( xy \notin E(G) \), \( |D_{xy} \cap V(B + xy)| < |D \cap V(B)| \).

**Proof.** Let \( c \) be the cut vertex of \( G \) such that \( A(G) \cap V(B) = \{c\} \). Note that \( D - V(B) \) and \( D \cap V(B) \) are disjoint as well as \( D_{xy} - V(B + xy) \) and \( D_{xy} \cap V(B + xy) \). We first establish the following claim.

**Claim.** \( |D_{xy} - V(B + xy)| \geq |D - V(B)| \).

**Proof.** Suppose to the contrary that \( |D_{xy} - V(B + xy)| < |D - V(B)| \). Clearly, \( |D_{xy} - V(B + xy)| = |D_{xy} - V(B)| \). Thus \( |D_{xy} - V(B)| < |D - V(B)| \). We will show that \( (D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G \). Firstly, we show that \( G[(D_{xy} - V(B)) \cup (D \cap V(B))] \) is connected. As \( D_{xy} \) is a \( \gamma_c \)-set of \( G + xy \), we must have that \( (G + xy)[D_{xy}] \) is connected. Since \( xy \in E(B + xy) \), we have that \( G[(D_{xy} - V(B + xy)) \cup \{c\}] \) is connected. Hence, \( G[(D_{xy} - V(B)) \cup \{c\}] \) is connected. Clearly, \( G[D \cap V(B)] \) is connected. Moreover, \( c \in D \cap V(B) \) by Lemma 9\( (3) \). Thus \( G[(D_{xy} - V(B)) \cup (D \cap V(B))] \) is connected.

We next show that \( (D_{xy} - V(B)) \cup (D \cap V(B)) \succ G \). Because \( D_{xy} \succ_c G + xy \) and \( xy \in E(B + xy) \), it follows that \( (D_{xy} - V(B)) \cup \{c\} \succ V(G) - V(B) \). It is easy to see that \( D \cap V(B) \succ V(B) \). So \( (D_{xy} - V(B)) \cup (D \cap V(B)) \succ G \). This implies that \( (D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G \). But

\[
|(D_{xy} - V(B)) \cup (D \cap V(B))| \leq |(D_{xy} - V(B))| + |(D \cap V(B))|
\]
\[
< |(D - V(B))| + |(D \cap V(B))|
\]
\[
= |(D - V(B)) \cup (D \cap V(B))| = |D|,
\]

contradicting the minimality of \( D \). Therefore \( |D_{xy} - V(B + xy)| \geq |D - V(B)| \) and this settles the claim. \( \square \)

We are now ready to prove this lemma. Suppose to the contrary that \( |D_{xy} \cap V(B + xy)| \geq |D \cap V(B)| \). Thus

\[
|D_{xy}| = |(D_{xy} - V(B + xy)) \cup (D_{xy} \cap V(B + xy))|
\]
\[
= |(D_{xy} - V(B + xy))| + |(D_{xy} \cap V(B + xy))|
\]
\[
\geq |(D - V(B))| + |(D_{xy} \cap V(B + xy))| \quad \text{(by the claim)}
\]
\[
\geq |(D - V(B))| + |(D \cap V(B))|
\]
\[
= |(D - V(B)) \cup (D \cap V(B))| = |D|,
\]
contradicting Lemma 7(1). Therefore, \(|D_{xy} \cap V(B + xy)| < |D \cap V(B)|\) and this completes the proof.

We now introduce four classes of graphs such that some graph in these classes is an end block of a \(k\)-\(γ_c\)-critical graph. For vertices \(c, z_1\) and \(z_2\), we let
\[
B_0 = \{c \lor K_{t_1} : \text{for an integer } t_1 \geq 1\},
\]
\[
B_1 = \{c \lor K_{t_2} \lor z_1 : \text{for an integer } t_2 \geq 1\}, \text{ and}
\]
\[
B_{2,1} = \{c \lor K_{t_3} \lor K_{t_4} \lor z_2 : \text{for integers } t_3, t_4 \geq 1\}.
\]

Before we construct the next class, it is worth to introduce a graph \(T\) which occurs in the characterization of \(k\)-\(γ_c\)-critical graphs with a maximum number of cut vertices. For positive integers \(l \geq 2, r\) and \(n_i\), we let \(S = \bigcup_{i=1}^{l} K_{1,n_i}\) and
\[
T = S \quad \text{or} \quad T = S \cup K_r.
\]

Then, for \(1 \leq i \leq l\), we let \(s^i_0, s^i_1, s^i_2, \ldots, s^i_{n_i}\) be the vertices of a star \(K_{1,n_i}\) centered at \(s^i_0\). We, further, let \(S = \bigcup_{i=1}^{l} \{s^i_1, s^i_2, \ldots, s^i_{n_i}\}\) and \(S' = \bigcup_{i=1}^{l} \{s^i_0\}\), moreover, let \(S'' = V(K_r)\) if \(T = S \cup K_r\) and \(S'' = \emptyset\) if \(T = S\). We note that
\[
T = S \quad \text{or} \quad T = S \cup K_r.
\]

That is, \(T\) can be obtained by removing the edges in the stars of \(S\) from a complete graph on \(S \cup S' \cup S''\). Throughout this paper, we are, in fact, using the complement of \(T\). We are ready to define the next class. Recall that, for graphs \(G_1\) and \(G_2\) such that \(G_2\) has \(H\) as a subgraph, the join \(G_1 \lor_H G_2\) is the graph constructed from the disjoint union of \(G_1\) and \(G_2\) by joining each vertex in \(G_1\) to each vertex in \(H\) with an edge.

\[
B_{2,2} = \{c \lor \Sigma|S| T : \text{for positive integers } l \geq 2, r\}.
\]

We note by the construction that, in \(T\), every vertex in \(S\) is adjacent to exactly \(|S' \cup S''| - 1\) vertices in \(S' \cup S''\). A graph in this class is illustrated in Figure 2.

According to the figure, an oval denotes a complete subgraph, double lines between subgraphs denote joining every vertex of one subgraph to every vertex of the other subgraph and a dash line denotes a removed edge.

It is worth noting that, for an end block \(B\) of a \(k\)-\(γ_c\)-critical graph having \(D\) as a \(γ_c\)-set, the number of vertices in \(D \cap V(B)\) can be as large as \(k\). We will give an example by using the graph \(T\). For an integer \(k \geq 5\), let \(K_{n_1}, \ldots, K_{n_k-3}\) be \(k - 3\) copies of complete graphs with \(n_1, \ldots, n_{k-3} \geq 2\) and let \(a_1\) and \(a_2\) be two isolated vertices. It is not difficult to see that the graph
\[a_1 \lor a_2 \lor K_{n_1} \lor \cdots \lor K_{n_{k-3}} \lor T[S]T\]

is a \(k\)-\(\gamma_c\)-critical graph having \(R = a_2 \lor K_{n_1} \lor \cdots \lor K_{n_{k-3}} \lor T[S]T\) as an end block. Clearly, \(|D \cap V(R)| = k\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{A graph \(G\) in the class \(B_{2,2}\).}
\end{figure}

In the following, we characterize an end block \(B\) such that \(|D \cap V(B)| \leq 3\).

Let \(c\) be the cut vertex of \(G\) in \(B\) and \(H\) be the component of \(G-c\) such that \(G[V(H) \cup \{c\}] = B\). We further let
\[W = N_H(c),\]
\[W' = \{w' \in V(H) - W : w'w \in E(G) \text{ for some } w \in W\}\]
\[W'' = V(H) - (W \cup W').\]

Note that \(W'\) or \(W''\) can be empty. Since \(c \in V(B)\), we have that \(|D \cap V(H)| = i\) if and only if \(|D \cap V(B)| = i+1\) for all \(i \geq 0\). Thus, \(|D \cap V(B)| \geq 1\).

**Lemma 16.** Let \(G\) be a \(k\)-\(\gamma_c\)-critical graph with a \(\gamma_c\)-set \(D\) and let \(B\) be an end block of \(G\). If \(|D \cap V(B)| = 1\), then \(B \in \mathcal{B}_0\).

**Proof.** In view of Lemma 9(2), \(G[W]\) is complete. Lemma 9(3) gives, further, that \(D \cap V(B) = \{c\}\). Since \(D > B\) and \(|(D \cap V(B)) - \{c\}| = 0\), it follows that \(W' \cup W'' = \emptyset\) and \(c \succ W\). So \(B \in \mathcal{B}_0\). This completes the proof.

**Lemma 17.** Let \(G\) be a \(k\)-\(\gamma_c\)-critical graph with a \(\gamma_c\)-set \(D\) and let \(B\) be an end block of \(G\). If \(|D \cap V(B)| = 2\), then \(B \in \mathcal{B}_1\).

**Proof.** Let \(\{y\} = (D \cap V(B)) - \{c\}\). By the connectedness of \(G[D]\), \(y \in W\). Thus \(W'' = \emptyset\) and \(V(H) = W \cup W'\). Suppose that there exist \(u, v \in V(H)\) such that \(uv \notin V(G)\). Consider \(G+uv\). Lemma 7(2) gives that \(D_{uv} \cap \{u, v\} \neq \emptyset\). Lemma 14 gives also that \(c \in D_{uv}\). Hence, \(|D_{uv} \cap V(B+uv)| \geq 2\) contradicting Lemma 15. Thus \(G[W \cup W']\) is complete. Let \(z_1 \in W'\). Consider \(G + cz_1\). Since \(|D \cap V(B)| = 2\), by Lemma 15, \(|D_{cz_1} \cap V(B + cz_1)| \leq 1\). Lemmas 9(3) and 14 yield that \(c \in D_{cz_1}\). So \(|D_{cz_1} \cap V(H)| = 0\). This implies that \(c \succ B + cz_1\). Since \(\{z_1\} = N_{G+cz_1}(c) \cap W', W' = \{z_1\}\). So \(B \in \mathcal{B}_1\) and this completes the proof.
Lemma 18. Let $G$ be a $k$-$\gamma_c$-critical graph with a $\gamma_c$-set $D$ and let $B$ be an end block of $G$. Suppose that $|D \cap V(B)| = 3$. Then $B \in B_{2,1}$ if $W'' \neq \emptyset$ and $B \in B_{2,2}$ if $W'' = \emptyset$. Consequently, $B \in B_{2,1} \cup B_{2,2}$.

Proof. Suppose that $|D \cap V(B)| = 3$. Lemma 9(2) implies that $G[W]$ is complete. We first establish the following claim.

Claim. For any non-adjacent vertices $u, v \in W \cup W' \cup W''$, we have $c \in D_{uv} \cap V(B + uv)$ and $|D_{uv} \cap W \cap \{u, v\}| = 1$.

Proof. Lemma 15 implies that $|D_{uv} \cap V(B + uv)| \leq 2$. In view of Lemmas 9(3) and 14, $c \in D_{uv} \cap V(B + uv)$. Thus $|D_{uv} \cap \{u, v\}| \leq 1$. Lemma 7(2) then gives that $|D_{uv} \cap \{u, v\}| = 1$. So $|D_{uv} \cap W \cap \{u, v\}| = 1$ because $(G + uv)[D_{uv}]$ is connected. This settles the claim.

Suppose there exist $u, v \in W' \cup W''$ such that $uv \notin E(G)$. Consider $G + uv$. By the claim $|D_{uv} \cap W \cap \{u, v\}| = 1$ contradicting $W \cap \{u, v\} = \emptyset$. Thus $G[W' \cup W'']$ is complete.

We first consider the case when $W'' \neq \emptyset$. Let $w \in W$ and $z_2 \in W''$. Consider $G + wz_2$. By the claim, $D_{wz_2} \cap V(B + wz_2) = \{c, w\}$. Since $\{z_2\} = W'' \cap N_{G+wz_2}(w)$, it follows that $W'' = \{z_2\}$. Suppose there exists $w' \in W'$ such that $ww' \notin E(G)$. Consider $G + ww'$. By the claim, $D_{ww'} \cap V(B + ww') = \{c, w\}$. Thus $D_{ww'}$ does not dominate $z_2$, a contradiction. Therefore $G[W \cup W']$ is complete and $B \in B_{2,1}$.

We finally consider the case when $W'' = \emptyset$. We will show that, for all $w \in W$, $|N_{W'}(w)| = |W'| - 1$. If $w \succ W'$, then $(D - V(H)) \cup \{w\} \succ G$. But $|(D - V(H)) \cup \{w\}| = k - 1$ contradicting $\gamma_c(G) = k$. Thus $|N_{W'}(w)| \leq |W'| - 1$. If $w$ is not adjacent to $x, y$ in $W'$, then consider $G + wx$. By the claim, $D_{wx} \cap V(B + wx) = \{c, w\}$. Clearly $D_{wx}$ does not dominate $y$, a contradiction. Hence, $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$. We now have that $G[W \cup W']$ is the complement of disjoint union of isolated vertices in $W'$ and stars whose centers are in $W'$ and all of end vertices are in $W$. It remains to show that there are at least two stars in $G[W \cup W']$. Suppose to the contrary that, in $G[W \cup W']$, there is exactly one star centered at $w'$. Because $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$, $w'$ is not adjacent to any vertex in $W$. So $w' \in W''$ contradicting $W'' = \emptyset$. Hence, there are at least two stars in $G[W \cup W']$. This completes the proof.

4. The Upper Bound of the Number of Cut Vertices

In this section, we establish the maximum number of cut vertices of $k$-$\gamma_c$-critical graphs. In view of Observation 2, it suffices to restrict our attention to the case $k \geq 3$. We begin this section by showing that $G$ does not have two end blocks in $B_0 \cup B_1$. 
Lemma 19. For \( k \geq 3 \), let \( G \) be a \( k-\gamma_c \)-critical graph. Then \( G \) contains at most one end block \( B \) such that \( B \in B_0 \cup B_1 \).

Proof. Suppose to the contrary that there exist two different end blocks \( U \) and \( R \) which are, respectively, in the classes \( B_i \) and \( B_j \) where \( \{i, j\} \subseteq \{0, 1\} \). Let \( u \) be the cut vertex of \( G \) in \( U \). If \( U \in B_0 \), then \( U = u \lor K_{t_1} \) for some integer \( t_1 \geq 1 \). If \( U \in B_1 \), then there exist an integer \( t_2 \geq 1 \) and a vertex \( z_1 \) of \( U \) such that \( U = u \lor K_{t_2} \lor z_1 \). Then, we choose

\[
X_1 = \begin{cases} 
\{u\} & \text{if } U \in B_0, \\
V(K_{t_2}) & \text{if } U \in B_1,
\end{cases}
\]

and we choose

\[
X = \begin{cases} 
V(K_{t_1}) & \text{if } U \in B_0, \\
\{z_1\} & \text{if } U \in B_1.
\end{cases}
\]

Clearly, \( U \) contains \( X \) and \( X_1 \) which satisfy the Properties (i) and (ii), respectively.

We now consider \( R \). Let \( r \) be the cut vertex of \( G \) in \( R \). If \( R \in B_0 \), then \( R = r \lor K_{t'_1} \) for some integer \( t'_1 \geq 1 \). But, if \( R \in B_1 \), then there exist an integer \( t'_2 \geq 1 \) and a vertex \( w_1 \) of \( R \) such that \( R = r \lor K_{t'_2} \lor w_1 \). Then, we choose

\[
Y_1 = \begin{cases} 
\{r\} & \text{if } R \in B_0, \\
V(K_{t'_2}) & \text{if } R \in B_1,
\end{cases}
\]

and we choose

\[
Y = \begin{cases} 
V(K_{t'_1}) & \text{if } R \in B_0, \\
\{w_1\} & \text{if } R \in B_1.
\end{cases}
\]

Clearly, \( R \) contains \( Y \) and \( Y_1 \) which satisfy the Properties (i) and (ii), respectively.

We observe that \( X, Y \) and \( Y_1 \) are pairwise disjoint because \( U \) and \( R \) are different blocks. Suppose that \( Y_1 \cap X_1 \neq \emptyset \). By the choice of \( X_1 \) and \( Y_1 \), if \( X_1 = V(K_{t_2}) \) or \( Y_1 = V(K_{t'_2}) \), then \( Y_1 \cap X_1 = \emptyset \) because \( U \) and \( R \) are different end blocks, contradicting the assumption that \( Y_1 \cap X_1 \neq \emptyset \). Hence, \( X_1 = \{u\} \) and \( Y_1 = \{r\} \). This implies that \( u = r \), moreover, \( U \) and \( R \) are both in \( B_0 \). Thus \( u \succ U \) and \( u \succ R \). Lemma 9(1) yields that \( G - u \) has \( U - u \) and \( R - u \) as the two components. We have that \( G = K_{t_1} \lor u \lor K_{t'_2} \). Clearly, \( u \succ_c G \) contradicting \( \gamma_c(G) \geq 3 \). Hence, \( Y_1 \cap X_1 = \emptyset \). So, \( G \) contains a bad subgraph contradicting Lemma 12. This completes the proof. \( \blacksquare \)

In the following, for a block \( B \) of \( G \), we let

\[
\mathcal{A}(B) = V(B) \cap A(G).
\]
We also let
\[ \zeta(G) = |\mathcal{A}(G)|, \zeta(B) = |\mathcal{A}(B)| \text{ and} \]
\[ \zeta_0(G) = \max \{\zeta(B) : B \text{ is a block of } G\}. \]
When no ambiguity can occur, we abbreviate \( \zeta_0(G) \) to \( \zeta_0 \). Clearly, \( \zeta_0 \leq \zeta(G) \). In the following lemma, we establish the existence of \( \zeta_0 \) end blocks.

**Lemma 20.** For any \( k-\gamma_c \)-critical graph \( G \), let \( B_0 \) be a block of \( G \) containing \( \zeta_0 \) cut vertices \( c_1, c_2, \ldots, c_{\zeta_0} \). Then there exist mutually disjoint end blocks \( B_1, B_2, \ldots, B_{\zeta_0} \).

**Proof.** In view of Lemma 9(1), \( G - c_i \) has only two components for \( 1 \leq i \leq \zeta_0 \). Let \( C_i \) be the component of \( G - c_i \) that does not contain any vertex of \( B_0 \). If graph \( G[[c_i] \cup V(C_i)] \) does not contain any cut vertex, then \( G[[c_i] \cup V(C_i)] \) is an end block and we let \( B_i = G[[c_i] \cup V(C_i)] \). If graph \( G[[c_i] \cup V(C_i)] \) contains a cut vertex, then, by Lemma 6, \( G[[c_i] \cup V(C_i)] \) has at least two end blocks. Therefore, at least one end block of \( G[[c_i] \cup V(C_i)] \) does not contain \( c_i \) and we let \( B_i \) be this end block. In both cases of the choice, \( B_i \) is an end block of \( G \). Obviously, \( B_1, B_2, \ldots, B_{\zeta_0} \) are mutually disjoint and this completes the proof. ■

**Lemma 21.** For \( k \geq 3 \), let \( G \) be a \( k-\gamma_c \)-critical graph with a \( \gamma_c \)-set \( D \) and let \( B_1, B_2, \ldots, B_{\zeta_0} \) be the end blocks of \( G \) from Lemma 20. Moreover, for \( 1 \leq i \leq \zeta_0 \), we let \( x_i \in \mathcal{A}(G) \cap V(B_i) \). Then at least \( \zeta_0 - 1 \) of the end blocks \( B_1, B_2, \ldots, B_{\zeta_0} \) satisfy \( |(D \cap V(B_i)) - \{x_i\}| \geq 2 \).

**Proof.** Lemma 19 gives that at least \( \zeta_0 - 1 \) blocks of \( \{B_i|1 \leq i \leq \zeta_0\} \) are not in \( B_j \) where \( j \in \{0,1\} \). Without loss of generality let \( B_1, B_2, \ldots, B_{\zeta_0} \) be such blocks. Hence
\[ |(D \cap V(B_i)) - \{x_i\}| \geq 2 \]
for \( 1 \leq i \leq \zeta_0 - 1 \) and this completes the proof. ■

We next let \( \overline{A} = \mathcal{A}(G) - \mathcal{A}(B_0) \) and \( \overline{\zeta} = |\overline{A}| \). That is, \( \overline{A} \) is the set of cut vertices which are not in \( B_0 \). Clearly,
\[ \zeta(G) = \overline{\zeta} + \zeta_0. \]
Recall that, for \( 1 \leq i \leq \zeta_0 \), \( C_i \) is the component of \( G - c_i \) which does not contain any vertex of \( B_0 \). We also let \( j_0 = \min \{|D \cap V(C_i)| : \text{for all } 1 \leq i \leq \zeta_0\} \).

The following theorem gives the relationship of \( \zeta_0, \overline{\zeta}, j_0 \) and \( k \).

**Theorem 22.** For \( k \geq 3 \), let \( G \) be a \( k-\gamma_c \)-critical graph. Then \( 3\zeta_0 - 2 + \overline{\zeta} + j_0 \leq k \).
Proof. Lemma 9(3) yields that $A(G) \subseteq D$. For each end block $B_i$ of $G$ which is a consequent of Lemma 20, $1 \leq i \leq \zeta_0$, let $x_i \in A(G) \cap V(B_i)$. Clearly $(D \cap V(B_1)) - \{x_1\}, (D \cap V(B_2)) - \{x_2\}, \ldots, (D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}$ and $A(G)$ are pairwise disjoint. These imply that

\[ \sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \leq k. \]

(4)

In view of Lemma 21, at least $\zeta_0 - 1$ end blocks of $B_1, B_2, \ldots, B_{\zeta_0}$ are not in $B_0 \cup B_1$. Without loss of generality let $B_1, B_2, \ldots, B_{\zeta_0-1}$ be such blocks. So $2 \leq |(D \cap V(B_i)) - \{x_i\}|$ for $1 \leq i \leq \zeta_0 - 1$. Therefore

\[ 2(\zeta_0 - 1) \leq \sum_{i=1}^{\zeta_0-1} |(D \cap V(B_i)) - \{x_i\}|. \]

(5)

By the minimality of $j_0$,

\[ 0 \leq j_0 \leq |(D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}|. \]

(6)

Therefore

\[ 3\zeta_0 - 2 + j_0 + \zeta = 2(\zeta_0 - 1) + j_0 + \zeta + \zeta_0 \leq \sum_{i=1}^{\zeta_0-1} |(D \cap V(B_i)) - \{x_i\}| + j_0 + \zeta(G) \quad \text{(by (3) and (5))} \]

\[ \leq \sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \quad \text{(by (6))} \]

\[ \leq k \quad \text{(by (4))}, \]

as required. \[ \blacksquare \]

Theorem 22 implies the following corollary.

Corollary 23. For $k \geq 3$, let $G$ be a $k$-$\gamma_c$-critical graph. Then $\zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor$.

Proof. Theorem 22 implies that $3\zeta_0 \leq k + 2 - \zeta - j_0$. As $\zeta, j_0 \geq 0$, we must have that

\[ \zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor \]

and this completes the proof. \[ \blacksquare \]
Note that Theorem 22 together with \( \zeta(G) = \bar{\zeta} + \zeta_0 \) give
\[
2\zeta_0 \leq k - \zeta(G) - j_0 + 2. \tag{7}
\]
We are now ready to establish Theorem 5. For completeness, we recall the statement of this theorem.

**Theorem 5.** For \( k \geq 3 \), let \( G \) be a \( k-\gamma_c \)-critical graph with \( \zeta(G) \) cut vertices. Then \( \zeta(G) \leq k - 2 \).

**Proof.** Suppose to the contrary that \( |A(G)| > k - 2 \). Lemma 9(3) gives that \( |A(G)| \leq k \). Thus either \( |A(G)| = k \) or \( |A(G)| = k - 1 \), in particular, \( k - \zeta(G) \leq 1 \). This implies by (7) that
\[
2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq 3.
\]
Therefore
\[
\zeta_0 \leq 1.
\]
If \( \zeta(G) \geq 2 \), then we always have a block containing more than one cut vertex. Thus \( \zeta_0 \geq 2 \), a contradiction. Therefore \( \zeta(G) \leq 1 \). As \( k - \zeta(G) \leq 1 \), we must have that
\[
k \leq 2,
\]
contradicting \( k \geq 3 \). Hence \( \zeta(G) \leq k - 2 \) and this completes the proof. \( \blacksquare \)

5. **Characterizations**

In this section, we characterize all \( k-\gamma_c \)-critical graphs \( G \) when \( \zeta(G) = k - 2 \). We first give the construction of a \( k-\gamma_c \)-critical graph with \( k - 2 \) cut vertices.

**The class \( \mathcal{F}(k) \)**

Let \( B \) be a graph in the class \( \mathcal{B}_{2,2} \) containing \( c, S, S' \) and \( S'' \) which are defined in \( \mathcal{B}_{2,2} \). We, further, let \( P_{k-1} = z_0, z_1, \ldots, z_{k-2} \) be a path of order \( k - 1 \). A graph \( G \) in the class \( \mathcal{F}(k) \) is constructed from the graphs \( B \) and \( P_{k-1} \) by identifying \( z_{k-2} \) with \( c \). A graph \( G \) in the class \( \mathcal{F}(k) \) is illustrated in Figure 3.

![Figure 3: A graph \( G \) in the class \( \mathcal{F}(k) \).](image-url)
Lemma 24. Let $G \in \mathcal{F}(k)$. Then $G$ is a $k$-$\gamma_c$-critical graph with $k - 2$ cut vertices.

Proof. Clearly, $z_1, z_2, \ldots, z_{k-2}$ are the $k - 2$ cut vertices of $G$. We observe that $\{z_1, z_2, \ldots, z_{k-2}, s_1, s_2\} \triangleright_c G$. Therefore $\gamma_c(G) \leq k$.

We next show that $\gamma_c(G) \geq k$. Let $D$ be a $\gamma_c$-set of $G$. Since $z_1$ is a cut vertex, by the connectedness of $G[D]$, $z_1 \in D$. We first suppose that $D \cap S'' \neq \emptyset$. As $z_1 \in D$, by the connectedness of $G[D]$, we must have $\{z_2, z_3, \ldots, z_{k-2}, y\} \subseteq D$ where $y \in D \cap S$. Thus $\gamma_c(G) = |D| \geq k$ and $\gamma_c(G) = k$. We now suppose that $D \cap S'' = \emptyset$. To dominate $B$, $|D \cap (S \cup S')| \geq 2$. Similarly, by the connectedness of $G[D]$, we have $\{z_2, z_3, \ldots, z_{k-2}\} \subseteq D$ and thus $\gamma_c(G) \geq k$. Therefore $\gamma_c(G) = k$.

We next establish the criticality. Let $u$ and $v$ be two non-adjacent vertices of $G$ and $S_1 = S \cup S' \cup S''$. We first consider the case when $\{u, v\} \subseteq S_1$. Thus $\{u, v\} = \{s_1, s_2\}$ for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, n_i\}$. Clearly $\{z_1, z_2, \ldots, z_{k-2}, s_1\} \triangleright_c G + uv$ and $\gamma_c(G + uv) \leq k - 1$.

We now consider the case when $|\{u, v\} \cap S_1| = 1$. Without loss of generality let $\{v\} = \{u, v\} \cap S_1$. If $u = z_{k-2}$, then $v \notin S_1$. Thus $\{z_1, z_2, \ldots, z_{k-2}, v\} \triangleright_c G + uv$. Therefore $\gamma_c(G + uv) \leq k - 1$. Since $l \geq 2$, there exists $v' \in S - \{v\}$ such that $\{v, v'\} \triangleright_c S_1$. Then, if $u \in \{z_1, z_2, \ldots, z_{k-3}\}$, we have $\{z_1, z_2, \ldots, u, \ldots, z_{k-3}, v, v'\} \triangleright_c G + uv$. Hence $\gamma_c(G + uv) \leq k - 1$. If $u = z_0$, then $\{z_2, z_3, \ldots, z_{k-2}, v, v'\} \triangleright_c G + uv$ and thus, $\gamma_c(G + uv) \leq k - 1$.

We finally consider the case when $|\{u, v\} \cap S_1| = 0$. Therefore $\{u, v\} \subseteq \{z_0, z_1, \ldots, z_{k-2}\}$. Thus $u = z_i$ and $v = z_j$ for some $i \neq j \in \{0, 1, 2, \ldots, k - 2\}$. Without loss of generality let $i < j$. Clearly $i + 2 \leq j$. Hence $(\{z_1, \ldots, z_{k-2}\} - \{z_i+1\}) \cup \{s_1, s_2\} \triangleright_c G + uv$.

So $\gamma_c(G + uv) \leq k - 1$. Thus $G$ is a $k$-$\gamma_c$-critical graph and this completes the proof.

Let $Z(k, \zeta)$ be the class of $k$-$\gamma_c$-critical graphs containing $\zeta$ cut vertices. As the graphs in these class have been characterized in [1] and [6] when $3 \leq k \leq 4$, we turn attention to the case when $k \geq 5$.

Lemma 25. For $k \geq 5$, let $G \in Z(k, \zeta)$ where $\zeta \in \{k - 3, k - 2\}$. Then $G$ has only two end blocks and the remaining blocks contain two cut vertices.

Proof. Clearly $\zeta(G) \geq k - 3$. We have by (7) that

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq k - (k - 3) - j_0 + 2 \leq 5.$$ 

That is $\zeta_0 \leq 2$. Lemma 9(1) implies that $G$ has only two end blocks and the other blocks contain two cut vertices. This completes the proof.
In view of Lemma 25, hereafter, $G$ has exactly two end blocks, $R_1, R_{k-1}$ say, and the other blocks $R_2, R_3, \ldots, R_{k-2}$ which contain two cut vertices. Without loss of generality let $z_1 \in V(R_1), z_{k-2} \in V(R_{k-1})$ and $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$ (see Figure 4).

**Lemma 26.** For $k \geq 5$, let $G \in \mathcal{Z}(k,k-2)$ and $R_1, R_{k-1}$ be two end blocks. Then $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2.$

![Figure 4. The structure of $G \in \mathcal{Z}(k,\zeta)$ where $\zeta \in [k-3,k-2]$.](image)

**Proof.** Lemma 9(3) yields that $\mathcal{A}(G) \subseteq D$. As $\zeta(G) = k-2$, we must have $|D - \mathcal{A}(G)| = 2$. Clearly $(D - \mathcal{A}(G)) \cap (V(R_1) \cup V(R_{k-1})) \neq \emptyset$, otherwise $R_1, R_{k-1} \in \mathcal{B}_0$ contradicting Lemma 19.

Without loss of generality let $|(D \cap V(R_1)) - \{z_1\}| \leq |(D \cap V(R_{k-1})) - \{z_{k-2}\}|.$ Suppose to the contrary that $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 1$. Thus $R_1, R_{k-1} \in \mathcal{B}_0 \cup \mathcal{B}_1$ contradicting Lemma 19. Hence $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$ and this completes the proof.

**Lemma 27.** For $k \geq 5$, let $G \in \mathcal{Z}(k,k-2)$ and $R_2, R_3, \ldots, R_{k-2}$ be blocks containing two cut vertices such that $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$. Then $\{z_{i-1}, z_i\} \triangleright_c R_i$ for $2 \leq i \leq k-1$, in particular, $z_{i-1}z_i \in E(G)$.

**Proof.** As $\zeta(G) = k-2$, by Lemma 26, we must have $D \cap V(R_i) = \{z_{i-1}, z_i\}$ for $2 \leq i \leq k-2$. Therefore $\{z_{i-1}, z_i\} \triangleright_c R_i$.

Clearly, $z_{i-1}z_i \in E(G)$ and this completes the proof.

**Lemma 28.** For $k \geq 5$, let $G \in \mathcal{Z}(k,k-2)$ and $R_i$ be a block of $G$ containing two cut vertices $z_{i-1}$ and $z_i$ for $2 \leq i \leq k-2$. Then $V(R_i) = \{z_{i-1}, z_i\}$.

**Proof.** By Lemma 26, $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Without loss of generality let $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Since $|D \cap \mathcal{A}(G)| = k-2$, $|(D \cap V(R_1)) - \{z_1\}| = 0$ and thus, Lemma 16 gives that $R_1 \in \mathcal{B}_0$.

We consider the case when $i = 2$. Let $z \in V(R_i) - \{z_1\}$. Suppose there exists $u \in V(R_2) - \{z_1, z_2\}$. Consider $G + uz$. We see that $z_2$ is a cut vertex of $G + uz$. 

Lemma 9(2) implies that \( z_2 \in D_u \). If \( |D_u \setminus V(R_1)| \leq k - 2 \), then, by Lemma 27, \( (D_u \setminus V(R_1)) \cup \{z_1\} \succ_G \) contradicting \( \gamma_c(G) = k \). Hence, by Lemma 7(1), \( |D_u \setminus V(R_1)| = k - 1 \). Since \( u \notin V(R_1) \), by Lemma 7(2), \( \{u\} = D_u \cap \{u, z\} \).

Lemma 27 implies that \( (D_u - \{u\}) \cup \{z_1\} \succ_G \). But \( |(D_u - \{u\}) \cup \{z_1\}| = k - 1 \) contradicting \( \gamma_c(G) = k \). Hence, \( V(R_2) = \{z_1, z_2\} \).

We consider the case when \( 3 \leq i \leq k - 2 \). Suppose to the contrary that \( R_i = V(R_i) - \{z_{i-1}, z_i\} \neq \emptyset \). Lemma 27 gives that \( \{z_{i-1}, z_i\} \succ_{R_i} \) and \( z_{i-1}z_i \in E(G) \). If there exists a vertex \( b' \in R_i \) which is not adjacent to \( z_j \) for some \( j \in \{i, i - 1\} \), then \( b'z_{2i-1-j} \in E(G) \). Note that \( b', z_j \in N_{R_i}(z_{2i-1-j}) \). Thus \( G[N_{R_i}(z_{2i-1-j})] \) is not a complete graph contradicting Lemma 9(2). Therefore, \( z_i \succ_{R_i} \) and \( z_{i-1} \succ_{R_i} \). We now have that \( z_i \succ V(R_i), z_{i-1} \succ V(R_i) \) and \( N[b'] \subseteq R_i \cup \{z_{i-1}, z_i\} \) for all \( b' \in R_i \). Moreover, we have that \( z_1 \succ R_1 \) and \( N[b] \subseteq V(R_1) \) for all \( b \in V(R_1) \).

Choose
\[
X_1 = \{z_1\}, X = V(R_1) - \{z_1\}, Y = R_i \setminus X, \text{ and } Y_1 = \{z_i, z_{i-1}\}.
\]

Clearly \( X, X_1, Y \) and \( Y_1 \) form a bad subgraph. This contradicts Lemma 12. Hence, \( R_i = \emptyset \) for all \( 2 \leq i \leq k - 3 \). This completes the proof.

The following theorem gives the characterization of the graphs in the class \( Z(k, k - 2) \).

**Theorem 29.** For \( k \geq 5 \), we have that \( Z(k, k - 2) = F(k) \).

**Proof.** Lemma 24 implies that \( F(k) \subseteq Z(k, k - 2) \). It suffices to show that a \( k - \gamma_c \)-critical graph with \( k - 2 \) cut vertices is in \( F(k) \). Let \( G \) be a \( k - \gamma_c \)-critical graph with \( k - 2 \) cut vertices. Lemma 25 implies that \( G \) has only two end blocks, \( R_1, R_{k-1} \) say, and the other blocks \( R_2, R_3, \ldots, R_{k-2} \) which contain two cut vertices. Let \( z_1 \in V(R_1), z_{k-2} \in V(R_{k-1}) \) and \( z_{i-1}, z_i \in V(R_i) \) for \( 2 \leq i \leq k - 2 \). Thus \( A(G) = \{z_1, z_2, \ldots, z_{k-2}\} \).

Lemma 26 implies that \( |(D \cap V(R_1)) - \{z_1\}| = 2 \) or \( |(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2 \). Without loss of generality let
\[
|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2.
\]

Thus \( |(D \cap V(R_1)) - \{z_1\}| = 0 \). By Lemma 16, \( R_1 \in B_0 \). Clearly, \( z_1 \succ R_1 \). As \( \gamma(G) = k - 2 \), we must have \( D \cap V(R_i) = \{z_{i-1}, z_i\} \) for \( 2 \leq i \leq k - 2 \) and \( D \cap V(R_1) = \{z_1\} \). By Lemma 27,
\[
\{z_{i-1}, z_i\} \succ_{R_i}.
\]

Let \( z_0 \in V(R_1) - \{z_1\} \). Clearly \( d(z_1, z_0) = 1 \). The following claim characterizes \( R_{k-1} \).

**Claim.** \( R_{k-1} \in B_{2,2} \).
Proof. Since $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$, there exists $w \in V(R_{k-1}) - \{z_{k-2}\}$ such that $d(w, z_{k-2}) \geq 2$. Thus

$$d(z_0, w) = d(z_0, z_1) + d(z_1, z_2) + \cdots + d(z_{k-2}, w) \geq k.$$ 

Lemma 10 gives that $d(z_0, w) = k$. Hence $d(z_{k-2}, w') \leq 2$ for all $w' \in V(R_{k-1}) - \{z_{k-2}\}$. So $R_{k-1} \notin B_{2,1}$. By Lemma 18, $R_{k-2} \in B_{2,2}$ and thus establishing the claim.

Lemma 28 implies that, for all $i \in \{2, 3, \ldots, k-2\}$, $V(R_i) = \{z_{i-1}, z_i\}$. So far, it remains to show that $V(R_1) = \{z_1, z_0\}$. Consider $G + z_2z_0$. Since $z_2$ is a cut vertex of $G + z_2z_0$, $z_2 \in D_{z_2z_0}$ by the connectedness of $(G + z_2z_0)[D_{z_2z_0}]$. We note by Lemma 27 that $z_1z_2 \in E(G)$. Then, if $|D_{z_2z_0} - V(R_1)| \leq k - 2$, we have that $(D_{z_2z_0} - V(R_1)) \cup \{z_1\} \nsubseteq G$ contradicting $\gamma_t(G) = k$. Therefore, by Lemma 7(1), $|D_{z_2z_0} - V(R_1)| = k - 1$. Thus $\{z_2\} = \{z_2, z_0\} \cap D_{z_2z_0}$ and this implies that $z_2 \nsubseteq R_1$ in $G + z_2z_0$. Since $V(R_1) \cap N_{G+z_2z_0}(z_2) = \{z_0\}$, $V(R_1) = \{z_1, z_0\}$ and this completes the proof.

6. Discussion

In this section, we discuss the related result on another type of domination critical graphs. For a graph $G$, a vertex subset $D$ of $G$ is a total dominating set of $G$ if every vertex of $G$ is adjacent to a vertex in $D$. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. A graph $G$ is said to be $k$-$\gamma_t$-critical if $\gamma_t(G) = k$ and $\gamma_t(G + uv) < k$ for any pair of non-adjacent vertices $u$ and $v$ of $G$. For $k = 3$, it was pointed out by Ananchuen in [1] that a graph $G$ is $3$-$\gamma_t$-critical if and only if $G$ is $3$-$\gamma_c$-critical. In [7], the authors established the similar result when $k = 4$. Therefore we have the following result.

Theorem 30 ([1] and [7]). For $k \in \{3, 4\}$, a connected graph $G$ is $k$-$\gamma_t$-critical if and only if $G$ is $k$-$\gamma_c$-critical.

For related results on $k$-$\gamma_t$-critical graphs, Hattingh et al. [4] established the upper bound of the number of end vertices of $k$-$\gamma_t$-critical graphs. They proved the following.

Theorem 31 [4]. For $k \geq 5$, every $k$-$\gamma_t$-critical graph has at most $k - 2$ end vertices.

They, further, established the existence of $k$-$\gamma_t$-critical graphs with prescribe end vertices according to the bound from Theorem 31.
**Theorem 32** [4]. For integers $k \geq 3$ and $0 \leq h \leq k - 2$ except only the case when $k = 4$ and $h = 2$, there exists a $k$-$\gamma_t$-critical graph with $h$ end vertices.

Hence, by Corollary 13 and Theorem 30, we can conclude that there is no $4$-$\gamma_t$-critical graph with two end vertices. This fulfills Theorem 32 in the following way.

**Corollary 33.** For integers $k \geq 3$ and $0 \leq h \leq k - 2$, there exists a $k$-$\gamma_t$-critical graph with $h$ end vertices if and only if $k \neq 4$ or $h \neq 2$.

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**REFERENCES**


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