A NOTE ON THE FAIR DOMINATION NUMBER IN OUTERPLANAR GRAPHS

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Abstract

For \( k \geq 1 \), a \( k \)-fair dominating set (or just \( k \)FD-set), in a graph \( G \) is a dominating set \( S \) such that \( |N(v) \cap S| = k \) for every vertex \( v \in V - S \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \)FD-set. A fair dominating set, abbreviated FD-set, is a \( k \)FD-set for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of \( G \) that is not the empty graph, is the minimum cardinality of an FD-set in \( G \). In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

Keywords: fair domination, outerplanar graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [13]. Specifically, let \( G \) be a simple graph with vertex set \( V(G) = V \) of order \( |V| = n \) and let \( v \) be a vertex in \( V \). The open neighborhood of \( v \) is \( N_G(v) = \{u \in V \mid uv \in E(G)\} \) and the closed neighborhood of \( v \) is \( N_G[v] = \{v\} \cup N_G(v) \). If the graph \( G \) is
clear from the context, then we simply write $N(v)$ rather than $N_G(v)$. The degree of a vertex $v$, is $\text{deg}(v) = |N(v)|$. A vertex of degree one is called a leaf and its neighbor a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \cup_{v \in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. The distance $d(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the minimum number of edges of a path from $u$ to $v$. A graph $G$ of order at least three is 2-connected if the deletion of any vertex does not disconnect the graph. A cut-vertex in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a block. A graph $G$ is outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph $G$ is Hamiltonian if there is a spanning cycle in $G$. For a subset $S$ of vertices of $G$, we denote by $G[S]$ the subgraph of $G$ induced by $S$.

A subset $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A vertex $v$ is said to be dominated by a set $S$ if $N[v] \cap S \neq \emptyset$.

Caro et al. [1] studied the concept of fair domination in graphs. For $k \geq 1$, a $k$-fair dominating set, abbreviated $k$FD-set, in $G$ is a dominating set $S$ such that $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The $k$-fair domination number of $G$, denoted by $fd_k(G)$, is the minimum cardinality of a $k$FD-set. A $k$FD-set of $G$ of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A fair dominating set, abbreviated FD-set, in $G$ is a $k$FD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of a graph $G$ that is not the empty graph is the minimum cardinality of an FD-set in $G$. An FD-set of $G$ of cardinality $fd(G)$ is called a $fd(G)$-set. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A perfect dominating set in a graph $G$ is a dominating set $S$ such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in $S$. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro et al. [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph $G$ of order $n$, and among open problems posed by Caro et al. [1], one asks to find $fd(G)$ for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block $K$ in an outerplanar graph $G$ a strong-block if $K$ contains
at least three vertices. We call a vertex $w$ in a strong-block $K$ of an outerplanar graph $G$ a special cut-vertex if $w$ belongs to a shortest path from $K$ to a strong-block $K' \neq K$. We call a strong-block $K$ in an outerplanar graph $G$ a leaf-block if $K$ contains exactly one special cut-vertex. We denote by $r(G)$ the number of strong-blocks of a graph $G$. The following is straightforward.

**Observation 1.** Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.

We make use of the following.

**Observation 2** (Caro et al. [1]). Every $1FD$-set in a graph contains all its strong support vertices.

**Theorem 3** (Leydolda et al. [14]). An outerplanar graph $G$ is Hamiltonian if and only if it is 2-connected.

**Theorem 4** (Hajian et al. [9]). If $G$ is a unicyclic graph of order $n$, then $fd(G) \leq (n + 1)/2.$

### 2. Main Result

**Theorem 5.** If $G$ is an outerplanar graph of order $n$ and size $m$ with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$. This bound is sharp.

**Proof.** Let $G$ be an outerplanar graph of order $n$ and size $m$ with $r \geq 1$ strong-blocks. We prove that $fd(G) \leq (4m - 3n + 3)/2 - r$. The result follows from Theorem 4 if $G$ is a unicyclic graph. Thus assume that $G$ is not a unicyclic graph. Suppose to the contrary that $fd(G) > (4m - 3n + 3)/2 - r$. Assume that $G$ has the minimum order, and among all such graphs, we may assume that the size of $G$ is as minimum as possible. Let $K_1, K_2, \ldots, K_r$ be the $r$ strong-blocks of $G$. By Theorem 3, $K_j$ is Hamiltonian, for $1 \leq j \leq r$. Let $C' = c_j^0c_1^j \cdots c_r^j c_0^j$ be a Hamiltonian cycle for $K_i$, for $1 \leq i \leq r$. We proceed with the following Claims 1 and 2.

**Claim 1.** For any $1 \leq i \leq r$, if $c_j^i$ is a vertex of $C'$, for some $j \in \{0, 1, \ldots, l_i\}$, such that $deg_G(c_j^i) = 2$, then $deg_G(c_{j+1}^i) \geq 3$ and $deg_G(c_{j-1}^i) \geq 3$, where the calculations in $j + 1$ and $j - 1$ are taken modulo $l_i$.

**Proof.** Assume that $deg_G(c_j^i) = 2$ for some $j \in \{0, 1, \ldots, l_i\}$. Suppose that $deg_G(c_{j+1}^i) = 2$. Let $G' = G - c_j^i c_{j+1}^i$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of $G$, $fd(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 3n + 3)/2 - (r - 1) = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd(G')$-set. If $|S' \cap \{c_j^i, c_{j+1}^i\}| \in \{0, 2\}$,
then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $|S' \cap \{c'_j, c'_{j+1}\}| = 1$.

Assume that $c'_j \in S'$. Then $c'_{j+1} \notin S'$, and $c'_{j+2} \in S'$, since $S'$ is a dominating set. Thus $\{c'_j, c'_{j+1}\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Next assume that $c'_{j+1} \in S'$.

Then $c'_j \notin S'$ and $c'_{j-1} \in S'$. Thus $\{c'_i\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$. So $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $\deg_G(c'_{j+1}) \geq 3$. Similarly, $\deg_G(c'_{j-1}) \geq 3$.

\begin{claim}
If $c'_j$ is a vertex of $C^i$, for some $j \in \{0, 1, \ldots, l_i\}$, such that $\deg_G(c'_j) = 2$, then non of $c'_{j+1}$ and $c'_{j-1}$ is a support vertex of $G$.
\end{claim}

\begin{proof}
Assume that $\deg_G(c'_j) = 2$ for some $j \in \{0, 1, \ldots, l_i\}$. Suppose that $c'_{j+1}$ is a support vertex of $G$. Let $G' = G - c'_j c'_{j+1}$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m - 1) - 3n + 3)/2 - r - 1 = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd_1(G')$-set.

By Observation 2, $c'_{j+1} \in S'$, since $c'_{j+1}$ is a strong support vertex of $G$. If $c'_{j-1} \notin S'$, then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $c'_{j-1} \in S'$ and so $\{c'_i\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $c'_{j+1}$ is not a support vertex of $G$. Similarly, $c'_{j-1}$ is not a support vertex of $G$.
\end{proof}

We consider the following cases.

\textbf{Case 1.} $r = 1$. First assume that $V(G) = \{c'_0, c'_1, \ldots, c'_{l_1}\}$ and so $n = l_1 + 1$.

By Claim 1, at least $[n/2]$ vertices of $C^1$ are of degree at least 3. Now, we can easily see that $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + [n/2]/2$. (Since $\delta(G) \geq 2$ and at least $[n/2]$ vertices of $G$ are of degree at least 3, we have $\sum_{v \in V(G)} \deg(v) \geq 2n + [n/2]$.)

Thus $m \geq n + [n/2]/2$. If $n$ is even, then $n \leq (4m - 3n)/2$ and if $n$ is odd, then $n \leq (4m - 3n - 1)/2$. We thus obtain that $n \leq (4m - 3n + 3)/2 - 1$. Now $V(G)$ is a 1FD-set in $G$ of cardinality $n$, and thus $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. We deduce that $V(G) \neq \{c'_0, c'_1, \ldots, c'_{l_1}\}$. Since $r = 1$, there is a vertex of degree one in $G$. Let $v_d$ be a leaf of $G$ such that $d(v_d, C^1)$ is maximum. Let $v_0 v_1 \cdots v_d$ be the shortest path from $v_d$ to a vertex $v_0 \in C^1$.

Clearly, $\{v_0, v_1, \ldots, v_d\} \cap V(C^1) = \{v_0\}$.

Assume that $d \geq 2$. Suppose that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 2) - 3(n - 2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most (4m - 3n + 3)/2 - 1
(4m − 3n + 3)/2 − 1 and so \( fd_1(G) \leq (4m − 3n + 3)/2 − 1 \), a contradiction. Thus assume that \( \deg_G(v_{d-1}) \geq 3 \). Clearly any vertex of \( N_G(v_{d-1}) - \{v_{d-2}\} \) is a leaf. Let \( G' \) be obtained from \( G \) by removing all leaves adjacent to \( v_{d-1} \). Clearly \( r(G') = r \). By the choice of \( G \), \( fd_1(G') \leq (4m(G') − 3n(G') + 3)/2 − r(G') \leq (4(m − 2) − 3(n − 2) + 3)/2 − 1 = (4m − 3n + 3)/2 − 2 \). Let \( S' \) be a \( fd_1(G') \)-set. If \( v_{d-1} \in S' \), then \( S' \) is a 1FD-set in \( G \) of cardinality at most \( (4m − 3n + 3)/2 − 2 \) and so \( fd_1(G) \leq (4m − 3n + 3)/2 − 2 \), a contradiction. Thus assume that \( v_{d-1} \notin S' \). Then \( v_{d-2} \in S' \). Now \( S' \cup \{v_{d-1}\} \) is a 1FD-set in \( G \) of cardinality at most \( (4m − 3n + 3)/2 − 1 \) and so \( fd_1(G) \leq (4m − 3n + 3)/2 − 1 \), a contradiction.

We next assume that \( d = 1 \). Let \( D_1 = \{c^j \mid \deg_G(c^j) = 2\} \) and \( D_2 = \{c^j \mid \deg_G(c^j) \geq 3\} \). By relabeling of the vertices of \( G \) we may assume that \( c^0 \) is a special cut-vertex of \( G \). Let \( G' \) be the graph obtained by removal of all edges \( c^0c^j \), with \( c^j \in \{c^1, \ldots, c^l\} \). Clearly \( G' \) has two components. Let \( G'_1 \) be the component of \( G' \) containing \( c^0 \), and \( G'_2 \) be the component of \( G' \) containing \( c^0 \). Clearly, \( \{c^1, c^2, \ldots, c^l\} \subseteq V(G'_1) \). We consider the following subcases.

Subcase 2.1. \( V(G'_1) = \{c^1, c^2, \ldots, c^l\} \). Let \( G'_1 = G[V(G'_1) \cup \{c^0\}] \). Clearly \( n(G'_1) = l + 1 \). By Claim 1, at least \( \lfloor l/2 \rfloor \) vertices of \( G' - c^0 \) are of degree at least 3.

Assume that \( l_i \) is even. Thus at least \( l_i/2 \) vertices of \( C^i - c^0 \) are of degree at least 3. Now, we can easily see that \( m(G'_1) = \frac{1}{2} \sum_{v \in V(G'_1)} \deg(v) \geq l_i + 1 \). Let \( G'_2 = G[V(G'_1) \cup \{c^1, c^i\}] - \{c^0, c^i\} \). Clearly \( n = n(G'_2) = l_i - 2 \), \( m = m(G'_2) + m(G'_1) - 2 \) and \( r(G'_2) = r - 1 \). By the choice of \( G \), \( fd_1(G'_2) \leq (4m(G'_2) - 3n(G'_2) + 3)/2 - r(G'_2) \). Let \( S'' \) be a \( fd_1(G'_2) \)-set. By Observation 2, \( c^0 \in S'' \), since \( c^0 \) is a strong support vertex of \( G'_2 \). Then \( S'' \cup \{c^1, c^2, \ldots, c^l\} \) is
a 1FD-set for $G$ of cardinality $|S''| + l_i$. On the other hand
\[
\frac{(4m - 3n + 3)}{2} - r \\
\geq (4m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3(l_i + 1) + 1)/2 - 1 \\
\geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\]
Thus $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

Assume next that $l_i$ is odd. Observe that at least $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3. Now, we can easily see that $m(G^*_1) = \frac{1}{2} \sum_{v \in V(G^*_1)} \deg(v) \geq l_i + 1 + (l_i - 1)/4$. We show that $m(G^*_1) = l_i + 1 + (l_i - 1)/4$. Suppose that $m(G^*_1) > l_i + 1 + (l_i - 1)/4$. Then $m(G^*_1) \geq l_i + 1 + (l_i - 1)/4 + 1/4$. Let $G^*_2 = G^*_1 \cup \{c_i^1, c_i^2\} - \{c_i^1\}$. Clearly $n = n(G^*_2) + l_i - 2$, $m = m(G^*_2) + m(G^*_1) - 2$ and $r(G^*_2) = r - 1$. By the choice of $G$, $fd_1(G^*_2) \leq (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2)$.

Let $S''$ be a $fd_1(G^*_2)$-set. By Observation 2, $c_0^i \in S''$, since $c_0^i$ is a strong support vertex of $G^*_2$. Then $S'' \cup \{c_1^i, c_2^i, \ldots, c_{l_i}^i\}$ is a 1FD-set for $G$ of cardinality $|S''| + l_i$. On the other hand
\[
\frac{(4m - 3n + 3)}{2} - r \\
\geq (4m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3(l_i + 1) + 1)/2 - 1 \\
\geq |S''| + (4(l_i + 1 + l_i/4 + 1/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\]
Thus $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. We thus obtain that $m(G^*_1) = l_i + 1 + (l_i - 1)/4$. Note that $|E(G^*_1) \cap E(C^i)| = l_i + 1$. Hence $|E(G^*_2) - E(C^i)| = (l_i - 1)/4$. Since $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3, we thus obtain that precisely $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree 3, and so $(l_i + 1)/2$ vertices of $C^i - c_0^i$ are of degree two. Now Claim 1 implies that $deg_G(c_1^i) = deg_G(c_{l_i}^i) = 2$. Thus we obtain that $deg_G(c_0^i) = 2$. Let $A_1 = \{c_j \mid deg_G(c_{l_i}^i) = 2 \text{ for } 1 \leq j \leq l_i\}$ and $A_2 = \{c_1^i, c_2^i, \ldots, c_{l_i}^i\} - A_1$. Clearly $|A_1| = (l_i + 1)/2$ and $|A_2| = (l_i - 1)/2$. Note that $|A_2|$ is even, since the number of odd vertices in every graph (here $G^*_1$) is even. Thus $|A_1|$ is odd, since $l_i$ is odd and $|A_1| + |A_2| = l_i$. Then $|A_1| \geq 3$, since $c_{l_i}^i \in A_1$. Now Claim 1 implies that $A_1 = \{c_1^i, c_3^i, \ldots, c_{l_i+1}^i/2, \ldots, c_{l_i}^i\}$ and $A_2 = \{c_2^i, c_4^i, \ldots, c_{l_i-1}^i\}$.

**Fact 1.** There are two adjacent vertices $c_s^i, c_t^i \in A_2$ such that $|s - t| = 2$.

**Proof.** Note that $l_i \equiv 1 \pmod{4}$, since $\frac{l_i - 1}{2}$ is even. If $l_i = 5$, then $c_2^i, c_4^i \in A_2$ are the desired vertices, since they are the only vertices of $G^*_1$ of degree three. Thus assume that $l_i \geq 9$. If $\{c_{l_i+1}^i + 1, c_{l_i+1}^i - 3\} \cap N(c_{l_i+1}^i - 1) \neq \emptyset$, then the desired pairs
are $c_{\frac{i}{2}+1}-3$ and the vertex of $\{c_{\frac{i}{2}+1}^t, c_{\frac{i}{2}+1}^h\} \cap N(c_{\frac{i}{2}+1}^t)$. Thus assume that $\{c_{\frac{i}{2}+1}^t, c_{\frac{i}{2}+1}^h\} \cap N(c_{\frac{i}{2}+1}^t) = \emptyset$. Clearly there is a vertex $c_i^t \in A_2$ such that $c_i^t$ is adjacent to $c_{\frac{i}{2}+1}^t$. Without loss of generality, assume that $t < \frac{i+1}{2} - 3$. Since $G$ is an outerplanar graph, $|A_2 \cap \{c_i^h : t+2 \leq h \leq \frac{i+1}{2} - 3\}|$ is even. Furthermore, since $G$ is an outerplanar graph, any vertex of $A_2 \cap \{c_i^h : t+2 \leq h \leq \frac{i+1}{2} - 3\}$ is adjacent to a vertex of $A_2 \cap \{c_i^h : t+2 \leq h \leq \frac{i+1}{2} - 3\}$. Consequently, there are two pairs $c_{h_1}, c_{h_2} \in A_2 \cap \{c_i^h : t+2 \leq h \leq \frac{i+1}{2} - 3\}$ such that $c_{h_1} \in N(c_{h_2})$ and $|h_1 - h_2| = 2$.

Let $c_i^t$ and $c_{i+2}^t$ be two adjacent vertices of $A_2$ according to Fact 1. Clearly, $deg(c_{i+1}^t) = 2$. Let $G^* = G - c_i^t c_{i+1}^t - c_{i+2}^t$. Clearly $n(G^*) = n$, $m(G^*) = m - 2$ and $r - 1 \leq r(G^*) \leq r$. By the choice of $G$, $f_{d_1}(G^*) \leq (4m(G^*) - 3n(G^*) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3$. Let $S^*$ be a $f_{d_1}(G^*)$-set. Since $c_{i+2}^t$ is a strong support vertex of $G^*$, by Observation 2, we have $c_{i+2}^t \in S^*$. If $c_{i-1}^t \notin S^*$, then $S^*$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 3$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r - 3$, a contradiction. Thus $c_{i-1}^t \in S^*$. Then $S^* \cup \{c_i^t, c_{i+1}^t\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction.

Subcase 2.2. $V(G_1) \neq \{c_1^t, c_2^t, \ldots, c_i^t\}$. Since $K_1$ is a leaf-block of $G$, $G_1 - c_i^t$ has some vertex of degree at most one. Let $v_d$ be a leaf of $G'_1$ such that $d(v_d, G_1 - c_i^t)$ is as maximum as possible, and the shortest path from $v_d$ to $C_i$ does not contain $c_d^t$. Let $G'$ be obtained from $G$ by removing all leaves adjacent to $v_d$. Clearly $r(G') = r$. By the choice of $G$, $f_{d_1}(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 3n + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $f_{d_1}(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. We deduce that $deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let $G''$ be obtained from $G$ by removing all leaves adjacent to $v_{d-1}$. Clearly $r(G'') = r$. By the choice of $G$, $f_{d_1}(G'') \leq (4m(G'') - 3n(G'') + 3)/2 - r(G'') = (4m - 3n + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let $S''$ be a $f_{d_1}(G'')$-set. If $v_{d-1} \notin S''$, then $S''$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus assume that $v_{d-1} \notin S''$. Then $v_{d-2} \in S''$. Now $S'' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $f_{d_1}(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

We thus assume that $d = 1$. Let $D_1 = \{c_j^t | deg_G(c_j^t) = 2\}$, $D_2 = \{c_j^t | c_j^t$
is a support vertex of $G$} and $D_3 = \{c^j_i \mid \deg_G(c^j_i) \geq 3$ and $c^j_i$ is not a support vertex of $G\}$. Clearly $|D_1| + |D_2| + |D_3| = l_i$. Observe that $|D_2| \geq 1$, since $d = 1$. Thus by Claims 1 and 2, $|D_1| \leq |D_3|$. Let $G_1^* = G[G_1^i \cup \{c^0_i\}]$. Observe that $m(G_1^i) = \frac{1}{2} \sum_{v \in V(G_1^i)} \deg(v) \geq n(G_1^i) + |D_3| / 2$. Then $n(G_1^i) \geq l_i + 1 + |D_2|$. Let $G_2^* = \{G_2 \cup \{c^1_i, c^2_i, \ldots, c^l_i\}\} - \{c^0_i\}$. Clearly $n = n(G_2^i) + n(G_1^i) - 3$, $m = m(G_2^i) + m(G_1^i) - 2$ and $r(G_2^i) = r - 1$. By the choice of $G$, $fd_1(G_2^i) \leq (4m(G_2^i) - 3n(G_2^i) + 3) / 2 - r(G_2^i)$. Let $S''$ be a $fd_1(G_2^i)$-set. By Observation 2, $c^0_i \in S''$, since $c^0_i$ is a strong support vertex of $G_2^i$. Then $S'' \cup \{c^1_i, c^2_i, \ldots, c^l_i\}$ is a 1FD-set for $G$ of cardinality $|S''| + l_i$. On the other hand

\[
(4m - 3n + 3)/2 - r \\
\geq (4(m(G_2^i) + m(G_1^i) - 2) - 3(n(G_2^i) + n(G_1^i) - 3) + 3) / 2 - r \\
= (4m(G_2^i) - 3n(G_2^i) + 3) / 2 - r(G_2^i) + (4m(G_1^i) - 3n(G_1^i) + 1) / 2 - 1 \\
\geq |S''| + (n(G_1^i) + 2|D_3| + 1)/2 - 1 \\
\geq |S''| + (l_i + 1 + |D_2| + 2|D_3| + 1)/2 - 1 \\
\geq (l_i + |D_2| + |D_3| + |D_1|)/2 \geq |S''| + l_i.
\]

Thus $fd_1(G) \leq |S''| + l_i \leq (4m - 3n + 3)/2 - r$, a contradiction.

To the sharpness, consider a cycle $C_5$.

3. Concluding Remarks

As it is noted, Caro et al. [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph $G$ of order $n$. They also proved that $fd(G) \leq n - 2$ for any connected graph $G$ of order $n \geq 3$. It is worth-noting that the bound of Theorem 5 improves the bound $n - 2$ when $4m < 5n + 2r - 7$. It is also known that every maximal outerplanar graph $G$ of order at least 3 is 2-connected [7], and thus $r(G) = 1$. Therefore, the bound of Theorem 5 improves the bound $17n/19$ when $4m < \frac{9n}{19} - 1$. We have the following conjecture.

**Conjecture 6.** If $G$ is a graph of order $n$ and size $m$ with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$.

**References**


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