Abstract

Let $H$ be a digraph (possibly with loops) and $D$ a digraph without loops whose arcs are colored with the vertices of $H$ ($D$ is said to be an $H$-colored digraph). For an arc $(x, y)$ of $D$, its color is denoted by $c(x, y)$. A directed path $W = (v_0, \ldots, v_n)$ in an $H$-colored digraph $D$ will be called $H$-path if and only if $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a directed walk in $H$. In $W$, we will say that there is an obstruction on $v_i$ if $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$ (if $v_0 = v_n$ we will take indices modulo $n$). A subset $N$ of $V(D)$ is said to be an $H$-kernel in $D$ if for every pair of different vertices in $N$ there is no $H$-path between them, and for every vertex $u$ in $V(D) \setminus N$ there exists an $H$-path in $D$ from $u$ to $N$. Let $D$ be an arc-colored digraph. The color-class digraph of $D$, $\mathcal{C}(D)$, is the digraph such that $V(\mathcal{C}(D)) = \{c(a) : a \in A(D)\}$ and $(i, j) \in A(\mathcal{C}(D))$ if and only if there exist two arcs, namely $(u, v)$ and $(v, w)$ in $D$, such that $c(u, v) = i$ and $c(v, w) = j$. The main result establishes that if $D = D_1 \cup D_2$ is an $H$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathcal{C}(D))$ with a property $P^*$ such that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. either $D[\{a \in A(D) : c(a) \in V_i\}]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in $\{1, \ldots, k\}$,
3. $D_i$ has no infinite outward path for every $i$ in $\{1, 2\}$,

1Corresponding author.
4. every cycle of length three in $D$ has at most two obstructions, then $D$ has an $H$-kernel.

Some results with respect to the existence of kernels by monochromatic paths in finite digraphs will be deduced from the main result.

**Keywords:** quasi-transitive digraph, kernel by monochromatic paths, alternating kernel, $H$-kernel, obstruction.

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1. Introduction

Let $H$ be a digraph possibly with loops and $D$ a digraph without loops. An $H$-arc coloring of $D$ is a function $c : A(D) \rightarrow V(H)$. $D$ is $H$-colored if $D$ has an $H$-arc coloring. A path $W = (v_0, \ldots, v_n)$ in $D$ is said to be an $H$-path if and only if $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a walk in $H$. We are going to consider that an arc is an $H$-path, that is to say, a singleton vertex is a walk in $H$. A subset $S$ of $V(D)$ is $H$-absorbent if for every $x$ in $V(D) \setminus S$ there is an $H$-path from $x$ to some point of $S$. A subset $I$ of $V(D)$ is $H$-independent if there is no $H$-path between any two distinct vertices of $I$. A subset $N$ of $V(D)$ is an $H$-kernel if $N$ is both $H$-absorbent and $H$-independent. The concept of $H$-kernel has its origins in the works carried out by Sands, Sauer and Woodrow [15], Linek and Sands [13] and Arpin and Linek [1]. In [15] Sands, Sauer and Woodrow proved that if the arcs of a finite tournament $T$ are colored with two colors, then there is always a vertex $v$ in $T$ such that for every $w$ in $V(T) \setminus \{v\}$ there exists a monochromatic path from $w$ to $v$. In [13] Linek and Sands gave an extension of the result of Sands, Sauer and Woodrow, in which the arcs of a tournament $T$ are colored with the elements of a partially ordered set $P$ and in their paper they give the first notion of $H$-path. In [1] Arpin and Linek work with $H$-colored digraphs and in their paper they introduce the concept of $H$-walk where an $H$-walk is a walk $(v_0, \ldots, v_n)$ in $D$ such that $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a walk in $H$. In [1] Arpin and Linek introduce the concept of $H$-independent set by walks as a subset of vertices $I$ of $D$ such that there is no $H$-walks between any two different vertices of $I$. They also define an $H$-sink as a subset of vertices $S$ of $D$ such that for any $u$ in $V(D) \setminus S$ there is $v$ in $S$ such that there exists an $H$-walk from $u$ to $v$. Galeana-Sánchez and Delgado-Escalante were inspired by the work of Arpin and Linek and in [6] they introduced the concept of $H$-kernels. A subset of vertices $N$ of $D$ is called $H$-kernel by walks if $N$ is both an $H$-independent set by walks and $N$ is an $H$-sink. Notice that the concept of $H$-kernel and the concept of $H$-kernel by walks are different because of that the existence of an $H$-walk between two vertices does not guarantee the existence of an $H$-path between those vertices and the concatenation of two $H$-paths is not always an $H$-walk, see Figure 1.
Figure 1. \((u, x, y, z, x, v)\) is a \(uv\)-\(H\)-walk in \(G\). The only one \(uv\)-path in \(G\) is \((u, x, v)\) but this path is not a \(uv\)-\(H\)-path in \(G\). \(\{v\}\) is an \(H\)-kernel by walks in \(G\). Every \(H\)-independent set in \(G\) consists only of one element but none of these is an \(H\)-kernel.

Notice that it follows from the definition of \(H\)-kernel that when \(A(H) = \emptyset\), an \(H\)-kernel is a kernel (a subset \(N\) of vertices of \(D\) such that (1) for every \(u\) and \(v\) in \(N\) it holds that \(\{(u, v), (v, u)\} \cap A(D) = \emptyset\) and (2) for every \(u\) in \(V(D) \setminus N\) there exists \(v\) in \(N\) such that \((u, v) \in A(D)\)) when \(A(H) = \{(u, u) : u \in V(H)\}\), an \(H\)-kernel is a kernel by monochromatic paths (\(mp\)-kernel) (a subset \(N\) of vertices of \(D\) such that (1) for every \(u\) and \(v\) in \(N\) there exists no monochromatic directed paths between \(u\) and \(v\) and (2) for every \(u\) in \(V(D) \setminus N\) there exists \(v\) in \(N\) such that there exists a monochromatic directed path from \(u\) to \(v\)) and when \(H\) has no loops, an \(H\)-kernel is an alternating kernel (a subset \(N\) of vertices of \(D\) such that (1) for every \(u\) and \(v\) in \(N\) it holds that there exists no directed path between \(u\) and \(v\) in which consecutive arcs have different colors and (2) for every \(u\) in \(V(D) \setminus N\) there exists \(v\) in \(N\) such that there is a directed path from \(u\) to \(v\) in which consecutive arcs have different colors). In each of these special cases for \(H\), sufficient conditions have been established in order to guarantee the existence of \(H\)-kernels, see for example [3, 5, 7, 9, 15]. Thus we can see that the concept of \(H\)-kernels is a generalization of the concepts of kernels, \(mp\)-kernels and alternating kernels.

Due to the difficulty of finding kernels, \(mp\)-kernels and alternating kernels in arc-colored digraphs, sufficient conditions for the existence of each of these \(H\)-kernels in arc-colored digraphs have been obtained mainly by study special classes of digraphs. A digraph \(D\) is quasi-transitive whenever \(\{(u, v), (v, w)\} \subseteq A(D)\) implies either \((u, w) \in A(D)\) or \((w, u) \in A(D)\). Quasi-transitive digraphs are of interest because these are a generalization of tournaments (due to Ghouilh-Houri [12]) and those digraphs are a special case of digraphs in which the existence of kernels, \(mp\)-kernels and alternating kernels has been studied.

In [10] Galeana-Sánchez and Rojas-Monroy proved that if \(D = D_1 \cup D_2\) (possibly \(A(D_1) \cap A(D_2) \neq \emptyset\)) where \(D_i\) is a quasi-transitive digraph which contains no asymmetric infinite outward path (in \(D_i\) for \(i \in \{1, 2\}\), and that every directed cycle of length 3 contained in \(D\) has at least two symmetric arcs, then \(D\) has a kernel.
A chromatic class of $D$ is the set of arcs of a same color. We say that a chromatic class $C$ is quasi-transitive if $D[C]$ is a quasi-transitive digraph. Let $D = D_1 \cup D_2$ be a digraph. We will say that $D$ is a union of asymmetric quasi-transitive digraphs if (1) $D_i$ is a quasi-transitive digraph for every $i \in \{1, 2\}$, (2) $D_i$ is asymmetric for every $i \in \{1, 2\}$ and (3) $A(D_1) \cap A(D_2) = \emptyset$.

In [11] Galeana-Sánchez et al. worked with a finite $m$-colored multidigraph (a digraph with parallel arcs) $D = D_1 \cup D_2$ which is a union of asymmetric quasi-transitive digraphs, and they proved that if $D$ satisfies that

1. every chromatic class induces a quasi-transitive digraph,
2. every chromatic class is contained in $D_i$ for some $i \in \{1, 2\}$ and
3. $D$ contains neither 3-colored directed triangles nor 3-colored transitive subtournaments of order 3,

then $D$ has an $mp$-kernel.

In [7], recently, Delgado-Escalante et al. proved the following.

**Theorem 1.** If $D$ is a finite $m$-colored quasi-transitive digraph such that every directed cycle of length 3 contained in $D$ is $3$-colored, then $D$ has an alternating kernel.

Basically the spirit of the conditions that guarantee the existence of kernels or $mp$-kernels in [10] and [11], respectively, arises from structural properties of 2-colored digraphs which were studied in [15] by Sands et al.

On the other hand, in [8] Galeana-Sánchez defined the color-class digraph $\mathcal{C}_C(D)$ of $D$ as the digraph whose vertices are the colors represented in the arcs of $D$ and $(i, j) \in A(\mathcal{C}_C(D))$ if and only if there exist two arcs, namely $(u, v)$ and $(v, w)$ in $D$, such that $(u, v)$ has color $i$ and $(v, w)$ has color $j$ (notice that $\mathcal{C}_C(D)$ can have loops by definition). Because of that in an $H$-colored digraph $D$, it holds that $V(\mathcal{C}_C(D)) \subseteq V(H)$, we can establish structural properties on $\mathcal{C}_C(D)$, with respect to $H$, in order to guarantee the existence of $H$-kernels.

Let $H$ be a digraph, $D$ an $H$-colored digraph and $(v_0, v_1, \ldots, v_n)$ a walk in $D$. We will say that there is an obstruction on $v_i$ if $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$ (if $v_0 = v_n$ we will take indices modulo $n$).

In this paper we continue with the study of the existence of $H$-kernels in unions of quasi-transitive digraphs and for this we will need the following definitions.

**Definition.** Let $H$ be a digraph, $D$ an $H$-colored digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathcal{C}_C(D))$. We will say that $\{V_1, \ldots, V_k\}$ has the property $P^*$ if the following conditions are satisfied.

1. $\mathcal{C}_C(D)[V_i]$ is a subdigraph of $H$ for every $i \in \{1, \ldots, k\}$.
2. If $(u, v) \in A(\mathcal{C}_C(D))$, for some $u$ in $V_i$ and for some $v$ in $V_j$ with $i \neq j$, then $(u, v) \notin A(H)$.
**Definition.** Let $H$ be a digraph, $D$ an $H$-colored digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(C(D))$. $V_i$ is said to be a quasi-transitive $V_i$-class if $D[\{a \in A(D) : c(a) \in V_i\}]$ is a quasi-transitive digraph for every $i$ in $\{1, \ldots, k\}$.

The main result establishes that if $H$ is a digraph, $D = D_1 \cup D_2$ is an $H$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(C(D))$ with the property $P^*$ such that
1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. either $D[\{a \in A(D) : c(a) \in V_i\}]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in $\{1, \ldots, k\}$,
3. $D_i$ has no infinite outward path for every $i$ in $\{1, 2\}$,
4. every directed cycle of length three in $D_i$ has at most two obstructions,
then $D$ has an $H$-kernel.

With the main result of this paper we show that the main result in [11] can be reduced for digraphs as follows.

Let $D = D_1 \cup D_2$ be a finite $m$-colored digraph which is a union of asymmetric quasi-transitive digraphs such that
1. every chromatic class is quasi-transitive,
2. if $C$ is a chromatic class, then $C \subseteq A(D_j)$ for some $j$ in $\{1, 2\}$ and
3. $D$ does not contain 3-colored directed cycles of length three.
Then $D$ has an $mp$-kernel.

In terms of $H$-kernels Theorem 1 says that if $H$ is a complete digraph without loops and $D$ is a finite $H$-colored quasi-transitive digraph such that every directed cycle of length 3 contained in $D$ has no obstructions, then $D$ has an $H$-kernel. However, the above is not true if $H$ is not complete; consider the directed cycle of length 3, $C_3$, whose arcs are colored with three different vertices of $H$, with $A(H) = \emptyset$, it is clear that $C_3$ has no $H$-kernel. In this paper we will deduce from the main result the following.

Let $H$ be a digraph (possibly with loops), $D$ an $H$-colored asymmetric quasi-transitive digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(C(D))$ with the property $P^*$. Suppose that
1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. $D$ has no infinite outward path,
3. every cycle of length three in $D$ has at most two obstructions.
Then $D$ has an $H$-kernel.

We will need the following result.

**Corollary 2** ([2], p. 53). If a quasi-transitive digraph $D$ has an $xy$-path but $(x, y) \notin A(D)$, then either $(y, x) \in A(D)$ or there exists vertices $u$ and $v$ in $V(D) \setminus \{x, y\}$ such that $(x, u, v, y)$ and $(y, u, v, x)$ are paths in $D$. 

2. Terminology and Notation

For general concepts we refer the reader to [2] and [4]. An arc of the form \((x, x)\) is a loop. An arc \((u, v)\) in \(A(D)\) is asymmetric if \((v, u) \notin A(D)\). We will say that a digraph \(D\) is asymmetric if every arc of \(D\) is asymmetric. We will say that two digraphs \(D_1\) and \(D_2\) are equal, denoted by \(D_1 = D_2\), if \(A(D_1) = A(D_2)\) and \(V(D_1) = V(D_2)\). A directed walk is a sequence \(W = (v_0, v_1, \ldots, v_n)\) such that \((v_i, v_{i+1}) \in A(D)\) for each \(i\) in \(\{0, \ldots, n - 1\}\). The number \(n\) is the length of the walk. We will say that the directed walk \((v_0, v_1, \ldots, v_n)\) is closed if \(v_0 = v_n\). If \(v_i \neq v_j\) for all \(i\) and \(j\) such that \(\{i, j\} \subseteq \{0, \ldots, n\}\) and \(i \neq j\), it is called a directed path. A directed cycle is a directed walk \((v_1, v_2, \ldots, v_n, v_1)\) such that \(v_i \neq v_j\) for all \(i\) and \(j\) such that \(\{i, j\} \subseteq \{1, \ldots, n\}\) and \(i \neq j\), this will be denoted by \(C_n\). If \(D\) is an infinite digraph, an infinite outward path is an infinite sequence \((v_1, v_2, \ldots)\) of distinct vertices of \(D\) such that \((v_i, v_{i+1}) \in A(D)\) for each \(i \in \mathbb{N}\). In this paper we are going to write walk, path, cycle, instead of directed walk, directed path, directed cycle, respectively. The union of walks will be denoted with \(\cup\). Let \(W = (v_0, v_1, \ldots, v_n)\) be a walk and \(\{v_i, v_j\} \subseteq V(W)\), with \(i < j\). Then the \(v_ivj\)-walk \((v_i, v_{i+1}, \ldots, v_{j-1}, v_j)\) contained in \(W\) will be denoted by \((v_i, W, v_j)\). For a subset \(S\) of \(V(D)\) the subdigraph of \(D\) induced by \(S\), denoted by \(D[S]\), has \(V(D[S]) = S\) and \(A(D[S]) = \{(u, v) \in A(D) : \{u, v\} \subseteq S\}\). A subset \(S\) of \(V(D)\) is said to be independent if the only arcs in \(D[S]\) are loops. For a subset \(B\) of \(A(D)\) the subdigraph of \(D\) induced by \(B\), denoted by \(D[B]\), has \(A(D[B]) = B\) and \(V(D[B]) = \{v \in V(D) : (u, v) \in B\text{ or } (v, u) \in B\text{ for some } u \in V(D)\}\). A pair of digraphs \(D\) and \(G\) are isomorphic if there exists a bijection \(f : V(D) \rightarrow V(G)\), such that \((x, y) \in A(D)\) if and only if \((f(x), f(y)) \in A(G)\) (\(f\) will be called isomorphism). We will say that a digraph \(D\) is complete if for every pair of different vertices \(u\) and \(v\) in \(V(D)\) it holds that \(\{(u, v), (v, u)\} \subseteq A(D)\).

A digraph \(D\) is said to be \(m\)-colored if the arcs of \(D\) are colored with \(m\) colors. A path is called monochromatic if all of its arcs are colored alike.

3. Previous Results

For the rest of the work \(H\) is a digraph possibly with loops and \(D\) is a, possibly infinite, digraph without loops.

We need to introduce some notation in order to present our proofs more compactly.

Let \(H\) be a digraph and \(D\) an \(H\)-colored digraph. Consider \(\{u, v\}\) and \(S\) two subsets of \(V(D)\). We will write \(u \xrightarrow{H}_D v\) if there exists a \(uv\)-\(H\)-path in \(D\); \(u \xrightarrow{H}_D S\) if there exists a \(uS\)-\(H\)-path in \(D\); \(u \xrightarrow{H}_D \overline{v}\) is the denial of \(u \xrightarrow{H}_D v\); \(u \xrightarrow{H}_D \overline{S}\) is
the denial of $u \xrightarrow{H} D S$.

We will start with some results which will be useful.

From now on, the set $\{a \in A(D) : c(a) \in V_i\}$ will be denoted by $B_i$ for every $i$ in $\{1, \ldots, k\}$.

**Lemma 3.** Let $H$ be a digraph and $D$ an $H$-colored digraph. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathcal{C}(D))$ with the property $P^*$. Then the following properties are satisfied.

1. Let $i$ be an index in $\{1, \ldots, k\}$. Every finite path in $D[B_i]$ is an $H$-path in $D[\mathcal{C}(D)]$.
2. If $P$ is a finite $H$-path in $D$, then there exists $i$ in $\{1, \ldots, k\}$ such that $P$ is contained in $D[B_i]$.

**Proof.** Let $P = (u_0, \ldots, u_m)$ be a path in $D[B_i]$. We will prove that $P$ is an $H$-path in $D$. It follows from the definition of color-class digraphs that $P' = (c(u_0, u_1), \ldots, c(u_{m-1}, u_m))$ is a walk in $\mathcal{C}(D)$. Since $c(u_j, u_{j+1}) \in V_i$ for every $j$ in $\{0, \ldots, m-1\}$, then $P'$ is a walk in $\mathcal{C}(D)[V_i]$, which implies that $P'$ is a walk in $H$ (because $\mathcal{C}(D)[V_i]$ is a subdigraph of $H$). Therefore $P$ is an $H$-path in $D[B_i]$.

On the other hand, let $P = (v_0, \ldots, v_n)$ be an $H$-path in $D$. Then, it follows from the definition of $H$-paths and the definition of color-class digraphs that $(c(v_{j-1}, v_j), c(v_j, v_{j+1})) \in A(H) \cap A(\mathcal{C}(D))$ for every $j$ in $\{1, \ldots, n-1\}$. Therefore, we get from 2 in definition of $P^*$ that there exists $i$ in $\{1, \ldots, k\}$ such that $c(v_j, v_{j+1}) \in V_i$ for every $j$ in $\{0, \ldots, n-1\}$. Thus $P$ is contained in $D[B_i]$.

**Lemma 4.** Let $H$ be a digraph, $D$ an $H$-colored digraph and $\{w, z\} \subseteq V(D)$. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathcal{C}(D))$ such that $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$. If there exists a $wz$-path in $D[B_j]$ and there exists no $zw$-path in $D[B_j]$ for some $j$ in $\{1, \ldots, k\}$, then $(w, z) \in A(D[B_j])$.

**Proof.** It follows from Corollary 2.

We can obtain an extension of Lemma 4 as follows.

**Lemma 5.** Let $H$ be a digraph and $D = D_1 \cup D_2$ an $H$-colored digraph. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathcal{C}(D))$ with the property $P^*$ such that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. either $D[B_i]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in $\{1, \ldots, k\}$.

Let $r$ be an index in $\{1, 2\}$. If $x \xrightarrow{H} z$ and $z \xrightarrow{H} x$, then $(x, z) \in A(D_r)$.
Proof. Let \( P \) be an \( xz\)-\( H \)-path in \( D_r \). It follows from Lemma 3 that there exists \( i \) in \( \{1, \ldots, k\} \) such that \( P \) is contained in \( D[B_i] \). The hypothesis implies that \( D[B_i] \) is a subdigraph of \( D_r \) (because \( P \) is in \( D_r \)). On the other hand, since 
\[
z \xrightarrow{H} z \quad 
\]
there exists no \( zx\)-\( H \)-path in \( D[B_i] \), which implies that there exists no \( zx\)-path in \( D[B_i] \) (by 1 in Lemma 3). Therefore, we get from Lemma 4 that 
\[
(x, z) \in A(D[B_i]). \quad \text{So, } (x, z) \in A(D_r). \quad \blacksquare
\]

The following result will be useful in what follows.

Lemma 6. Let \( H \) be a digraph, \( D = D_1 \cup D_2 \) an \( H \)-colored digraph and \( \{V_1, \ldots, V_k\} \) a partition of \( V(\mathcal{C}_C(D)) \) with the property \( P^* \). If either \( D[B_i] \) is a subdigraph of \( D_1 \) or it is a subdigraph of \( D_2 \) for every \( i \) in \( \{1, \ldots, k\} \), then there exists a partition of \( V(\mathcal{C}_C(D_r)) \) with the property \( P^* \) for every \( r \) in \( \{1, 2\} \).

Proof. Suppose that \( \{V_1, \ldots, V_k\} \) is such that \( D[B_i] \) is a subdigraph of \( D_1 \) for every \( i \) in \( \{1, \ldots, t\} \) and \( \{V_{t+1}, \ldots, V_k\} \) is such that \( D[B_j] \) is a subdigraph of \( D_2 \) for every \( j \) in \( \{t+1, \ldots, k\} \). Then, considering that \( D_1 \) and \( D_2 \) are also \( H \)-colored digraphs, it follows that \( \{V_1, \ldots, V_t\} \) is a partition of \( V(\mathcal{C}_C(D_1)) \) with the property \( P^* \) and \( \{V_{t+1}, \ldots, V_k\} \) is a partition of \( V(\mathcal{C}_C(D_2)) \) with the property \( P^* \) (this follows from the fact that \( \mathcal{C}_C(D_r) \) is a subdigraph of \( \mathcal{C}_C(D) \) for every \( r \) in \( \{1, 2\} \) and the fact that either \( V_i \subseteq V(\mathcal{C}_C(D_1)) \) or \( V_i \subseteq V(\mathcal{C}_C(D_2)) \) for every \( i \) in \( \{1, \ldots, k\} \)). \( \blacksquare \)

Proposition 7. Let \( H \) be a digraph, \( D = D_1 \cup D_2 \) an \( H \)-colored digraph which is a union of asymmetric quasi-transitive digraphs, \( r \) an index in \( \{1, 2\} \), \( \{x, y\} \subseteq V(D) \) and \( \{V_1, \ldots, V_k\} \) a partition of \( V(\mathcal{C}_C(D)) \) with the property \( P^* \). Suppose that

1. \( V_i \) is a quasi-transitive \( V_i \)-class for every \( i \) in \( \{1, \ldots, k\} \),
2. either \( D[B_i] \) is a subdigraph of \( D_1 \) or it is a subdigraph of \( D_2 \) for every \( i \) in \( \{1, \ldots, k\} \),
3. every cycle of length three in \( D_i \) has at most two obstructions for every \( i \) in \( \{1, 2\} \),
4. \( x \xrightarrow{H} D_r y \) and \( y \xrightarrow{H} D_r x \).

If \( z \) is a vertex in \( V(D) \) such that \( y \xrightarrow{H} D_r z \), then \( (x, z) \in A(D_r) \); if \( z \xrightarrow{H} D_r x \), then \( (z, y) \in A(D_r) \).

Proof. Notice that it follows from Lemma 5 and hypothesis 4 of this proposition that \( (x, y) \in A(D_r) \).

If \( y \xrightarrow{H} D_r z \), then let \( (y = w_0, \ldots, w_m = z) \) be a \( yz\)-\( H \)-path in \( D_r \). We will prove that \( (x, z) \in A(D_r) \) by induction on \( m \).
If $m = 0$, it is clear that $(x, z) \in A(D_r)$ (because in this case $y = w_0 = z$).

Suppose that if $(y = w_0, \ldots, u_{m-1})$ is a $yu_{m-1}$-$H$-path in $D_r$ with length $m - 1$, then $(x, u_{m-1}) \in A(D_r)$.

Let $P = (y = a_0, \ldots, a_m)$ be a $ya_m$-$H$-path in $D_r$ with length $m$. We will prove that $(x, a_m) \in A(D_r)$. Since $(y, P, a_{m-1})$ is a $ya_{m-1}$-$H$-path in $D_r$ with length $m - 1$, it follows from the induction hypothesis that $(x, a_{m-1}) \in A(D_r)$. Since $\{(x, a_{m-1}), (a_{m-1}, a_m)\} \subseteq A(D_r)$ and $D_r$ is a quasi-transitive digraph, it follows that $\{(x, a_m), (a_m, x)\} \cap A(D_r) \neq \emptyset$. If $(a_m, x) \in A(D_r)$, then $\gamma = (x, a_{m-1}, a_m, x)$ is a cycle of length three in $D_r$ which has at most two obstructions by hypothesis 3 of this proposition. If there is no obstruction on $a_{m-1}$, we have that $P' = (x, a_{m-1}, a_m)$ is an $H$-path in $D_r$, then we get by Lemma 6 and by Lemma 3 that there exists $i$ in $\{1, \ldots, k\}$ such that $D[B_i]$ is a subdigraph of $D_r$ and $P'$ is contained in $D[B_i]$, respectively. Since $D[B_i]$ is a quasi-transitive digraph, $P'$ is a path with length two in $D[B_i], (a_m, x) \in A(D_r)$ and $D_i$ is an asymmetric digraph, we get that $(a_m, x) \in A(D[B_i])$. This implies that $\gamma$ is contained in $D[B_i]$. In the same way, we can conclude that $\gamma$ is contained in $D[B_j]$ for some $j$ in $\{1, \ldots, k\}$ if either there is no obstruction on $x$ or there is no obstruction on $a_m$. Therefore, in particular, we get from Lemma 3 that $(a_{m-1}, a_m, x)$ is an $H$-path in $D_r$, that is $(c(a_{m-1}, a_m), c(a_m, x)) \in A(H)$. Thus $P \cup (a_m, x)$ is an $yx$-$H$-path in $D_r$, a contradiction with hypothesis 4 of this proposition. Therefore, $(a_m, x) \notin A(D_r)$, which implies that $(x, a_m) \notin A(D_r)$.

If $z \xrightarrow{H} D_r \xrightarrow{H} x$, then we can consider the converse of $D$ and the converse of $H$ (where the converse of a digraph $G$ is the digraph $\bigtriangleup G$ which one obtains from $G$ by reversing all arcs). It is clear that the digraph $\overline{D} = \overline{D_1} \cup \overline{D_2}$ is an $\overline{H}$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(\overline{C}(\overline{D}))$ with the property $P^*$, with respect to $\overline{H}$. In addition, the hypothesis 1, 2 and 3 fulfill in this digraph $\overline{H}$-colored, in the context of the new related digraphs associated, and hypothesis 4 says that $y \xrightarrow{\overline{H}} \xrightarrow{\overline{H}} x$ and $x \xrightarrow{\overline{H}} \xrightarrow{\overline{H}} y$. Since in $\overline{D}$ we have that $x \xrightarrow{\overline{H}} \xrightarrow{\overline{H}} z$, then we conclude from the previous case that $(y, z) \in A(\overline{D_r})$. Therefore, $(z, y) \in A(D_r)$.

Notice that, since an arc $(w, t)$ in $D_r$ defines a $wt$-$H$-path in $D_r$, we also can conclude from Proposition 7 that $x \xrightarrow{H} D_r \xrightarrow{H} z$ if $(x, z) \in A(D_r)$ or $z \xrightarrow{H} y \xrightarrow{H} y$ if $(z, y) \in A(D_r)$.

**Proposition 8.** Let $H$ be a digraph, $D = D_1 \cup D_2$ an $H$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ a partition of $V(C(D))$ with the property $P^*$. Suppose that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. either $D[B_i]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in \{1, …, $k$\},

3. every cycle of length three in $D_i$ has at most two obstructions.

Then there exists no cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in $D_r$, with $r$ in \{1, 2\}, such that $u_{i+1} \xrightarrow{H} u_i$ for every $i$ in \{0, …, $n$\} (indices modulo $n + 1$).

**Proof.** Proceeding by contradiction, suppose that there exists a cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in $D_r$, for some $r$ in \{1, 2\}, of minimum length such that $u_{i+1} \xrightarrow{H} u_i$ for every $i$ in \{0, …, $n$\} (indices modulo $n + 1$). Notice that there exists $j_0$ in \{0, …, $n$\} such that there is an obstruction on $u_{j_0}$ in $\gamma$, otherwise $u_{i+1} \xrightarrow{H} u_i$ for every $i$ in \{0, …, $n$\} (indices modulo $n + 1$), which is a contradiction. Suppose without loss of generality that there is an obstruction on $u_1$, that is $(c(u_0, u_1), c(u_1, u_2)) \notin A(H)$. Since $u_0 \xrightarrow{H} u_1$ (because $(u_0, u_1) \in A(D_r)$), $u_1 \xrightarrow{H} u_0$ and $u_1 \xrightarrow{H} u_2$ (because $(u_1, u_2) \in A(D_r)$), we get from Proposition 7 that $(u_0, u_2) \in A(D_r)$. Because of that $\gamma' = (u_0, u_2) \cup (u_2, \gamma, u_0)$ is a cycle with length less than the length of $\gamma$, it follows from the choice of $\gamma$ that $u_2 \xrightarrow{H} u_0$. Therefore, since $u_1 \xrightarrow{H} u_2$ (because $(u_1, u_2) \in A(D_r)$), $u_2 \xrightarrow{H} u_1$ and $u_2 \xrightarrow{H} u_0$, we get from Proposition 7 that $(u_1, u_0) \in A(D_r)$, which is a contradiction.

Therefore, there exists no cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in $D_r$, with $r$ in \{1, 2\}, such that $u_{i+1} \xrightarrow{H} u_i$ for every $i$ in \{0, …, $n$\} (indices modulo $n + 1$). \[\square\]

**Definition.** Let $H$ be a digraph, $D$ an $H$-colored digraph and $G$ a subdigraph of $D$. We will say that a subset $S$ of $V(D)$ is an $H$-semikernel modulo $G$ in $D$ if

1. $S$ is an $H$-independent set in $D$,
2. if some vertex $x$ in $V(D) \setminus S$ is such that $x \xrightarrow{H} u$ in $S$, then there exists $s$ in $S$ such that $x \xrightarrow{H} s$.

**Proposition 9.** Let $H$ be a digraph, $D = D_1 \cup D_2$ an $H$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ a partition of $V(G(C)(D))$ with the property $P^*$. Suppose that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in \{1, …, $k$\},
2. either $D[B_i]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in \{1, …, $k$\},
3. $D_i$ has no infinite outward path for every $i$ in \{1, 2\},
4. every cycle of length three in $D_i$ has at most two obstructions.
Then there exists $x$ in $V(D)$ such that $\{x\}$ is an $H$-semikernel modulo $D_r$ in $D$, with $r$ in $\{1, 2\}$.

**Proof.** Suppose without loss of generality that $r = 1$. Proceeding by contradiction, suppose that for every $w$ in $V(D)$ there exists $v_w$ in $V(D) \setminus \{w\}$ such that $w \overset{H}{D_2} v_w$ and $v_w \overset{H}{D} w$. Therefore, for every $n$ in $\mathbb{N}$ given $w_n$ in $V(D)$ there exists $w_{n+1}$ in $V(D) \setminus \{w_n\}$ such that $w_n \overset{H}{D_2} w_{n+1}$ and $w_{n+1} \overset{H}{D} w_n$. So, it follows from Lemma 5 that $(w_n, w_{n+1}) \in A(D_2)$ for every $n$ in $\mathbb{N}$. If $w_i \neq w_j$ for every $i$ different from $j$, then $(w_n)_{n \in \mathbb{N}}$ is an infinite outward path in $D_2$ which is not possible. Therefore, there exist $w_i$ and $w_j$, with $i < j$, such that $w_i = w_j$, which implies that $(w_i, w_{i+1}, \ldots, w_j = w_i)$ is a closed walk in $D_2$ which contains a cycle $\gamma = (w_{i_0}, w_{i_1}, \ldots, w_{i_0})$ such that $w_{i+1} \overset{H}{D_2} w_{i_s}$ for every $s$ in $\{0, \ldots, t\}$ (indices modulo $t + 1$), a contradiction with Proposition 8. Therefore, there exists $x$ in $V(D)$ such that $\{x\}$ is an $H$-semikernel modulo $D_1$ in $D$. 

### 4. Main Result

**Theorem 10.** Let $H$ be a digraph, $D = D_1 \cup D_2$ an $H$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathbb{C}_C(D))$ with the property $P^*$. Suppose that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. either $D[B_i]$ is a subdigraph of $D_1$ or it is a subdigraph of $D_2$ for every $i$ in $\{1, \ldots, k\}$,
3. $D_i$ has no infinite outward path for every $i$ in $\{1, 2\}$,
4. every cycle of length three in $D_i$ has at most two obstructions for every $i$ in $\{1, 2\}$.

Then $D$ has an $H$-kernel.

**Proof.** $\mathcal{I} = \{S \subseteq V(D) : S$ is $H$-independent in $D\}$ and $\mathcal{L} = \{S \in \mathcal{I} : S$ is an $H$-semikernel modulo $D_1$ in $D\}$.

Since $\{w\}$ is an $H$-independent set for every $w$ in $V(D)$, it follows that $\mathcal{I} \neq \emptyset$; by Proposition 9 we get that $\mathcal{L} \neq \emptyset$.

For sets $S, T$ in $\mathcal{L}$, put $S \lessdot T$ if for all $s$ in $S$ there exists $t$ in $T$ such that either $s = t$, or $s \overset{H}{D_1} t$ and $t \overset{H}{D_1} s$.

**Claim 1.** $(\mathcal{L}, \lessdot)$ is a poset.

**Proof.** Consider $\{S, T, R\}$ a subset of $\mathcal{L}$.
(1.1) \( \leq \) is reflexive. Clearly \( S \leq S \) for every \( S \) in \( \mathcal{L} \).

(1.2) \( \leq \) is antisymmetric. Suppose that \( S \leq T \) and \( T \leq S \). We will prove that \( S = T \). Let \( t \) be a vertex in \( T \) and suppose that \( t \notin S \). Then since \( T \leq S \), we have that there exists \( s \) in \( S \) such that \( t \xrightarrow{H \, D_1} s \) and \( s \xrightarrow{H \, D_1} t \). Because of that \( s \notin T \) (\( T \) is \( H \)-independent) and \( S \leq T \), it follows that there exists \( t' \) in \( T \setminus \{ t \} \) such that \( s \xrightarrow{H \, D_1} t' \) and \( t' \xrightarrow{H \, D_1} s \).

Thus, we get from Proposition 7 that \( t \xrightarrow{H \, D_1} t' \) which contradicts that \( T \) is an \( H \)-independent set in \( D \). Therefore, \( T \subseteq S \) and in the same way we can deduce that \( S \subseteq T \).

(1.3) \( \leq \) is transitive. Suppose that \( S \leq T \) and \( T \leq R \). We will prove that \( S \leq R \), that is, for all \( s \) in \( S \) there exists \( r \) in \( R \) such that either \( s = r \) or \( s \xrightarrow{H \, D_1} r \) and \( r \xrightarrow{H \, D_1} s \). Let \( s \) be a vertex in \( S \). Then \( S \leq T \) implies that there exists \( t \) in \( T \) such that either \( s = t \) or \( s \xrightarrow{H \, D_1} t \) and \( t \xrightarrow{H \, D_1} s \); because of \( T \leq R \) we get that for \( t \) in \( T \) there exists \( r \) in \( R \) such that \( t = r \) or \( t \xrightarrow{H \, D_1} r \) and \( r \xrightarrow{H \, D_1} s \). If \( s = t \), then we have that \( s = r \) or \( s \xrightarrow{H \, D_1} r \) and \( r \xrightarrow{H \, D_1} s \). If \( s \neq t \) and \( t = r \), then \( s \xrightarrow{H \, D_1} r \) and \( r \xrightarrow{H \, D_1} s \).

If \( s \neq t \) and \( t \neq r \), then \( s \xrightarrow{H \, D_1} t \), \( t \xrightarrow{H \, D_1} s \), \( t \xrightarrow{H \, D_1} r \) and Proposition 7 implies that \( s \xrightarrow{H \, D_1} r \). Because of \( t \xrightarrow{H \, D_1} s \) and \( s \xrightarrow{H \, D_1} t \) it follows from Proposition 7 that \( r \xrightarrow{H \, D_1} s \) (because \( r \xrightarrow{H \, D_1} t \)).

Claim 2. \( (\mathcal{L}, \leq) \) has maximal elements.

Proof. (2.1) Any chain in \( \mathcal{L} \) has an upper bound in \( \mathcal{L} \).

Let \( \mathcal{C} \) be a chain in \( (\mathcal{L}, \leq) \), consider the following sets.

For \( S \) in \( \mathcal{C} \), let \( N_S \) be the set defined as \( \{ T \in \mathcal{C} : S \leq T \} \). Notice that \( N_S \neq \emptyset \) because \( S \in N_S \).

\[ \mathcal{I}^\infty = \{ s \in \bigcup_{A \in \mathcal{C}} A : \text{there exists } S \in \mathcal{C} \text{ such that } s \in T \text{ for every } T \in N_S \} \]

(2.2) \( \mathcal{I}^\infty \neq \emptyset \).

Proceeding by contradiction, suppose that \( \mathcal{I}^\infty = \emptyset \). Let \( S_0 \) be in \( \mathcal{C} \) and \( s_0 \in S_0 \). Since \( s_0 \notin \mathcal{I}^\infty \), there exists \( S_1 \) in \( N_{S_0} \) such that \( s_0 \notin S_1 \). Because of \( S_0 \leq S_1 \) we get that there exists \( s_1 \) in \( S_1 \) such that \( s_0 \xrightarrow{H \, D_1} s_1 \) and \( s_1 \xrightarrow{H \, D_1} s_0 \). Since \( s_1 \notin \mathcal{I}^\infty \), there exists \( S_2 \) in \( N_{S_1} \) such that \( s_1 \notin S_2 \). Thus, \( S_1 \leq S_2 \) implies that there exists
s_2 \text{ in } S_2 \text{ such that } s_1 \xrightarrow{H_{D_1}} s_2 \text{ and } s_2 \xrightarrow{H_{D_1}} s_1. \text{ Therefore, for every } n \in \mathbb{N} \text{ given } S_n \text{ in } \mathcal{C} \text{ and } s_n \text{ in } S_n \text{ there exist } S_{n+1} \text{ in } N_{S_n} \text{ and } s_{n+1} \text{ in } S_{n+1} \text{ such that } s_n \notin S_{n+1}, s_n \xrightarrow{H_{D_1}} s_{n+1} \text{ and } s_{n+1} \xrightarrow{H_{D_1}} s_n. \text{ Then for every } n \in \mathbb{N} \text{ it follows from Lemma 5 that } (s_n, s_{n+1}) \in A(D_1). \text{ If } s_i \neq s_j \text{ for every } i \text{ different from } j, \text{ then } (s_n)_{n \in \mathbb{N}} \text{ is an infinite outward path in } D_1 \text{ which is not possible. Therefore, there exist } s_i \text{ and } s_j, \text{ with } i < j, \text{ such that } s_i = s_j, \text{ which implies that } (s_i, s_{i+1}, \ldots, s_j = s_i) \text{ is a closed walk in } D_1 \text{ which contains a cycle } \gamma = (s_{i_0}, s_{i_1}, \ldots, s_{i_t}, s_{i_0}) \text{ such that } s_{i+t+1} \xrightarrow{H_{D_2}} s_i \text{ for every } s \text{ in } \{0, \ldots, t\} \text{ (indices modulo } t+1\text{), a contradiction with Proposition 8. Therefore, } \mathcal{I}^\infty \neq \emptyset.

(2.3) \mathcal{I}^\infty \text{ is an } H\text{-independent set in } D. \text{ Proceeding by contradiction, suppose that there exists a subset } \{u, v\} \text{ of } \mathcal{I}^\infty, u \neq v, \text{ such that } u \xrightarrow{H_{D_1}} v. \text{ Since } \{u, v\} \subseteq \mathcal{I}^\infty, \text{ there exists a subset } \{S_0, T_0\} \text{ of } \mathcal{C} \text{ such that } u \in S \text{ for every } S \in N_{S_0} \text{ and } v \in T \text{ for every } T \in N_{T_0}. \text{ Since } \mathcal{C} \text{ is a chain, we can suppose without loss of generality that } S_0 \subseteq T_0. \text{ Thus, because of } T_0 \subseteq N_{S_0} \text{ we get that } u \in T_0, \text{ which contradicts that } T_0 \text{ is an } H\text{-independent set in } D \text{ (because } v \in T_0). \text{ Therefore, } \mathcal{I}^\infty \text{ is an } H\text{-independent set in } D.

(2.4) \mathcal{I}^\infty \in \mathcal{L}. \text{ Suppose that there exist } u \text{ in } V(D) \setminus \mathcal{I}^\infty \text{ and } s \text{ in } \mathcal{I}^\infty \text{ such that } s \xrightarrow{H_{D_1}} u. \text{ We will prove that there exists } w \text{ in } \mathcal{I}^\infty \text{ such that } u \xrightarrow{H_{D_1}} w. \text{ Proceeding by contradiction, suppose that } u \xrightarrow{H_{D_1}} \mathcal{I}^\infty.

\text{Consider } S_1 \text{ in } \mathcal{C} \text{ such that } s \in S_1. \text{ Since } S_1 \in \mathcal{L}, \text{ there exists } s_1 \in S_1 \text{ such that } u \xrightarrow{H_{D_1}} s_1. \text{ Because of } u \xrightarrow{H_{D_1}} \mathcal{I}^\infty \text{ we get that } s_1 \notin \mathcal{I}^\infty; \text{ it follows from the fact } s \xrightarrow{H_{D_2}} u, \text{ the fact that } S_1 \text{ is an } H\text{-independent set, and by Proposition 7 that } u \xrightarrow{H_{D_2}} s_1, \text{ which implies that } u \xrightarrow{H_{D_1}} s_1. \text{ Since } s_1 \notin \mathcal{I}^\infty, \text{ there exists } S_2 \text{ in } N_{S_1} \text{ such that } s_1 \notin S_2. \text{ Thus, } S_1 \subseteq S_2 \text{ implies that there exists } s_2 \in S_2 \text{ such that } s_1 \xrightarrow{H_{D_1}} s_2 \text{ and } s_2 \xrightarrow{H_{D_1}} s_1. \text{ Then, we get from Proposition 7 that } u \xrightarrow{H_{D_1}} s_2 \text{ (because } u \xrightarrow{H_{D_1}} s_1), \text{ which implies that } s_2 \notin \mathcal{I}^\infty \text{ (because } u \xrightarrow{H_{D_1}} \mathcal{I}^\infty). \text{ Hence, since } s_2 \notin \mathcal{I}^\infty, \text{ we get that there exists } S_3 \text{ in } N_{S_2} \text{ such that } s_2 \notin S_3; \text{ the fact } S_2 \subseteq S_3 \text{ implies that there exists } s_3 \in S_3 \text{ such that } s_2 \xrightarrow{H_{D_1}} s_3 \text{ and } s_3 \xrightarrow{H_{D_1}} s_2. \text{ Then, from Proposition 7 and the fact } u \xrightarrow{H_{D_1}} s_2, \text{ we get that } u \xrightarrow{H_{D_1}} s_3, \text{ which implies that } s_3 \notin \mathcal{I}^\infty \text{ (because } u \xrightarrow{H_{D_1}} \mathcal{I}^\infty). \text{ With this procedure we have that for every } n \in \mathbb{N} \text{ given } S_n \text{ in } \mathcal{C} \text{ and } s_n \text{ in } V(D) \setminus \mathcal{I}^\infty \text{ such that } s_n \in S_n \text{ there exist}
$S_{n+1}$ in $N_S$, $s_{n+1}$ in $S_{n+1}$ such that $s_{n+1} \notin \mathcal{J}^\infty$, $s_n \xrightarrow{H} s_{n+1}$, $s_{n+1} \xrightarrow{H} s_n$ and $u \xrightarrow{H} s_{n+1}$. Therefore, we get from Lemma 5 that $(s_n, s_{n+1}) \in A(D_1)$ and since $(s_n)_{n \in \mathbb{N}}$ cannot be an infinite outward path in $D_1$, there exist $s_i$ and $s_j$, with $i < j$, such that $s_i = s_j$, which implies that $(s_i, s_{i+1}, \ldots, s_j = s_i)$ is a closed walk in $D_1$ which contains a cycle $\gamma = (s_{i_0}, s_{i_1}, \ldots, s_{i_k}, s_{i_0})$ such that $s_{i+1} \xrightarrow{H} s_{i_1}$ for every $s$ in $\{0, \ldots, t\}$ (indices modulo $t + 1$), a contradiction with Proposition 8. Therefore, there exists $w$ in $\mathcal{J}^\infty$ such that $u \xrightarrow{H} w$.

(2.5) $S \subseteq \mathcal{J}^\infty$ for every $S$ in $\mathcal{C}$. Let $S$ be in $\mathcal{C}$ and $u$ in $S$. We will prove that there exists $w$ in $\mathcal{J}^\infty$ such that $u = w$ or $[u \xrightarrow{H} w$ and $w \xrightarrow{H} u]$. Suppose that $u \notin \mathcal{J}^\infty$. Then there exists $S_1$ in $N_S$ such that $u \notin S_1$; $S \subseteq S_1$ implies that there exists $s_1$ in $S_1$ such that $u \xrightarrow{H} s_1$ and $s_1 \xrightarrow{H} u$. If $s_1 \in \mathcal{J}^\infty$, then we are done; otherwise since $s_1 \notin \mathcal{J}^\infty$, there exists $S_2$ in $N_{S_1}$ such that $s_1 \notin S_2$. Thus, $S_1 \subseteq S_2$ implies that there exists $s_2$ in $S_2$ such that $s_1 \xrightarrow{H} s_2$ and $s_2 \xrightarrow{H} s_1$. Then, we get from Proposition 7 that $u \xrightarrow{H} s_2$ (because $u \xrightarrow{H} s_1$). Therefore, proceeding in the same way as in (2.4) and considering that both $D_1$ has no infinite outward paths and $D_1$ has no cycle as the cycle in Proposition 8, we conclude that there exists a sequence of vertices $s_1, s_2, \ldots, s_n$, for some $n$ in $\mathbb{N}$, such that $s_n$ in $\mathcal{J}^\infty$, $u \xrightarrow{H} s_n$; for every $i$ in $\{1, \ldots, n - 1\}$ $s_i \xrightarrow{H} s_{i+1}$, $s_{i+1} \xrightarrow{H} s_i$, $s_i \notin \mathcal{J}^\infty$ and $u \xrightarrow{H} s_i$. It remains to prove that $s_n \xrightarrow{H} u$. Proceeding by contradiction, suppose that $s_n \xrightarrow{H} u$. Then in this case considering that $s_i \xrightarrow{H} s_{i+1}$ and $s_{i+1} \xrightarrow{H} s_i$ for every $i$ in $\{1, \ldots, n - 1\}$, we can apply $n - 1$ times Proposition 7 and conclude that $s_j \xrightarrow{H} u$ for every $j$ in $\{1, \ldots, n - 1\}$, in particular $s_1 \xrightarrow{H} u$, which is not possible. Therefore, $s_n \xrightarrow{H} u$.

Therefore, we have proved that any chain in $\mathcal{L}$ has an upper bound in $\mathcal{L}$, and so, by Zorn’s Lemma, it follows that $(\mathcal{L}, \leq)$ contains a maximal element. □

Let $N$ be a maximal element of $(\mathcal{L}, \leq)$.

**Claim 3.** $N$ is an $H$-kernel of $D$.

**Proof.** Since $N$ is an $H$-independent set in $D$, it remains to prove that $N$ is an $H$-absorbent set in $D$. Proceeding by contradiction, suppose that $N$ is not an $H$-absorbent set in $D$. Then the set $X = \{x \in V(D) \setminus N : x \xrightarrow{H} N\}$ is not empty.
(3.1) There exists \(x_0\) in \(X\) such that if \(x_0 \xrightarrow{H}{D_2} y\), for some \(y\) in \(X\), then \(y \xrightarrow{H}{D} x_0\). The proof of (3.1) is similar to the proof given in Proposition 9.

Consider the sets \(T = \{v \in N : v \xrightarrow{H}{D_1} x_0\}\), \(B = N \setminus T\) and \(K = B \cup \{x_0\}\).

(3.2) \(K\) is \(H\)-independent in \(D\).

Since \(B\) is \(H\)-independent in \(D\) and \(x_0 \xrightarrow{H}{D} B\), it remains to prove that \(B \xrightarrow{H}{D} x_0\). It follows from the definition of \(B\) that \(B \xrightarrow{H}{D_1} x_0\). On the other hand, since \(N\) is an \(H\)-semikernel modulo \(D_1\) in \(D\) (because \(N \in \mathcal{L}\)), we get that \(B \xrightarrow{H}{D_2} x_0\) (because \(x_0 \xrightarrow{H}{D} N\)). Therefore, \(B \xrightarrow{H}{D} x_0\) (recall 2 in Lemma 3 and 2 of this theorem).

(3.3) \(K \notin \mathcal{L}\).

Proceeding by contradiction, suppose that \(K \in \mathcal{L}\). We will see that \(N \subseteq K\). Let \(u\) be in \(N\) and suppose that \(u \notin K\). We will prove that there exists \(t\) in \(K\) such that \(u \xrightarrow{H}{D_1} t\) and \(t \xrightarrow{H}{D_1} u\). Since \(u \in T\) (because \(N = T \cup B\)), we get that \(u \xrightarrow{H}{D_1} x_0\), and because of \(x_0 \xrightarrow{H}{D} N\), we have that \(x_0 \xrightarrow{H}{D_1} u\). Therefore, \(x_0\) is the vertex desired. Hence \(N \subseteq K\), which is not possible because \(K \neq N\) and \(N\) is maximal.

Since \(K \notin \mathcal{L}\) and \(K\) is \(H\)-independent in \(D\), it follows from the definition of \(\mathcal{L}\) that \(K\) is not an \(H\)-semikernel modulo \(D_1\) in \(D\), that is, there exist \(v\) in \(K\) and \(w\) in \(V(D) \setminus K\) such that \(v \xrightarrow{H}{D_2} w\) and \(w \xrightarrow{H}{D} K\). Notice that \((v, w) \in A(D_2)\) (by Lemma 5).

(3.4) \(v = x_0\).

Proceeding by contradiction, suppose that \(v \neq x_0\). The fact \(v \in B\) implies that \(w \notin N\). Since \(N\) is an \(H\)-semikernel modulo \(D_1\) in \(D\) and \(w \xrightarrow{H}{D} K\), it follows that there exists \(t\) in \(T\) such that \(w \xrightarrow{H}{D} t\). The fact \(t \in T\) implies that \(t \xrightarrow{H}{D_1} x_0\) and since \(x_0 \xrightarrow{H}{D_1} t\), we get from Proposition 7 that \(w \xrightarrow{H}{D_1} t\) (because \(w \xrightarrow{H}{D} x_0\)), which implies that \(w \xrightarrow{H}{D_1} t\). Therefore, \(v \xrightarrow{H}{D_2} w\), \(w \xrightarrow{H}{D_2} v\) and \(w \xrightarrow{H}{D_2} t\) implies that \(v \xrightarrow{H}{D_2} t\) (by Proposition 7), which contradicts that \(N\) is an \(H\)-independent set in \(D\).

Since \(v = x_0\), it follows from the choice of \(x_0\) that \(w \notin X\) (because \(w \xrightarrow{H}{D} x_0\)). Notice that \(w \notin N\) by definition of \(X\) and because \(x_0 \in X\). Since \(w \in V(D) \setminus (N \cup X)\), we get from the definition of \(X\) that there exists \(t\) in \(T\) such that \(w \xrightarrow{H}{D} t\).
The fact $t \in T$ implies that $t \xrightarrow{H_{D_1}} x_0$, and since $x_0 \xrightarrow{H} x_1$, we get from Proposition 7 that $w \xrightarrow{H_{D_1}} t$ (because $w \xrightarrow{H} x_0$), which implies that $w \xrightarrow{H} t$. Therefore, $v \xrightarrow{H_{D_2}} w$, $w \xrightarrow{H_{D_2}} v$ and $w \xrightarrow{H} t$ implies that $v \xrightarrow{H} t$ (by Proposition 7), which contradicts that $x_0 \xrightarrow{H} N$.

Therefore, $N$ is $H$-absorbent in $D$. Thus, $N$ is an $H$-kernel of $D$. \hfill $\Box$

5. Some Consequences of Theorem 10

**Corollary 11.** Let $D = D_1 \cup D_2$ be a finite $m$-colored which is a union of asymmetric quasi-transitive digraphs such that

1. every chromatic class is quasi-transitive,
2. if $C$ is a chromatic class, then $C \subseteq A(D_j)$ for some $j$ in $\{1, 2\}$, and
3. $D$ has no $3$-colored $C_3$.

Then $D$ has an $m$-kernel.

**Corollary 12.** Let $H$ be a digraph, $D$ an $H$-colored asymmetric quasi-transitive digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathcal{E}_C(D))$ with the property $P^*$. Suppose that

1. $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, k\}$,
2. $D$ has no infinite outward path,
3. every cycle of length three in $D$ has at most two obstructions.

Then $D$ has an $H$-kernel.

**Proof.** Let $D^*$ be an asymmetric quasi-transitive digraph such that $D^*$ and $D$ are isomorphic, with $V(D) \cap V(D^*) = \emptyset$, and let $H^*$ be a digraph such that $H^*$ and $H$ are isomorphic, with $V(H) \cap V(H^*) = \emptyset$. Consider $f : V(D) \to V(D^*)$ and $g : V(H) \to V(H^*)$ two isomorphisms. Suppose that $D^*$ is an $H^*$-colored digraph such that $(u, v)$ has color $i$ in $D$ if and only if $(f(u), f(v))$ has color $g(i)$ in $D^*$. Therefore, it follows that $D^*$ holds the same hypotheses as $D$.

Let $D' = D \cup D^*$. Notice that $D'$ is an $H'$-colored digraph (with $V(H') = V(H) \cup V(H^*)$ and $A(H') = A(H) \cup A(H^*)$) which is a union asymmetric of quasi-transitive digraphs. If $V_i^* = \{g(j) : j \in V_i\}$ for every $i$ in $\{1, \ldots, k\}$, then $\{V_1', \ldots, V_k'\}$ is a partition of $V(\mathcal{E}_C(D^*))$ which has the property $P^*$ and so $\{V_1', \ldots, V_k'\}$ is a partition of $V(\mathcal{E}_C(D'))$ which has the property $P^*$. Now consider that the hypotheses of Theorem 10 fulfill on $D'$ by the definition of $D^*$, the definition of $H^*$, the $H^*$-coloring of $D^*$ and the hypotheses on $D$. 
Therefore, it follows from Theorem 10 that $D'$ has an $H'$-kernel, say $N$. Therefore, it follows from the definition of $D'$ that $N \cap V(D)$ is an $H$-kernel of $D$.

**Corollary 13.** Let $D$ be a finite $m$-colored asymmetric quasi-transitive digraph such that
1. every chromatic class is quasi-transitive,
2. $D$ has no 3-colored $C_3$.

Then $D$ has an $mp$-kernel.

**Proof.** Notice that in this case the arcs of $D$ are colored with the vertices of $H$, where $V(H) = \{1, \ldots, m\}$ and $A(H) = \{(u, u) : u \in V(H)\}$. Since (a) $\{V_1 = \{1\}, \ldots, V_m = \{m\}\}$ is a partition of $V(C_3(D))$ which has the property $P^*$, (b) $V_i$ is a quasi-transitive $V_i$-class for every $i$ in $\{1, \ldots, m\}$ (because $D[B_i]$ is a chromatic class of $D$), and (c) every cycle with length three in $D$ has at most two obstructions (by hypothesis in 2), it follows from Corollary 12 that $D$ has an $H$-kernel which is an $mp$-kernel.

**Corollary 14.** Let $T$ be a finite $m$-colored tournament such that
1. every chromatic class is quasi-transitive,
2. $T$ has no 3-colored $C_3$.

Then $T$ has an $mp$-kernel.

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**References**


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