GRAPH EXPONENTIATION AND NEIGHBORHOOD RECONSTRUCTION

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Abstract

Any graph \( G \) admits a neighborhood multiset \( \mathcal{N}(G) = \{N_G(x) \mid x \in V(G)\} \) whose elements are precisely the open neighborhoods of \( G \). We say \( G \) is neighborhood reconstructible if it can be reconstructed from \( \mathcal{N}(G) \), that is, if \( G \cong H \) whenever \( \mathcal{N}(G) = \mathcal{N}(H) \) for some other graph \( H \). This note characterizes neighborhood reconstructible graphs as those graphs \( G \) that obey the exponential cancellation \( G^K_2 \cong H^K_2 \Rightarrow G \cong H \).

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Our graphs are finite and may have loops, but not parallel edges. The open neighborhood of a vertex \( x \) of a graph \( G \) is \( N_G(x) := \{y \in V(G) \mid xy \in E(G)\} \). Notice that \( x \in N_G(x) \) if and only if \( xx \in E(G) \), that is, there is a loop at \( x \).

To any graph \( G \) there is an associated neighborhood multiset \( \mathcal{N}(G) = \{N_G(x) \mid x \in V(G)\} \) whose elements are the open neighborhoods of \( G \). It is possible that \( \mathcal{N}(G) = \mathcal{N}(H) \) but \( G \not\cong H \). Figure 1 shows the simplest instance of this. Here \( G \not\cong H \) but \( \mathcal{N}(G) = \{\{0\}, \{1\}\} = \mathcal{N}(H) \). Figure 2 shows a more complex and interesting example.

Figure 1. Two non-isomorphic graphs with the same neighborhood multiset.
A graph $G$ is called neighborhood reconstructible if $\mathcal{N}(G) = \mathcal{N}(H)$ implies $G \cong H$ for any graph $H$ with $V(H) = V(G)$. Figure 2 shows that the Petersen graph is not neighborhood reconstructible. Aigner and Triesch [1] attribute the neighborhood reconstruction problem to Sós [9]. They note that deciding if a graph is not neighborhood reconstructible is NP-complete.

Given graphs $G$ and $K$, the graph exponential $G^K$ is the graph whose vertex set is the set of all functions $V(K) \to V(G)$, where two functions $f, g$ are adjacent precisely if $f(x)g(y) \in E(G)$ for all $xy \in E(K)$. (See [6, 8].) If $V(K) = \{v_1, \ldots, v_n\}$, then a function $f : V(K) \to V(G)$ can be identified with an $n$-tuple $f = (x_1, \ldots, x_n) \in V(G)^n$ signifying $f(v_i) = x_i$.

We are interested exclusively in $G^{K_2}$. Note $V(G^{K_2}) = V(G) \times V(G)$, and two functions $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if and only if $x_1y_2 \in E(G)$ and $x_2y_1 \in E(G)$. That is,

$$E(G^{K_2}) = \{(x_1, x_2)(y_1, y_2) \mid x_1y_2 \in E(G) \text{ and } x_2y_1 \in E(G)\}.$$ 

See Figure 3, which shows that $G^K \cong H^K$ does not necessarily imply $G \cong H$.

![Figure 3. Two exponentials $G^{K_2}$ and $H^{K_2}$. This shows $G^K \cong H^K$ may not imply $G \cong H$.](image-url)

Actually, the conditions under which $G^K \cong H^K$ implies $G \cong K$ are not fully understood today. (The issue is further complicated by the fact that there are at least two definitions of graph exponentiation; compare [4].) This note links one instance of this exponential cancellation to neighborhood reconstruction. Our
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main result is that $G$ is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2}$ implies $G \cong H$ for all graphs $H$. To understand why we might expect this, consider Proposition 1 below, whose proof is almost automatic. (Figures 1 and 3 illustrate Proposition 1.)

**Proposition 1.** If $G$ and $H$ are two graphs on the same vertex set and $\mathcal{N}(G) = \mathcal{N}(H)$, then $G^{K_2} \cong H^{K_2}$.

**Proof.** Say $\mathcal{N}(G) = \mathcal{N}(H)$. As $G$ and $H$ have the same neighborhood multiset, there is a bijection $\varphi : V(G) \rightarrow V(H)$ for which $N_G(x) = N_H(\varphi(x))$ for each $x \in V(G)$. (Such map $\varphi$ is unique if no two vertices of $G$ have the neighborhood; otherwise there is more than one $\varphi$.) The bijection $\lambda : V(G^{K_2}) \rightarrow V(H^{K_2})$ where $\lambda(x, y) = (\varphi(x), y)$ is an isomorphism. Indeed,

$$
(x, y)(u, v) \in E(G^{K_2}) \iff (\varphi(x), y)(\varphi(u), v) \in E(H^{K_2}) \iff \lambda(x, y)(\varphi(u), v) \in E(H^{K_2}).
$$

We will use this proposition in the proof of our main result. We will also need the direct product of graphs: $G \times H$ is the graph whose vertex set is the set $V(G) \times V(H)$, and whose edges are

$$E(G \times H) = \{(x, y)(x', y') \mid xx' \in E(G) \text{ and } yy' \in E(H)\}.$$ 

See Chapter 8 of [2] for a survey of the direct product.

For a positive integer $k$, the direct power $G^k$ is $G \times \cdots \times G$ ($k$ factors). Any square $G^2$ admits a mirror automorphism $\mu : G^2 \rightarrow G^2$ of order 2, where $\mu(x, y) = (y, x)$. From the definitions it is immediate that

1. $(x, y)(u, v) \in E(G^2)$ if and only if $(x, y)\mu(u, v) \in E(G^{K_2})$,
2. $(x, y)(u, v) \in E(G^{K_2})$ if and only if $\mu(x, y)(u, v) \in E(G^2)$.

Recall the following two results (by Lovász) concerning direct powers and products. (They are Theorems 2 and 5, respectively, in [7].)

**Proposition 2.** If $G^k \cong H^k$ for a positive integer $k$, then $G \cong H$.

**Proposition 3.** If $G \times K \cong H \times K$, then there is an isomorphism $G \times K \rightarrow H \times K$ of form $(x, y) \mapsto (\lambda(x, y), y)$ for some map $\lambda : G \times K \rightarrow H$.

Actually, we will only need a weaker instance of Proposition 3, one that is easy to prove from scratch. If $G \times K_2 \cong H \times K_2$, then there exists an isomorphism $G \times K_2 \rightarrow H \times K_2$ of form $(x, y) \mapsto (\lambda(x, y), y)$. 
We are ready for our main theorem.

**Theorem 4.** A graph $G$ is neighborhood reconstructible if and only if the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds for any graph $H$.

**Proof.** Say the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds. Let $\mathcal{N}(G) = \mathcal{N}(H)$ for a graph $H$ with $V(H) = V(G)$. Proposition 1 yields $G^{K_2} \cong H^{K_2}$, whence $G \cong H$. Thus $G$ is neighborhood reconstructible.

Conversely, suppose $G$ is neighborhood reconstructible. Say $G^{K_2} \cong H^{K_2}$ for some graph $H$. We must show $G \cong H$.

Put $V(K_2) = \{0, 1\}$. Take an isomorphism $\varphi : G^{K_2} \rightarrow H^{K_2}$. Using (1) and (2), observe that

\[
(x, y)(u, v) \in E(G^2) \iff (x, y) \mu(u, v) \in E(G^{K_2})
\]

\[
\iff \varphi(x, y) \varphi(u, v) \in E(H^{K_2})
\]

\[
\iff \mu \varphi(x, y) \varphi(u, v) \in E(H^2).
\]

From this we get an isomorphism $\Theta : G^2 \times K_2 \rightarrow H^2 \times K_2$ defined as

\[
\Theta((x, y), \varepsilon) = \begin{cases} 
(\varphi \mu(x, y), \varepsilon) & \text{if } \varepsilon = 0, \\
(\mu \varphi(x, y), \varepsilon) & \text{if } \varepsilon = 1.
\end{cases}
\]

From $G^2 \times K_2 \cong H^2 \times K_2$ we get $G^2 \times K_2 \times K_2 \cong H^2 \times K_2 \times K_2$, yielding $(G \times K_2)^2 \cong (H \times K_2)^2$. By Proposition 2 we have $G \times K_2 \cong H \times K_2$. Then Proposition 3 guarantees an isomorphism $\theta : G \times K_2 \rightarrow H \times K_2$ having form

\[
\theta(x, \varepsilon) = \begin{cases} 
(\lambda_0(x), \varepsilon) & \text{if } \varepsilon = 0, \\
(\lambda_1(x), \varepsilon) & \text{if } \varepsilon = 1
\end{cases}
\]

for two bijections $\lambda_0, \lambda_1 : V(G) \rightarrow V(H)$, which (by definition of the direct product) necessarily satisfy $xy \in E(G)$ if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$.

Now form a graph $H'$ on $V(G)$ whose edges are precisely $\lambda_1^{-1}(u)\lambda_1^{-1}(v)$ for each $uv \in E(H)$. Thus $\lambda_1^{-1} : H \rightarrow H'$ is an isomorphism.

We claim that $N_G(x) = N_{H'}(\lambda_1^{-1}\lambda_0(x))$ for each $x \in V(G) = V(H')$. Note $y \in N_G(x)$ if and only if $xy \in E(G)$, if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$, if and only if $\lambda_1^{-1}\lambda_0(x)\lambda_1^{-1}\lambda_1(y) \in E(H')$, if and only if $\lambda_1^{-1}\lambda_0(x)\lambda_1^{-1}\lambda_1(y) \in E(H')$, if and only if $y \in N_{H'}(\lambda_1^{-1}\lambda_0(x))$. Thus indeed $N_G(x) = N_{H'}(\lambda_1^{-1}\lambda_0(x))$.

Consequently $\mathcal{N}(G) = \mathcal{N}(H')$, so $G \cong H'$ because $G$ is neighborhood reconstructible. But $H' \cong H$, so $G \cong H$.

The present note is a sequel to [3], which characterizes neighborhood reconstructible graphs as those graphs $G$ which obey the cancellation law $G \times K \cong H \times K \Rightarrow G \cong K$ for all graphs $H$ and $K$. 

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References


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