NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES\footnote{This work was supported by NSFC (No.11671232), HNSF(No.A2015202301) and HUSTP (No.ZD2015106, QN2017044).}

XUE ZHAO

AND

CHANGQING XU\footnote{Corresponding author.}

School of Science,
Hebei University of Technology
Tianjin 300401, P.R. China

e-mail: zhaoxhxy@163.com
chqxn@hebut.edu.cn

Abstract

For a given graph $G = (V(G), E(G))$, a proper total coloring $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ is neighbor sum distinguishing if $f(u) \neq f(v)$ for each edge $uv \in E(G)$, where $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. The smallest integer $k$ in such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak first introduced this coloring and conjectured that $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$ for any graph with maximum degree $\Delta(G)$. In this paper, by using the discharging method, we prove that for any planar graph $G$ without 5-cycles, $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 10\}$. The bound $\Delta(G) + 2$ is sharp. Furthermore, we get the exact value of $\chi''_{\Sigma}(G)$ if $\Delta(G) \geq 9$.

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph $G$, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If $G$ is a planar graph embedded in the plane, we use $F(G)$ to denote its face set. A vertex $v$ is a $t$-vertex, $t^-$-vertex, $t^+$-vertex if $d_G(v) = t$, $d_G(v) \leq t$, $d_G(v) \geq t$ in $G$, respectively. A $t$-face is defined similarly. An $t$-face $v_1v_2 \cdots v_l$ is a $(b_1, b_2, \ldots, b_l)$-face, where $v_j$ is a $b_j$-vertex, for $i = 1, 2, \ldots, l$. Let $d'_G(v)$ denote the number of $t$-vertices adjacent to $v$ in $G$. Let $n^d_G(v)$ denote the number of $d$-faces incident with $v$ in $G$. A configuration $F$ is reducible to $G$, if it cannot be a configuration of $G$.

Given a graph $G$, set $n_i(G) = |\{v \in V(G) : d_G(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph $G'$ is smaller than $G$ if one of the following holds:

1. $|E(G')| < |E(G)|$,
2. $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_1(G))$ with respect to the standard lexicographic order, where $t = \max \{\Delta(G), \Delta(G')\}$.

A graph is minimum for a property if no smaller graph satisfies it.

Given a graph $G$ and a positive integer $k$, a proper total $k$-coloring of $G$ is a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements $x$, $y \in V(G) \cup E(G)$. Let $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. If $f(u) \neq f(v)$ for each edge $uv \in E(G)$, then $\phi$ is a neighbor sum distinguishing total $k$-coloring, or $k$-tnsd-coloring for simplicity. The smallest number $k$ is the neighbor sum distinguishing total chromatic number of $G$, denoted by $\chi'^n_k(G)$. For $k$-tnsd-coloring, Pilśniak and Woźniak gave the following conjecture.

**Conjecture 1** [11]. For any graph $G$, $\chi'^n_k(G) \leq \Delta(G) + 3$.

Pilśniak and Woźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong et al. [3] showed that Conjecture 1 holds for some sparse graphs. Yao et al. [21, 22] considered tnsd-coloring of degenerate graphs. Li et al. [9] proved that Conjecture 1 holds for $K_4$-minor free graphs. Song et al. [15] determined $\chi'^n_k(G)$ for $K_4$-minor free graph $G$ with $\Delta(G) \geq 5$. For planar graph, it was proved that this conjecture holds with $\Delta(G) \geq 13$ by Li et al. [7] and $\Delta(G) \geq 11$ by Qu et al. [12]. For planar graph, it was proved that $\chi'^n_k(G) \leq \Delta(G) + 2$ holds with $\Delta(G) \geq 14$ by Cheng et al. [2], $\Delta(G) \geq 12$ by Song et al. [14] and $\Delta(G) \geq 11$ by Yang et al. [20]. The bound $\Delta(G) + 2$ is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].
Recently, Ge et al. [6] got the following result.

**Theorem 2** [6]. Let $G$ be a planar graph without 5-cycles. Then
\[
\chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 3, 10 \}.
\]

In this paper, we prove the following results.

**Theorem 3.** Let $G$ be a planar graph without 5-cycles. Then
\[
\chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 2, 10 \}.
\]

**Theorem 4.** Let $G$ be a planar graph without 5-cycles and without adjacent $\Delta(G)$-vertices. Then $\chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 1, 10 \}$.

Clearly, $\chi_{\Sigma}^n(G) \geq \Delta(G) + 1$ for any graph $G$. If $G$ has adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) \geq \Delta(G) + 2$. Thus we get the following corollary.

**Corollary 5.** Let $G$ be a planar graph without 5-cycles and $\Delta(G) \geq 9$. If $G$ has no adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) = \Delta(G) + 1$, otherwise $\chi_{\Sigma}^n(G) = \Delta(G) + 2$.

## 2. The Proof of Theorem 3

We will prove it by contradiction. Let $G$ be a minimum counterexample to Theorem 3 which is embedded in the plane. Set $k = \max \{ \Delta(G) + 2, 10 \}$. By the choice of $G$, any planar graph $G'$ without 5-cycles which is smaller than $G$ has a $k$-tnsd-coloring $\phi'$. In the following, we will choose some $G'$ and extend the coloring $\phi'$ of $G$ to a desired coloring $\phi$ of $G$ to get a contradiction. 

Unless otherwise stated, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$, set $\phi(x) = \phi'(x)$.

In the following proof, we will omit the coloring of all $3^-$-vertices. Since they have at most 9 forbidden colors and $k \geq 10$, they can be colored easily.

In Figure 1, we draw a vertex $x$ in black if it has no other neighbors than the ones already depicted, and a vertex $x$ in white if it might have more neighbors than the ones shown in the figure.

**Claim 1.** These configurations of $F_1$, $F_2$, $F_3$, and $F_4$ in Figure 1 are reducible.

**Proof.** (1) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F_1'$ in Figure 1). Thus $G'$ is a planar graph without 5-cycles which is smaller than $G$. Hence $G'$ admits a $k$-tnsd-coloring $\phi'$. We can stick $u_i$, $v_i$ together properly for $i = 1, 2$ (if necessary, exchange the colors of $uv_1$ and $uv_2$), and then recolor $u_i$, $v_i$, thus we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(2) Suppose to the contrary that $G$ contains configuration $F_2$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F_2'$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$.
(i) If $\phi'(wu_1) \neq \phi'(uu_2)$ or $\phi'(wu_1) = \phi'(uu_2) \notin \{\phi'(vv_1), \phi'(vv_2)\}$, then we can stick $u_i$, $v_i$ together for $i = 1, 2$ (if necessary, exchange the colors of $vv_1$ and $vv_2$).

(ii) If $\phi'(wu_1) = \phi'(uu_2) \in \{\phi'(vv_1), \phi'(vv_2)\}$, without loss of generality, suppose that $\phi'(wu_2) = \phi'(vv_1)$. Exchange the colors of $vv_1$ ($wu_2$) and $uv$. Therefore, we can stick $u_i$, $v_i$ together for $i = 1, 2$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(3) Suppose to the contrary that $G$ contains configuration $F_3$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}$, $v_{i2}$ for $i = 1, 3$ (see $F'_3$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$.

(i) If $\phi'(wu_{12}) \neq \phi'(vv_{32})$ or $\phi'(wu_{12}) = \phi'(vv_{32}) \notin \{\phi'(vv_{11}), \phi'(vv_{31})\}$, then we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 3$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{31}$).

(ii) If $\phi'(wu_{12}) = \phi'(vv_{32}) \in \{\phi'(vv_{11}), \phi'(vv_{31})\}$, without loss of generality, suppose that $\phi'(wu_{12}) = \phi'(vv_{11})$. Then we exchange the colors of $wu_{12}$ and $uv$. Therefore, we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 3$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(4) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}$, $v_{i2}$ for $i = 1, 4$ (see $F'_4$ in Figure 1) without producing 5-cycles. Thus $G'$ admits a $k$-tnsd-coloring $\phi'$.

(i) If $\phi'(uw_{12}) \neq \phi'(zv_{42})$ or $\phi'(uw_{12}) = \phi'(zv_{42}) \notin \{\phi'(vv_{11}), \phi'(vv_{41})\}$, then we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 4$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{41}$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of Claim 1.}
\end{figure}
(ii) If $\phi'(uv_{12}) = \phi'(zv_{42}) \in \{ \phi'(vv_{11}), \phi'(vv_{41}) \}$, without loss of generality, suppose that $\phi'(uv_{12}) = \phi'(zv_{42}) = \phi'(vv_{11})$. Since $\phi'(uv_{2}) \neq \phi'(vv_{3})$, suppose that $\phi'(uv_{2}) \neq \phi'(uv_{12})$. We exchange the colors of $uv_{12}$ and $uv_{2}$. Therefore, we can stick $v_{i1}, v_{i2}$ together for $i = 1, 4$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

It is easy to see that the following claim given in [16] also holds with the graph $G$ in our proof.

**Claim 2** [16]. In the graph $G$, the following results holds.

1. Each $t$-vertex is not adjacent to any $(7-t)$-vertex, where $t = 4, 5$.
2. For each vertex $v \in V(G)$, if $d_{G}^{1}(v) \geq 1$, then $d_{G}^{2}(v) = 0$; if $d_{G}^{1}(v) \geq 2$, then $d_{G}^{2}(v) = 0$.
3. If $d_{G}(v) = 5$, then $d_{G}^{2}(v) \leq 1$.
4. If $d_{G}(v) = 6$, then $d_{G}^{2}(v) \leq 2$. Furthermore, if $d_{G}^{2}(v) \geq 1$, then $d_{G}^{3}(v) \leq 1$.
5. If $d_{G}(v) = 7$, then $d_{G}^{2}(v) \leq 2$. Furthermore, if $d_{G}^{2}(v) \geq 1$, then $d_{G}^{3}(v) \leq 2$.
6. If $d_{G}(v) = t (t \geq 8)$, then $d_{G}^{1}(v) < \left[ \frac{t}{2} \right]$.
7. If $d_{G}(v) = t (t \geq 8)$ and $d_{G}^{2}(v) \geq 1$, then $d_{G}^{2}(v) + d_{G}^{3}(v) \leq t - 1$.
8. Each 3-face in $G$ is a $(2, 6^{+}, 6^{+})$-face, a $(3, 5^{+}, 5^{+})$-face or a $(4^{+}, 4^{+}, 5^{+})$-face.

**Claim 3.** Each 4-face in $G$ is a $(2, 6^{+}, 3^{+}, 6^{+})$-face, a $(3, 6^{+}, 3, 6^{+})$-face, a $(3, 5^{+}, 4, 5^{+})$-face or a $(4^{+}, 4^{+}, 4, 4^{+})$-face.

**Proof.** Let $T = v_{1}v_{2}v_{3}v_{4}$ be a 4-face of $G$, and assume that $d_{G}(v_{1}) \leq d_{G}(v_{i})$, where $i = 2, 3, 4$. If $d_{G}(v_{1}) = 2$, by Claim 2(1), $d_{G}(v_{2}) \geq 6, d_{G}(v_{4}) \geq 6$. By Claim 1, $F_{1}$ is reducible, thus $T$ is a $(2, 6^{+}, 3^{+}, 6^{+})$-face. If $d_{G}(v_{1}) = d_{G}(v_{4}) = 3$, by Claim 2(1) and Claim 2(3), $d_{G}(v_{2}) \geq 6$ and $d_{G}(v_{3}) \geq 6$, thus $T$ is a $(3, 6^{+}, 3, 6^{+})$-face. If $d_{G}(v_{1}) = 3$ and $d_{G}(v_{3}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 5$ and $d_{G}(v_{4}) \geq 5$, thus $T$ is a $(3, 5^{+}, 4, 5^{+})$-face. If $d_{G}(v_{1}) \geq 4$ and $d_{G}(v_{3}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 4$ and $d_{G}(v_{4}) \geq 4$, thus $T$ is a $(4^{+}, 4^{+}, 4^{+}, 4^{+})$-face.

Let $H$ be the graph obtained from $G$ by removing all 1-vertices. By Claims 1–3, we have the following facts.

**Fact 1.** For the graph $H$, we have $\delta(H) \geq 2; d_{H}(v) = d_{G}(v)$, for $2 \leq d_{G}(v) \leq 5$. If $d_{G}(v) \geq 6$, then $d_{H}(v) \geq 5$.

**Fact 2.**

1. In the graph $H$, each 3-vertex is not adjacent to any 4-vertex.
2. If $d_{H}(v) = 5$, then $d_{H}^{2}(v) = 0$ and $d_{H}^{3}(v) \leq 1$.
3. If $d_{H}(v) = 6$, then $d_{H}^{2}(v) \leq 1$; furthermore, if $d_{H}^{2}(v) = 1$, then $d_{H}^{3}(v) = 0$; if $d_{H}^{2}(v) = 0$, then $d_{H}^{3}(v) \leq 2$. 

If \( d_H(v) = 7 \), then \( d^2_H(v) \leq 2 \); furthermore, if \( d^2_H(v) = 2 \), then \( d^3_H(v) = 0 \); if \( d^2_H(v) = 1 \), then \( d^3_H(v) \leq 1 \).

If \( d_H(v) = l \) (\( l \geq 8 \)), then \( d^2_H(v) \leq l - 1 \).

**Fact 3.**

1. Each 3-face in \( H \) is a \((2, 6^+, 6^+)-face\), a \((3, 5^+, 5^+)-face\) or a \((4^+, 4^+, 5^+)-face\).
2. Each 4-face in \( H \) is a \((2, 6^+, 3^+, 6^+)-face\), a \((3, 6^+, 3^+, 6^+)-face\), a \((3, 5^+, 4^+, 5^+)-face\) or a \((4^+, 4^+, 4^+, 4^+)-face\).

A \((2, 6^+, 6^+)-face\) or a \((3, 5^+, 5^+)-face\) is called a *bad* 3-face. A \((4^+, 5^+, 5^+)-face\) is called a *normal* 3-face. A \((2, 6^+, 3^+, 6^+)-face\) or a \((3, 6^+, 3^+, 6^+)-face\) is called a *bad* 4-face, and other 4-face is a *normal* 4-face. We use \( n'_i(v) \), \( n''_i(v) \) to denote the number of bad \( i \)-faces and the number of normal \( i \)-faces incident with \( v \) in \( H \), respectively, \( i = 3, 4 \).

Since \( G \) has no 5-cycles, we have the following fact.

**Fact 4.** These configurations are reducible to \( H \):

1. a 5-face,
2. a 3-face adjacent to two 3-faces,
3. a 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

**Fact 5.** If \( d_H(v) = l \) and \( n'_3(v) > 0 \), then \( n^2_3(v) + n^4_3(v) \leq l - 2 \).

By Euler’s formula, we have

\[
\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.
\]

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: \( w(v) = 2d_H(v) - 6 \) for each \( v \in V(H) \); \( w(f) = d_H(f) - 6 \) for each \( f \in F(H) \). Next, we will design some discharging rules. Let \( w' \) be the new charge after the discharging process. It suffices to show that \( w'(x) \geq 0 \) for each \( x \in V(H) \cup F(H) \), which leads to a contradiction.

In the following, a \( k \)-face means a \( k \)-face in \( H \), the discharging rules are defined as follows.

- **R1** Every 2-vertex \( v \) in \( H \) takes 1 from each neighbor.
- **R2** Every 4-vertex \( v \) in \( H \) gives 1 to each incident 3-face, gives \( \frac{1}{2} \) to each incident 4-face.
- **R3** Every \( 5^+ \)-vertex \( v \) in \( H \) gives \( \frac{3}{2} \) to each incident bad 3-face, gives 1 to each incident normal 3-face.
R4 Every $5^+$-vertex $v$ in $H$ gives 1 to each incident bad 4-face, gives $\frac{3}{4}$ to each incident normal 4-face.

We will verify the new charge of each $x \in V(H) \cup F(H)$. In the following, we use $d(v)$, $d_i(v)$, $n_i(v)$ and $d(f)$ to denote $d_H(v)$, $d_i^H(v)$, $n_i^H(v)$ and $d_H(f)$, respectively. We first consider the new charge of each $f \in F(H)$.

- $d(f) = 3$. If $f$ is a bad 3-face, by R3, $w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0$; otherwise, by R2 and R3, $w'(f) = 3 - 6 + 1 \cdot 3 = 0$.

- $d(f) = 4$. If $f$ is a bad 4-face, by R4, $w'(f) = 4 - 6 + 1 \cdot 2 = 0$. If $f$ is a $(2, 6^+, 4^+, 6^+)$-face or a $(3, 5^+, 4^+, 5^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{3}{2} \cdot 2 + \frac{1}{2} = 0$. If $f$ is a $(4^+, 4^+, 4^+, 4^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0$.

- $d(f) = t$ ($t \geq 6$). $w'(f) = w(f) = t - 6 \geq 0$.

Next we will consider the new charge of each $v \in V(H)$.

- $d(v) = 2$. By R1, $w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0$.

- $d(v) = 3$. No rule applies to $v$, $w'(v) = 2 \cdot 3 - 6 = 0$.

- $d(v) = 4$. By Fact 2(1), $d_2(v) = d_3(v) = 0$. If $n_3(v) = 0$, by R2, $w'(v) = 2 \cdot 4 - 6 - \frac{1}{2} \cdot n_4(v) \geq 2 - \frac{1}{2} \cdot 4 = 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 2$.

By R2, $w'(v) = 2 \cdot 4 - 6 - 1 \cdot n_3(v) - \frac{1}{2} \cdot n_4(v) \geq 2 - 1 \cdot 2 = 0$.

- $d(v) = 5$. By Fact 2(2), $d_2(v) = d_3(v) = 0$, $d_3(v) \leq 1$, so we have $n_3''(v) \leq 2$ and $n_4''(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{4} \cdot n_4''(v) \geq 4 - \frac{3}{4} \cdot 5 = \frac{1}{4} > 0$.

If $n_3(v) > 0$, by Fact 5, $n_3''(v) + n_4''(v) \leq 3$. By R3 and R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n_3''(v) - 1 \cdot n_4''(v) - \frac{3}{4} \cdot n_4''(v) \geq 4 - \frac{3}{4} \cdot 2 - 1 = 0$.

- $d(v) = 6$. By Fact 2(3), $d_2(v) \leq 1$.

(a) $d_2(v) = 1$. By Fact 2(3), $d_3(v) = 0$, so we have $n_3''(v) \leq 1$ and $n_4''(v) = 0$.

If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_4''(v) \geq 6 - 1 - \frac{3}{2} \cdot 6 = \frac{1}{2} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3''(v) + n_4''(v) \leq 4$. By R1, R3 and R4, $w''(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_3''(v) - 1 \cdot n_4''(v) - \frac{3}{4} \cdot n_4''(v) \geq 6 - 1 \cdot 1 - 1 - 1 - 3 = \frac{1}{2} > 0$.

(b) $d_2(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) \geq 2 \cdot 6 - 6 - 1 \cdot n_4(v) \geq 6 - 1 \cdot 6 = 0$.

If $n_3(v) > 0$, by Fact 5, $n_3''(v) + n_4(v) \leq 4$. By R3 and R4, $w'(v) \geq 2 \cdot 6 - 6 - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 6 - \frac{3}{2} \cdot 4 = 0$.

- $d(v) = 7$. By Fact 2(4), $d_2(v) \leq 2$.

(a) $d_2(v) = 2$. By Fact 2(4), $d_3(v) = 0$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3''(v) = n_4''(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_4''(v) \geq 8 - 2 - \frac{3}{2} \cdot 7 = \frac{3}{2} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 5$. Noting that $n_3''(v) = n_4''(v) = 0$, by R1, R3 and R4, $w''(v) = 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_3''(v) - \frac{3}{4} \cdot n_4''(v) \geq 8 - 2 - 1 \cdot 5 > 0$.

(b) $d_2(v) \leq 1$. If $n_3(v) = 0$, by R1 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_4(v) \geq 8 - 1 \cdot 7 = 0$. If $n_3(v) > 0$, by Fact 4 and Fact 5, $n_3(v) \leq 4$ and $n_3(v) + n_4(v) \leq 5$.

By R1, R3 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 8 - 1 \cdot \frac{3}{2} \cdot 4 = 0$.

- $d(v) = l$ ($l \geq 8$), by Fact 2(5), $d_2(v) \leq l - 1$. 
(a) \(d_2(v) = l - 1\). By Claim 1, \(F_1\) and \(F_2\) are reducible, so we have \(n_3(v) = 0\) and \(n_4(v) \leq 2\). By R1 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 1) - 1 \cdot 2 = l - 7 > 0\).

(b) \(d_2(v) = l - 2\).

(b1) \(n_3(v) = 0\). By Claim 1, \(F_1\) is reducible, so we have \(n_4(v) \leq 4\). By R1 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 2) - 4 = l - 8 \geq 0\).

(b2) \(n_3(v) > 0\). By Claim 1, \(F_1\) and \(F_2\) are reducible, and by Fact 4, we have \(n_3(v) = 1\) and \(n_4(v) = 0\). By R1 and R3, \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 2) - \frac{3}{2} = l - \frac{11}{2} > 0\).

(c) \(d_2(v) = l - 3\).

(c1) \(n_3(v) = 0\). By Claim 1, \(F_1\) is reducible, so we have \(n_4(v) \leq 6\).

If \(n_4(v) = 6\), by Claim 1, \(F_3\) is reducible, so we have \(n_4'(v) = 0\). By R1 and R4, \(w'(v) = 2l - 6 - d_2(v) - \frac{3}{4} \cdot n_4'(v) = 2l - 6 - (l - 3) - \frac{3}{4} \cdot 6 = l - \frac{15}{2} > 0\).

If \(n_4(v) \leq 5\), by R1 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - 1 \cdot 5 = l - 8 \geq 0\).

(c2) \(n_3(v) > 0\). By Claim 1, \(F_2\) is reducible, so we have \(n_3(v) \leq 2\). By Claim 1, \(F_1\) is reducible, and by Fact 4, we have \(n_4(v) \leq 2\). By R1, R3 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - \frac{3}{2} \cdot 2 - 2 = l - 8 \geq 0\).

(d) \(d_2(v) = l - 4\).

(d1) \(n_3(v) = 0\). By Claim 1, \(F_1\) is reducible, so we have \(n_4(v) \leq 8\).

If \(n_4(v) = i (i = 7, 8)\). By Claim 1, \(F_3\) is reducible, so we have \(n_4'(v) \leq 8 - i\). By R1 and R4, \(w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_4'(v) \geq 2l - 6 - (l - 4) - 1 \cdot (8 - i) - \frac{3}{4} \cdot (i - (8 - i)) = l - 4 - \frac{i}{2} \geq 0\).

\(n_4(v) \leq 6\). By R1 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot 6 = l - 8 \geq 0\).

(d2) \(n_3(v) > 0\). By Claim 1, \(F_2\) is reducible, so each 2-neighbor of \(v\) is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have \(n_3'(v) \leq 2\).

\(n_3(v) = i (i = 1, 2)\). By Claim 1, \(F_1\) and \(F_2\) are reducible, and note that each 3-face is not adjacent to a 4-face, we have \(n_4(v) \leq 6 - 2i\). By R1, R3 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - \frac{3}{2} \cdot i - 1 \cdot (6 - 2i) = l - 8 + \frac{i}{2} > 0\).

(e) \(d_2(v) = l - 5\).

(e1) \(n_3(v) = 0\). If \(n_4(v) \leq l - 1\), by R1 and R4, \(w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot (l - 1) = 0\). Now suppose that \(n_4(v) = l\). By Claim 1, \(F_1\) is reducible, so we have \(d_2(v) \leq \lfloor \frac{l}{2} \rfloor\). Noting that \(d_2(v) = l - 5\), we have \(8 \leq l \leq 10\). By Claim 1, \(F_1\), \(F_3\) and \(F_4\) are reducible, so we have \(n_4'(v) \leq 4\). By R1 and R4, \(w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_4'(v) \geq 2l - 6 - (l - 5) - 1 \cdot 4 - \frac{3}{4} \cdot (l - 4) = \frac{l}{4} - 2 > 0\).

(e2) \(n_3(v) > 0\). By Claim 1, \(F_2\) is reducible, and by Fact 4, we have \(n_3(v) \leq 3\).
\( n_3(v) = 3 \). By Claim 1, \( F_1 \) is reducible, and by Fact 4, we have \( n_4(v) = 0 \). By R1 and R3, \( w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot 3 = l - \frac{11}{2} > 0. \)

\( n_3(v) = i \) \((i = 1, 2)\). By Claim 1, \( F_1 \) is reducible, and by Fact 4, we have \( n_4(v) \leq 8 - 2i \). By Claim 1, \( F_3 \) is reducible. So if \( n_4(v) = 8 - 2i \), we have \( n'_4(v) = 0 \). By R1, R3 and R4, \( w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - \frac{3}{2} \cdot n'_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - \frac{3}{4} \cdot (8 - 2i) = l - 7 > 0 \). If \( n_4(v) \leq 7 - 2i \), by R1, R3 and R4, \( w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - 1 \cdot (7 - 2i) = l + \frac{2}{3} - 8 > 0. \)

\((f)\) \( d_2(v) \leq l - 6 \). Set \( t = \left\lceil \frac{2l - d_2(v) - 1}{3} \right\rceil \). By Claim 1, \( F_2 \) is reducible, and by Fact 4, we have \( n_3(v) \leq t, n_4(v) \leq l \) and if \( n_3(v) > 0 \), then \( n_3(v) + n_4(v) \leq l - 2. \)

\((f1)\) \( n_3(v) = 0 \), by R1 and R4, \( w'(v) \geq 2l - 6 - d_2(v) - l \geq 2l - 6 - (l - 6) - l = 0. \)

\((f2)\) \( n_3(v) > 0 \), by R1, R3 and R4, \( w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - n_4(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - (l - 2 - n_3(v)) \geq l - 4 - d_2(v) - \frac{1}{2} \cdot l = l - 4 - d_2(v) - \frac{1}{2} \left\lceil \frac{2l - d_2(v) - 1}{3} \right\rceil \geq 0. \)

Now we get that for each \( x \in V(H) \cup F(H) \), \( w'(x) \geq 0 \), which is a contradiction. This completes the proof of Theorem 3.

### 3. The Proof of Theorem 4

The proof of Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let \( G \) be a minimum counterexample to Theorem 4 which is embedded in the plane. Set \( k = \max\{\Delta(G) + 1, 10\} \). By the choice of \( G \), any planar graph \( G' \) without 5-cycles and without adjacent \( \Delta(G) \)-vertices which is smaller than \( G \) has a \( k \)-tsud-coloring \( \phi' \). Similarly, we will choose some \( G' \) and extend the coloring \( \phi' \) of \( G' \) to a desired coloring \( \phi \) of \( G \) to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

### References


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