DOMINATING VERTEX COVERS: THE VERTEX-EDGE DOMINATION PROBLEM

WILLIAM F. KLOSTERMeyer

School of Computing
University of North Florida
Jacksonville, FL 32224 USA
e-mail: wkloster@unf.edu

MARGARET-ELLEN MESSINGER

Department of Mathematics and Computer Science
Mount Allison University, Sackville, NB, Canada
e-mail: mmessinger@mta.ca

AND

ANDERS YEO

Department of Mathematics and Computer Science
University of Southern Denmark
Odense, Denmark
and Department of Mathematics
University of Johannesburg
Auckland Park, 2006 South Africa
e-mail: yeo@imada.sdu.dk

Abstract

The vertex-edge domination number of a graph, $\gamma_{ve}(G)$, is defined to be the cardinality of a smallest set $D$ such that there exists a vertex cover $C$ of $G$ such that each vertex in $C$ is dominated by a vertex in $D$. This is motivated by the problem of determining how many guards are needed in a graph so that a searchlight can be shone down each edge by a guard either incident to that edge or at most distance one from a vertex incident to the edge. Our main result is that for any cubic graph $G$ with $n$ vertices, $\gamma_{ve}(G) \leq 9n/26$. We also show that it is $NP$-hard to decide if $\gamma_{ve}(G) = \gamma(G)$ for bipartite graph $G$.

Keywords: cubic graph, dominating set, vertex cover, vertex-edge dominating set.

2010 Mathematics Subject Classification: 05C69.
1. Introduction

Let $G = (V, E)$ be an undirected graph with $n$ vertices. A dominating set of graph $G$ is a set $D \subseteq V$ such that for each $u \in V \setminus D$, there exists an $x \in D$ adjacent to $u$. A vertex $u$ is said to dominate a vertex $v$ if either $u = v$ or $u$ is adjacent to $v$. The minimum cardinality amongst all dominating sets of $G$ is the domination number, denoted $\gamma(G)$. A vertex cover of graph $G$ is a set $C \subseteq V$ such that for each edge $uv \in E$, at least one of $u, v$ is an element of $C$. The minimum cardinality amongst all vertex covers of $G$ is the vertex cover number, denoted $\tau'(G)$.

A number of recent papers have studied problems associated with defending or searching a finite, undirected graph $G = (V, E)$. These problems sometimes refer to protecting the graph with guards. A variety of graph protection problems and models have been considered in the literature of late, see the survey [5]. In the usual protection model, each attack in a sequence of attacks is defended by a mobile guard that is sent to the attacked vertex from a neighboring vertex or, in the case when edges are attacked, by sending a guard across the attacked edge (as introduced in [4]). A dominating set can then be viewed as a static positioning of guards which protect the vertices of the graph, while a vertex cover can be viewed a static positioning of guards which protect the edges of the graph.

A number of other papers have considered so-called searchlight problems which, inspired by the famous art gallery problem, attempt to use searchlights to find an intruder in a graph or a polygon. See for example [2] and [12]. In this paper, we study a variation on the searchlight problem. We shall consider the problem in which the guards, each of whom holds a searchlight, must shine a searchlight down some edge (where they think there might be an intruder). The problem is formally defined below and was initially defined by Peters in [10]. The problem was also studied in [1,7–9,11].

We now define what one may informally think of as a vertex-cover-dominating-set, or what is called a vertex-edge dominating set, for simplicity. The parameter $\gamma_{ve}(G)$ is called the vertex-edge domination number of $G$ (see [10]) and is defined to be equal to the cardinality of a smallest set $D$ such that there exists a vertex cover $C$ of $G$ such that each vertex in $C$ is dominated by a vertex in $D$. Alternatively, a set $D$ is a vertex-edge dominating set if and only if the set of vertices not dominated by $D$ form an independent set.

We shall say that an edge $uv$ is protected if there is a guard on $u, v$, or any neighbor of $u, v$. As examples, observe that $\gamma_{ve}(P_4) = 1$ and $\gamma_{ve}(C_5) = 2$. It is clear that $\tau'(G) \geq \gamma(G) \geq \gamma_{ve}(G)$ for any graph $G$ without isolated vertices.

Informally, we wish to place guards on the vertices of a graph so that any edge is “close” to any guard; that is, each edge is incident to a vertex with a guard or incident with a vertex adjacent to a vertex with a guard. Following the
art gallery metaphor, one may suppose that an alarm is triggered on edge $uv$. A guard must be able to quickly view $uv$ to determine whether there is an intruder on the edge or a false alarm. Thus, if guards occupy the vertices of a vertex-edge dominating set and an alarm is triggered on edge $uv$, there is a guard nearby: on an endpoint of $u, v$ or on a vertex adjacent to $u$ or $v$. Such a guard can shine a flashlight down incident edge $uv$ to check for an intruder or move to one of $u, v$ and shine a flashlight down incident edge $uv$. As a simple example, consider the graph $G$ shown in Figure 1 with a guard located on vertex $y$. Suppose an alarm is triggered on some edge $e$ of $G$. If $e$ is incident with $y$, the guard simply shines a flashlight down edge $e$. Otherwise, the guard moves to $x$ or $z$ and shines a flashlight down edge $e$.

With respect to the formal definition of the vertex-edge domination number, observe that $C = \{x, z\}$ is a vertex cover of graph $G$ shown in Figure 1. It is clear that $D = \{y\}$ is a set of minimum cardinality such that each vertex of $C$ is dominated by $D$. Thus, $\gamma_{ve}(G) = 1$.

![Figure 1. A graph $G$ with $\gamma_{ve}(G) = 1$.](image)

Upper bounds on the vertex-edge domination number of graphs of order $n$ were presented in [1] for non-trivial connected graphs (upper bound of $\gamma_{ve}(G) \leq n/2$) and connected $C_5$-free graphs (upper bound of $\gamma_{ve}(G) \leq n/3$).

In this paper, we present results on the vertex-edge domination number of some graphs. Our main result is shown in Section 2: $\gamma_{ve}(G) \leq 9n/26$ for any cubic graph $G$ with $n$ vertices. In Section 3, we show that it is $NP$-hard to determine whether a bipartite graph, $B$, satisfies $\gamma_{ve}(B) = \gamma(B)$. We start with a simple result.

**Proposition 1.** Let $G$ be a connected graph of order at least 2. Then $\gamma_{ve}(G) = \tau'(G)$ if and only if $\tau'(G) = 1$.

**Proof.** As $G$ is a connected graph of order at least 2 we have $\tau'(G) \geq 1$. If $\tau'(G) = 1$, then the proposition follows, as $\tau'(G) \geq \gamma_{ve}(G)$ for all $G$.

Now suppose $\tau'(G) > 1$. Let $C$ be a minimum vertex cover of $G$. We construct a vertex-edge dominating set $D$ with fewer vertices than $C$. Initially, let $D = C$. If any two vertices in $C$ are adjacent, then one of them can be removed from $D$. So suppose no two vertices in $C$ are adjacent. If there exist two vertices in $C$ that are distance two apart, then these two vertices can be replaced in $D$ by the vertex that lies on the path of length two between them. If there are no such
vertices of distance two apart in \( C \), then it follows that the closest pair of vertices in \( C \) are distance at least three apart and thus \( C \) cannot be a vertex cover, as there is an edge on the shortest path between any two vertices in \( C \) that is not covered by any vertex in \( C \).

\[ \blacksquare \]

2. Cubic Graphs

Kostochka and Stocker proved that the domination number of a cubic graph with \( n \) vertices is at most \( 5n/14 \), see [6]. There exists a cubic graph on 14 vertices where the domination number is 5, so the bound is tight. Thus, trivially, for any cubic graph \( G \), \( \gamma_{\text{ve}}(G) \leq 5n/14 \approx 0.35714n \). In this section, we prove our main result, that for any cubic graph \( G \), \( \gamma_{\text{ve}}(G) \leq 9n/26 \approx 0.34615n \).

In Section 2.1, we define a useful class of hypergraphs and state two useful hypergraph results. In Section 2.2, we state and prove our main result, Theorem 4.

### 2.1. Main result on hypergraphs from [3]

For the hypergraph \( H \), let \( n(H) \) denote the number of vertices in \( H \), \( m(H) \) denote the number of edges in \( H \) and \( e_i(H) \) denote the number of edges in \( H \) of size \( i \).

For hypergraph \( H \) with the vertex set \( V \), a smallest subset of \( V \) that contains vertices from every edge is called a transversal and its cardinality is denoted by \( \tau(H) \).

In order to state the main result from [3], we need to define a particular class of hypergraphs \( B \). Let \( B \) be the class of bad hypergraphs defined as exactly those that can be generated using the operations (A)–(D) below.

(A) Let \( H_2 \) be the hypergraph with two vertices \( \{x, y\} \) and one edge \( \{x, y\} \) and let \( H_2 \) belong to \( B \).

(B) Given any \( B' \in B \) containing a 2-edge \( \{u, v\} \), define \( B \) as follows. Let \( V(B) = V(B') \cup \{x, y\} \) and let \( E(B) = E(B') \cup \{\{u, v, x\}, \{u, v, y\}, \{x, y\}\} \setminus \{u, v\} \). Now add \( B \) to \( B \).

(C) Given any \( B' \in B \) containing a 3-edge \( \{u, v, w\} \), define \( B \) as follows. Let \( V(B) = V(B') \cup \{x, y\} \) and let

\[
E(B) = E(B') \cup \{\{u, v, w, x\}, \{u, v, w, y\}, \{x, y\}\} \setminus \{u, v, w\}.
\]

Now add \( B \) to \( B \).

(D) Given any \( B_1, B_2 \in B \), such that \( B_i \) contains a 2-edge \( \{u_i, v_i\} \), for \( i = 1, 2 \), define \( B \) as follows.

Let \( V(B) = V(B_1) \cup V(B_2) \cup \{x\} \) and let \( E(B) = E(B_1) \cup E(B_2) \cup \{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \setminus \{\{u_1, v_1\}, \{u_2, v_2\}\} \). Now add \( B \) to \( B \).
**Definition 1.** For any hypergraph $H$, let $b(H)$ denote the number of connected components in $H$ that belong to $\mathcal{B}$. Further, let $b^1(H)$ denote the maximum number of vertex disjoint subhypergraphs in $H$ which are isomorphic to hypergraphs in $\mathcal{B}$ and which are intersected by exactly one other edge in $H$.

**Theorem 2** [3]. If $H$ is a hypergraph whose all edges have size 2, 3, or 4, and $\Delta(H) \leq 3$, then

$$24\tau(H) \leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H).$$

Using Theorem 2, we can prove the following result, which is implicit in [3]; therefore we include a short proof for completeness.

**Theorem 3.** Let $H$ be a hypergraph whose all edges have size 3 or 4, and $\Delta(H) \leq 3$ and every 4-edge contains a vertex that does not belong to any 3-edge. Then $12\tau(H) \leq 3n(H) + 2e_4(H) + 3e_3(H)$.

**Proof.** Assume that $R \in \mathcal{B}$ and that $R$ contains no 2-edge. In this case we note that the last operation carried out in the construction of $R$ is operation (D) (see Subsection 2.1), as operations (A)–(C) all create 2-edges. Therefore there exist five vertices $\{u_1, v_1, u_2, v_2, x\}$ in $R$ where $\{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \subseteq E(R)$. However then $R$ is not a subgraph of $H$ as the edge $\{u_1, v_1, u_2, v_2\}$ contains no vertex that does not belong to a 3-edge. Therefore $b(H) = b^1(H) = 0$ and by Theorem 2 we have the following, which completes the proof of the theorem.

$$24\tau(H) \leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H)$$

$$= 6n(H) + 4e_4(H) + 6e_3(H).$$

**2.2. Upper bound for cubic graphs**

The bound that we shall present in Theorem 4 cannot be improved to anything below $n/3$, due to the graph in Figure 2. We leave it as an open problem to either find larger connected cubic graphs with $\gamma_{ve}(G) = n/3$ or show that the graph in Figure 2 is the only one; for instance, it does not appear easy to combine copies of the graph in Figure 2 in some way to arrive at another such example.

The open neighborhood of a vertex $v \in V(G)$ is $N(v) = \{u \in V \mid uv \in E(G)\}$ and its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$.

**Theorem 4.** If $G$ is a cubic graph, then $\gamma_{ve}(G) \leq 9n/26$.

**Proof.** Let $S$ be a maximal independent set in $G$ and assume that $|S| = (5/14 - \varepsilon_1)n$, where $n = |V(G)|$ (\varepsilon_1 may be positive or negative). Let $T$ be the set of all vertices in $S = V(G) \setminus S$ that have exactly one neighbor in $S$ and let $\varepsilon_2 = (|S| - |T|)/n$. We will now prove the following two claims.
Claim A. $\gamma_{ve}(G) \leq |S| - \varepsilon_2 n/4 = (\frac{5}{14} - \frac{\varepsilon_1}{4}) n$.

Proof. Let $U$ be a maximal subset of $S$ such that $S \setminus U$ dominates $\overline{S}$. As $\overline{S}$ is a vertex cover of $G$, we note that $\gamma_{ve}(G) \leq |S \setminus U|$.

We will now show that $|U| \geq \varepsilon_2 n/4$, which will complete the proof of Claim A. Clearly this is true if $\varepsilon_2 \leq 0$, so assume that $\varepsilon_2 > 0$. For the sake of contradiction assume that $|U| < \varepsilon_2 n/4$ and let $T'$ be the set of all vertices not in $T$ that have a unique neighbor in $S \setminus U$; note that $T' \subseteq N(U)$. As $G$ is cubic, we must have $|T'| \leq 3|U|$, which implies the following inequality.

$$|S \setminus U| = |S| - |U| \geq (|T| + \varepsilon_2 n) - |U| > (|T| + 4|U|) - |U| = |T| + 3|U| \geq |T| + |T'|.$$ 

As $|T| + |T'| < |S \setminus U|$, we note that some vertex in $s \in S \setminus U$ is not adjacent to a vertex in $T \cup T'$ (as each vertex in $T \cup T'$ is adjacent to at most one vertex in $S \setminus U$). This is a contradiction to the maximality of $U$, as $s$ could have been added to $U$. This completes the proof of Claim A. □

Claim B. $\frac{12}{14} \gamma_{ve}(G) \leq (\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_1}{14}) n$.

Proof. We will first construct a 4-uniform hypergraph $H$ as follows. Let $V(H) = V(G)$ and for every vertex $s \in S$ add $N_G[s]$ as a hyperedge in $H$. This completes the definition of $H$. As $G$ is cubic, we note that $H$ is 4-uniform with $n = |V(G)|$ vertices and $m_H = |S|$ edges.

Note that $\Delta(H) \leq 3$ as for all $x \in V(G)$ at least one vertex in $N[x]$ belongs to $S$ and therefore at most three vertices from $N[x]$ belong to $\overline{S}$ (which are the vertices that give rise to edges containing $x$). Furthermore, no 4-edge in $H$ has all its vertices in $S$.

Let $Q_1 \subseteq V(H)$ be all degree one vertices in $H$. Note that every vertex in $Q_1$ belongs to $\overline{S}$ and it has all its neighbors in $S$. Let $H'$ be the hypergraph obtained from $H$ by deleting all vertices in $Q_1$ (by deleting a vertex $v$, we mean deleting $v$ and shrinking every edge, $e$, containing $v$ such that it contains the vertex set $V(e) \setminus \{v\}$ instead of $V(e)$). Note that all edges in $H'$ have size three or four and if $e$ is a 3-edge, then all vertices in $e$ belong to $S$. As no 4-edge is completely contained in $S$ we note that every 4-edge contains a vertex (in $\overline{S}$) which does not
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belong to any 3-edge. Therefore the following holds by Theorem 3.

\[ 12\tau(H') \leq 3n(H') + 2e_4(H') + 3e_3(H') \leq 3(n - |Q_1|) + 2(m_H - |Q_1|) + 3|Q_1|. \]

Next, as \( m_H = n - |S| \), this implies the following

\[ 12\tau(H') \leq 5n - 2|S| - 2|Q_1|. \]

We will first show that \( \gamma_{ve}(G) \leq \tau(H') \) and then evaluate \( 5n - 2|S| - 2|Q_1| \).

Let \( R \) be a transversal in \( H' \) with \( |R| = \tau(H') \). As \( R \) contains a vertex from \( N[y] \) for all \( y \in \overline{S} \), we note that \( R \) dominates all vertices in \( \overline{S} \). As \( \overline{S} \) is a vertex cover of \( G \), we get that \( \gamma_{ve}(G) \leq |R| = \tau(H') \) as desired.

We will now evaluate \( 5n - 2|S| - 2|Q_1| \). Let \( Q_2 \) be the vertices in \( \overline{S} \) of degree 2 in \( H \) and let \( Q_3 \) be the vertices in \( \overline{S} \) of degree 3 in \( H \). In \( G \) the vertices in \( Q_1 \) have 3 neighbors in \( S \), the vertices in \( Q_2 \) have 2 neighbors in \( S \), and the vertices in \( Q_3 \) have 1 neighbor in \( S \). By double counting the number of edges between \( S \) and \( \overline{S} \) we get the following

\[ 3|S| = 3|Q_1| + 2|Q_2| + 1|Q_3|. \]

Recall that \( Q_3 = T \) and \( |S| - |T| = \varepsilon_2 n \) (and therefore \( |S| - \varepsilon_2 n = |T| \)), and thus we obtain the following

\[ 3|S| = 3|Q_1| + 2|Q_2| + (|S| - \varepsilon_2 n). \]

As \( Q_1 \cup Q_2 = \overline{S} \setminus T \) we also note that the following holds

\[ |Q_1| + |Q_2| = |\overline{S}| - |T| \leq (n - |S|) - (|S| - \varepsilon_2 n). \]

Next, the above two equations can be rewritten as follows

\[ 3|Q_1| + 2|Q_2| = 2|S| + \varepsilon_2 n \quad 2|Q_1| + 2|Q_2| = 2n - 4|S| + 2\varepsilon_2 n. \]

Subtracting the second equation from the first, one obtains the following

\[ |Q_1| = 6|S| - 2n - \varepsilon_2 n. \]

Now since \( |S| = (5n/14 - \varepsilon_1) \), we get the following equality

\[ 5n - 2|S| - 2|Q_1| = 5n - 2|S| - 2(6|S| - 2n - \varepsilon_2 n) = 9n - 14|S| + 2\varepsilon_2 n \]
\[ = 9n - 14(5/14 - \varepsilon_1)n + 2\varepsilon_2 n = n(4 + 14\varepsilon_1 + 2\varepsilon_2). \]

Therefore \( 12\tau(H') \leq n(4 + 14\varepsilon_1 + 2\varepsilon_2) \), which completes the proof of Claim B (by dividing both sides by 14). \( \square \)
Adding the results in Claim A and Claim B, we get the following inequality
\[ \gamma_{ve}(G) + \frac{12}{14} \gamma_{ve}(G) \leq \left( \frac{5}{14} - \varepsilon_1 - \frac{\varepsilon_2}{4} \right) n + \left( \frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14} \right) n \]
which implies
\[ \frac{26}{14} \gamma_{ve}(G) \leq \left( \frac{9}{14} - \frac{7\varepsilon_2 - 4\varepsilon_2}{28} \right) n. \]

Therefore if \( \varepsilon_2 \geq 0 \), then we have \( \gamma_{ve}(G) \leq \frac{9}{26} n \), as desired. If \( \varepsilon_2 < 0 \), then we note that \( S \) is a dominating set in \( G \) and therefore \( \gamma_{ve}(G) \leq |S| = (5/14 - \varepsilon_1) n \). Combining this with Claim B results in the following inequality
\[ \gamma_{ve}(G) + \frac{12}{14} \gamma_{ve}(G) \leq \left( \frac{5}{14} - \varepsilon_1 \right) n + \left( \frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14} \right) n. \]

Analogously to above this implies the following
\[ \frac{26}{14} \gamma_{ve}(G) \leq \left( \frac{9}{14} + \frac{2\varepsilon_2}{28} \right) n. \]

This again implies \( \gamma_{ve}(G) \leq \frac{9}{26} n \), as desired. \( \blacksquare \)

Following the example shown in Figure 2, we leave open the following question.

**Question 1.** Is it true that for any cubic graph \( G \) of order \( n \), \( \gamma_{ve}(G) \leq n/3 \)?

In fact, a stronger open problem was stated in [1]. Namely, is it true that \( \gamma_{ve}(G) \leq n/3 \) for all connected graphs of order \( n \geq 6 \)?

3. NP-Hardness

Recall that a support vertex in a tree is a vertex that is adjacent to a leaf in the tree. The trees, \( T \), satisfying \( \gamma_{ve}(T) = \gamma(T) \) were characterized by Theorem 32 of [9]. This result states that \( \gamma_{ve}(T) = \gamma(T) \) if and only if \( T \) has an efficient dominating set \( S \) such that each vertex of \( S \) is a support vertex of \( T \). A simple corollary of the result in [9] is the following.

**Corollary 5.** We can decide if \( \gamma_{ve}(T) = \gamma(T) \) in polynomial time for all trees \( T \).

We now consider the case when we want to decide whether \( \gamma_{ve}(G) = \gamma(G) \) for bipartite graphs \( G \).

**Theorem 6.** It is NP-hard to decide whether \( \gamma_{ve}(G) = \gamma(G) \) for a bipartite graph \( G \).
**Proof.** Recall that if \( H = (V, E) \) is a hypergraph, then we denote the cardinality of a smallest subset of \( V \) that contains vertices from every edge (called a transversal) by \( \tau(H) \).

We will reduce from the \( NP \)-hard problem of deciding whether a 3-uniform hypergraph, \( H = (V, E) \), has a transversal of size at most \( k \). That is, the hypergraph \( H = (V, E) \), where \( V \) is the vertex set of \( H \) and each edge \( e \in E \) is a set containing three vertices. We then want to decide whether there is a subset, \( X \subseteq V \), of size at most \( k \) that contains at least one vertex of every edge of \( H \).

The idea is to construct a graph, \( G \), such that \( \gamma_{ve}(G) < \gamma(G) \) if and only if \( \tau(H) \leq k \). Start the construction of graph \( G \) with vertex set \( V \). To this, for each edge \( e \in E \), we add the vertex set \( V_e = \{v_i^e \mid i = 1, 2, \ldots, k\} \) and the edges from each vertex in \( V_e \) to the three vertices in \( V \) that belong to \( e \). Then we add the vertices \( W = \{w_1, w_2, \ldots, w_k\} \) and for all \( i = 1, 2, \ldots, k \) add all edges from \( w_i \) to \( v_i^e \) for all \( e \in E \). Finally, we add the two new vertices \( x \) and \( y \) and all edges from \( x \) to \( V \cup \{y\} \). This completes the construction of \( G \).

We will show that \( \tau(H) \leq k \) if and only if \( \gamma_{ve}(G) < \gamma(G) \). Let \( S_i = w_i \cup \{v_i^e \mid e \in E\} \). Note that \( \gamma(G) = k + 1 \), as any dominating set in \( G \) must contain at least one vertex from each \( S_i \) (in order to dominate \( w_i \)) and a vertex from \( \{x, y\} \) (in order to dominate \( y \)) and \( W \cup \{x\} \) is a dominating set in \( G \).

If \( \tau(H) \leq k \), then let \( T \) be a transversal of \( H \) of size \( \tau(H) \). Note that \( T \subseteq V \) and \( T \) is a vertex-edge-dominating set in \( G \) (as the only vertices not dominated by \( T \) in \( G \) are \( \{w_1, w_2, \ldots, w_k, y\} \) which form an independent set). Therefore \( \gamma_{ve}(G) \leq |T| \leq k < k + 1 = \gamma(G) \).

Now assume that \( \gamma_{ve}(G) < \gamma(G) \). For the sake of contradiction assume that \( \tau(H) > k \). Let \( Q \) be a vertex-edge dominating set in \( G \) of size \( \gamma_{ve}(G) \). As \( |Q| = \gamma_{ve}(G) < \gamma(G) = k + 1 \) and \( \tau(H) > k \) we note that \( Q \cap V \) is not a transversal in \( H \). Therefore some edge \( e \in E \) is not covered by \( Q \cap V \). Due to the edge \( v_i^e \), we note that \( Q \) must contain at least one vertex from each \( S_i \), \( i = 1, 2, \ldots, k \). As \( |Q| \leq k \), we therefore note that the edge \( xy \) is not covered by \( Q \), a contradiction. Therefore \( \tau(H) > k \), which completes the proof. \( \blacksquare \)

**Acknowledgements**

The authors thank the anonymous referees for their helpful comments, especially pointing out some useful references.

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Received 4 January 2018
Revised 25 August 2018
Accepted 25 August 2018