NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES\(^1\)

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Abstract

For a given graph \(G = (V(G), E(G))\), a proper total coloring \(\phi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}\) is neighbor sum distinguishing if \(f(u) \neq f(v)\) for each edge \(uv \in E(G)\), where \(f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v), v \in V(G)\). The smallest integer \(k\) in such a coloring of \(G\) is the neighbor sum distinguishing total chromatic number, denoted by \(\chi''_{\Sigma}(G)\). Pilśniak and Woźniak first introduced this coloring and conjectured that \(\chi''_{\Sigma}(G) \leq \Delta(G) + 3\) for any graph with maximum degree \(\Delta(G)\). In this paper, by using the discharging method, we prove that for any planar graph \(G\) without 5-cycles, \(\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 10\}\). The bound \(\Delta(G) + 2\) is sharp. Furthermore, we get the exact value of \(\chi''_{\Sigma}(G)\) if \(\Delta(G) \geq 9\).

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph $G$, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If $G$ is a planar graph embedded in the plane, we use $F(G)$ to denote its face set. A vertex $v$ is a $t$-vertex, $t^-$-vertex, $t^+$-vertex if $d_G(v) = t$, $d_G(v) \leq t$, $d_G(v) \geq t$ in $G$, respectively. A $t$-face is defined similarly. An $t$-face $v_1v_2\ldots v_l$ is a $(b_1,b_2,\ldots,b_l)$-face, where $v_i$ is a $b_i$-vertex, for $i = 1, 2, \ldots, l$. Let $d_G^t(v)$ denote the number of $t$-vertices adjacent to $v$ in $G$. Let $n^t_{G}(v)$ denote the number of $d$-faces incident with $v$ in $G$. A configuration $F$ is reducible to $G$, if it cannot be a configuration of $G$.

Given a graph $G$, set $n_i(G) = |\{v \in V(G) : d_G(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph $G'$ is smaller than $G$ if one of the following holds:

1. $|E(G')| < |E(G)|$,
2. $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_1(G))$ with respect to the standard lexicographic order, where $t = \max \{\Delta(G), \Delta(G')\}$.

A graph is minimum for a property if no smaller graph satisfies it.

Given a graph $G$ and a positive integer $k$, a proper total $k$-coloring of $G$ is a mapping $\phi: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. Let $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. If $f(u) \neq f(v)$ for each edge $uv \in E(G)$, then $\phi$ is a neighbor sum distinguishing total $k$-coloring, or $k$-tnsd-coloring for simplicity. The smallest number $k$ is the neighbor sum distinguishing total chromatic number of $G$, denoted by $\chi'^k_{\Sigma}(G)$. For $k$-tnsd-coloring, Pilśniak and Wóźniak gave the following conjecture.

**Conjecture 1** [11]. For any graph $G$, $\chi'^k_{\Sigma}(G) \leq \Delta(G) + 3$.

Pilśniak and Wóźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong et al. [3] showed that Conjecture 1 holds for some sparse graphs. Yao et al. [21, 22] considered tnsd-coloring of degenerate graphs. Li et al. [9] proved that Conjecture 1 holds for $K_4$-minor free graphs. Song et al. [15] determined $\chi'^k_{\Sigma}(G)$ for $K_4$-minor free graph $G$ with $\Delta(G) \geq 5$. For planar graph, it was proved that this conjecture holds with $\Delta(G) \geq 13$ by Li et al. [7] and $\Delta(G) \geq 11$ by Qu et al. [12]. For planar graph, it was proved that $\chi'^k_{\Sigma}(G) \leq \Delta(G) + 2$ holds with $\Delta(G) \geq 14$ by Cheng et al. [2], $\Delta(G) \geq 12$ by Song et al. [14] and $\Delta(G) \geq 11$ by Yang et al. [20]. The bound $\Delta(G) + 2$ is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].
Recently, Ge et al. [6] got the following result.

**Theorem 2** [6]. Let $G$ be a planar graph without 5-cycles. Then
\[ \chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 3, 10 \}. \]

In this paper, we prove the following results.

**Theorem 3.** Let $G$ be a planar graph without 5-cycles. Then
\[ \chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 2, 10 \}. \]

**Theorem 4.** Let $G$ be a planar graph without 5-cycles and without adjacent $\Delta(G)$-vertices. Then $\chi_{\Sigma}^n(G) \leq \max \{ \Delta(G) + 1, 10 \}$.

Clearly, $\chi_{\Sigma}^n(G) \geq \Delta(G) + 1$ for any graph $G$. If $G$ has adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) \geq \Delta(G) + 2$. Thus we get the following corollary.

**Corollary 5.** Let $G$ be a planar graph without 5-cycles and $\Delta(G) \geq 9$. If $G$ has no adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) = \Delta(G) + 1$, otherwise $\chi_{\Sigma}^n(G) = \Delta(G) + 2$.

## 2. The Proof of Theorem 3

We will prove it by contradiction. Let $G$ be a minimum counterexample to Theorem 3 which is embedded in the plane. Set $k = \max \{ \Delta(G) + 2, 10 \}$. By the choice of $G$, any planar graph $G'$ without 5-cycles which is smaller than $G$ has a $k$-tnsd-coloring $\phi'$. In the following, we will choose some $G'$ and extend the coloring $\phi'$ of $G'$ to a desired coloring $\phi$ of $G$ to get a contradiction. Unless otherwise stated, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$, set $\phi(x) = \phi'(x)$.

In the following proof, we will omit the coloring of all 3-vertices. Since they have at most 9 forbidden colors and $k \geq 10$, they can be colored easily.

In Figure 1, we draw a vertex $x$ in black if it has no other neighbors than the ones already depicted, and a vertex $x$ in white if it might have more neighbors than the ones shown in the figure.

**Claim 1.** These configurations of $F_1$, $F_2$, $F_3$ and $F_4$ in Figure 1 are reducible.

**Proof.** (1) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F_1'$ in Figure 1). Thus $G'$ is a planar graph without 5-cycles which is smaller than $G$. Hence $G'$ admits a $k$-tnsd-coloring $\phi'$. We can stick $u_i$, $v_i$ together properly for $i = 1, 2$ (if necessary, exchange the colors of $uu_1$ and $uu_2$), and then recolor $u_i$, $v_i$, thus we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(2) Suppose to the contrary that $G$ contains configuration $F_2$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F_2'$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$. 


(i) If $\phi'(wu_1) \neq \phi'(uu_2)$ or $\phi'(wu_1) = \phi'(uu_2) \notin \{\phi'(vv_1), \phi'(vv_2)\}$, then we can stick $u_i$, $v_i$ together for $i = 1, 2$ (if necessary, exchange the colors of $vv_1$ and $vv_2$).

(ii) If $\phi'(wu_1) = \phi'(uu_2) \in \{\phi'(vv_1), \phi'(vv_2)\}$, without loss of generality, suppose that $\phi'(uu_2) = \phi'(vv_1)$. Exchange the colors of $vv_1$ ($uu_2$) and $uv$. Therefore, we can stick $u_i$, $v_i$ together for $i = 1, 2$. Thus, by recoloring, we can obtain a $k$-tsnd-coloring $\phi$ of $G$, a contradiction.

(3) Suppose to the contrary that $G$ contains configuration $F_3$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}$, $v_{i2}$ for $i = 1, 3$ (see $F'_3$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tsnd-coloring $\phi'$.

(i) If $\phi'(wu_{12}) \neq \phi'(vw_{32})$ or $\phi'(wu_{12}) = \phi'(vw_{32}) \notin \{\phi'(vv_{11}), \phi'(vv_{31})\}$, then we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 3$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{31}$).

(ii) If $\phi'(wu_{12}) = \phi'(vw_{32}) \in \{\phi'(vv_{11}), \phi'(vv_{31})\}$, without loss of generality, suppose that $\phi'(wu_{12}) = \phi'(vv_{11})$. Then we exchange the colors of $wu_{12}$ and $uv_2$. Therefore, we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 3$. Thus, by recoloring, we can obtain a $k$-tsnd-coloring $\phi$ of $G$, a contradiction.

(4) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}$, $v_{i2}$ for $i = 1, 4$ (see $F'_4$ in Figure 1) without producing 5-cycles. Thus $G'$ admits a $k$-tsnd-coloring $\phi'$.

(i) If $\phi'(uw_{12}) \neq \phi'(zv_{42})$ or $\phi'(uw_{12}) = \phi'(zv_{42}) \notin \{\phi'(vv_{11}), \phi'(vv_{41})\}$, then we can stick $v_{i1}$, $v_{i2}$ together for $i = 1, 4$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{41}$).

Figure 1. Illustration of Claim 1.
(ii) If \( \phi'(uv_{12}) = \phi'(zv_{42}) \in \{ \phi'(vv_{11}), \phi'(vv_{41}) \} \), without loss of generality, suppose that \( \phi'(uv_{12}) = \phi'(zv_{42}) = \phi'(vv_{11}) \). Since \( \phi'(uv_{2}) \neq \phi'(uv_{3}) \), suppose that \( \phi'(uv_{2}) \neq \phi'(uv_{12}) \). We exchange the colors of \( uv_{12} \) and \( uv_{2} \). Therefore, we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 4 \). Thus, by recoloring, we can obtain a \( k \)-tsd-coloring \( \phi \) of \( G \), a contradiction.

It is easy to see that the following claim given in [16] also holds with the graph \( G \) in our proof.

**Claim 2** [16]. In the graph \( G \), the following results holds.

1. Each \( t^- \)-vertex is not adjacent to any \( (7-t^-) \)-vertex, where \( t = 4, 5 \).
2. For each vertex \( v \in V(G) \), if \( d_{1}^G(v) \geq 1 \), then \( d_{2}^G(v) = 0 \); if \( d_{1}^G(v) \geq 2 \), then \( d_{3}^G(v) = 0 \).
3. If \( d_{G}(v) = 5 \), then \( d_{3}^G(v) \leq 1 \).
4. If \( d_{G}(v) = 6 \), then \( d_{3}^G(v) \leq 2 \). Furthermore, if \( d_{3}^G(v) \geq 1 \), then \( d_{3}^G(v) \leq 1 \).
5. If \( d_{G}(v) = 7 \), then \( d_{3}^G(v) \leq 2 \). Furthermore, if \( d_{3}^G(v) \geq 1 \), then \( d_{3}^G(v) \leq 2 \).
6. If \( d_{G}(v) = l \) \((l \geq 8)\), then \( d_{3}^G(v) < \left\lfloor \frac{l}{2} \right\rfloor \).
7. If \( d_{G}(v) = l \) \((l \geq 8)\) and \( d_{3}^G(v) \geq 1 \), then \( d_{3}^G(v) + d_{3}^G(v) \leq l - 1 \).
8. Each 3-face in \( G \) is a \((2,6^+,6^+)\)-face, a \((3,5^+,5^+)\)-face or a \((4^+,4^+,5^+)\)-face.

**Claim 3.** Each 4-face in \( G \) is a \((2,6^+,3^+,6^+)\)-face, a \((3,6^+,3,6^+)\)-face, a \((3,5^+,4^+,5^+)\)-face or a \((4^+,4^+,4^+,4^+)\)-face.

**Proof.** Let \( T = v_{1}v_{2}v_{3}v_{4}v_{1} \) be a 4-face of \( G \), and assume that \( d_{G}(v_{1}) \leq d_{G}(v_{4}) \), where \( i = 2, 3, 4 \). If \( d_{G}(v_{1}) = 2 \), by Claim 2(1), \( d_{G}(v_{2}) \geq 6, d_{G}(v_{4}) \geq 6 \). By Claim 1, \( F_{1} \) is reducible, thus \( T \) is a \((2,6^+,3^+,6^+)\)-face. If \( d_{G}(v_{1}) = d_{G}(v_{4}) = 3 \), by Claim 2(1) and Claim 2(3), \( d_{G}(v_{2}) \geq 6 \) and \( d_{G}(v_{4}) \geq 6 \), thus T is a \((3,6^+,3,6^+)\)-face. If \( d_{G}(v_{1}) = 3 \) and \( d_{G}(v_{4}) \geq 4 \), by Claim 2(1), \( d_{G}(v_{2}) \geq 5 \) and \( d_{G}(v_{4}) \geq 5 \), thus T is a \((3,5^+,4^+,5^+)\)-face. If \( d_{G}(v_{1}) \geq 4 \) and \( d_{G}(v_{3}) \geq 4 \), by Claim 2(1), \( d_{G}(v_{2}) \geq 4 \) and \( d_{G}(v_{4}) \geq 4 \), thus \( T \) is a \((4^+,4^+,4^+)\)-face.

Let \( H \) be the graph obtained from \( G \) by removing all 1-vertices. By Claims 1–3, we have the following facts.

**Fact 1.** For the graph \( H \), we have \( \delta(H) \geq 2; d_{H}(v) = d_{G}(v) \), for \( 2 \leq d_{G}(v) \leq 5 \). If \( d_{G}(v) \geq 6 \), then \( d_{H}(v) \geq 5 \).

**Fact 2.**

1. In the graph \( H \), each \( 3^- \)-vertex is not adjacent to any \( 4^- \)-vertex.
2. If \( d_{H}(v) = 5 \), then \( d_{3}^{H}(v) = 0 \) and \( d_{4}^{H}(v) \leq 1 \).
3. If \( d_{H}(v) = 6 \), then \( d_{3}^{H}(v) \leq 1 \); furthermore, if \( d_{H}(v) = 1 \), then \( d_{3}^{H}(v) = 0 \); if \( d_{3}^{H}(v) = 0 \), then \( d_{3}^{H}(v) \leq 2 \).
(4) If \( d_H(v) = 7 \), then \( d_2^2 H(v) \leq 2 \); furthermore, if \( d_2^2 H(v) = 2 \), then \( d_3^3 H(v) = 0 \); if \( d_2^2 H(v) = 1 \), then \( d_3^3 H(v) \leq 1 \).

(5) If \( d_H(v) = l \) \((l \geq 8)\), then \( d_2^2 H(v) \leq l - 1 \).

**Fact 3.**

1. Each 3-face in \( H \) is a \((2, 6^+, 6^+)-face, a (3, 5^+, 5^+)-face or a (4^+, 4^+, 5^+)-face.

2. Each 4-face in \( H \) is a \((2, 6^+, 3^+, 6^+)-face, a (3, 6^+, 3, 6^+)-face, a (3, 5^+, 4^+, 5^+)-face or a (4^+, 4^+, 4^+, 4^+)-face.

A \((2, 6^+, 6^+)-face or a (3, 5^+, 5^+)-face is called a bad 3-face. A \((4^+, 5^+)-face is called a normal 3-face. A \((2, 6^+, 3^+, 6^+)-face or a (3, 6^+, 3, 6^+)-face is called a bad 4-face, and other 4-face is a normal 4-face. We use \( n'_i(v), n''_i(v) \) to denote the number of bad \( i \)-faces and the number of normal \( i \)-faces incident with \( v \) in \( H \), respectively, \( i = 3, 4 \).

Since \( G \) has no 5-cycles, we have the following fact.

**Fact 4.** These configurations are reducible to \( H \):

1. a 5-face,
2. a 3-face adjacent to two 3-faces,
3. a 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

**Fact 5.** If \( d_H(v) = l \) and \( n^3_H(v) > 0 \), then \( n^3_H(v) + n^4_H(v) \leq l - 2 \).

By Euler’s formula, we have

\[
\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.
\]

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: \( w(v) = 2d_H(v) - 6 \) for each \( v \in V(H) \); \( w(f) = d_H(f) - 6 \) for each \( f \in F(H) \). Next, we will design some discharging rules. Let \( w' \) be the new charge after the discharging process. It suffices to show that \( w'(x) \geq 0 \) for each \( x \in V(H) \cup F(H) \), which leads to a contradiction.

In the following, a \( k \)-face means a \( k \)-face in \( H \), the discharging rules are defined as follows.

**R1** Every 2-vertex \( v \) in \( H \) takes 1 from each neighbor.

**R2** Every 4-vertex \( v \) in \( H \) gives 1 to each incident 3-face, gives \( \frac{1}{2} \) to each incident 4-face.

**R3** Every 5\(^+\)-vertex \( v \) in \( H \) gives \( \frac{3}{2} \) to each incident bad 3-face, gives 1 to each incident normal 3-face.
R4 Every 5+-vertex \( v \) in \( H \) gives 1 to each incident bad 4-face, gives \( \frac{3}{4} \) to each incident normal 4-face.

We will verify the new charge of each \( x \in V(H) \cup F(H) \). In the following, we use \( d(v), d_i(v), n_i(v) \) and \( d(f) \) to denote \( d_H(v), d'_H(v), n'_H(v) \) and \( d_H(f) \), respectively. We first consider the new charge of each \( f \in F(H) \).

- \( d(f) = 3 \). If \( f \) is a bad 3-face, by R3, \( w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0 \); otherwise, by R2 and R3, \( w'(f) = 3 - 6 + 1 \cdot 3 = 0 \).

- \( d(f) = 4 \). If \( f \) is a bad 4-face, by R4, \( w'(f) = 4 - 6 + 1 \cdot 2 = 0 \). If \( f \) is a \((2,6^+,4^+,6^+)\)-face or a \((3,5^+,4^+,5^+)\)-face, by R2 and R4, \( w'(f) \geq 4 - 6 + \frac{3}{4} \cdot 2 + \frac{1}{2} = 0 \). If \( f \) is a \((4^+,4^+,4^+,4^+)\)-face, by R2 and R4, \( w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0 \).

- \( d(f) = t (t \geq 6) \). \( w'(f) = w(f) = t - 6 \geq 0 \).

Next we will consider the new charge of each \( v \in V(H) \).

- \( d(v) = 2 \). By R1, \( w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0 \).

- \( d(v) = 3 \). No rule applies to \( v, w'(v) = 2 \cdot 3 - 6 = 0 \).

- \( d(v) = 4 \). By Fact 2(1), \( d_2(v) = d_3(v) = 0 \). If \( n_3(v) = 0 \), by R2, \( w'(v) = 2 \cdot 4 - 6 - \frac{1}{2} \cdot n_4(v) \geq 2 - \frac{1}{2} \cdot 4 = 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n_4(v) \leq 2 \). By R2, \( w'(v) = 2 \cdot 4 - 6 - 1 \cdot n_3(v) - \frac{1}{2} \cdot n_4(v) \geq 2 - 1 \cdot 2 = 0 \).

- \( d(v) = 5 \). By Fact 2(2), \( d_2(v) = d_3(v) = 0, d_4(v) \leq 1 \), so we have \( n'_3(v) \leq 2 \) and \( n'_4(v) = 0 \). If \( n_3(v) = 0 \), by R4, \( w'(v) = 2 \cdot 5 - 6 - \frac{3}{4} \cdot n'_4(v) \geq 4 - \frac{3}{4} \cdot 5 = \frac{1}{4} > 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n'_4(v) \leq 3 \). By R3 and R4, \( w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n'_3(v) - 1 \cdot n'_4(v) \geq 4 - \frac{3}{2} \cdot 2 - 1 = 0 \).

- \( d(v) = 6 \). By Fact 2(3), \( d_2(v) \leq 1 \).
  
  (a) \( d_2(v) = 1 \). By Fact 2(3), \( d_3(v) = 0 \), so we have \( n'_3(v) \leq 1 \) and \( n'_4(v) = 0 \). If \( n_3(v) = 0 \), by R1 and R4, \( w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n'_4(v) \geq 6 - 1 - \frac{3}{2} \cdot 6 = \frac{1}{2} > 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n'_4(v) \leq 4 \). By R1, R3 and R4, \( w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n'_3(v) - 1 \cdot n'_4(v) \geq 6 - 1 - \frac{3}{2} \cdot 1 - 1 \cdot 3 = \frac{1}{2} > 0 \).

  (b) \( d_2(v) = 0 \). If \( n_3(v) = 0 \), by R4, \( w'(v) = 2 \cdot 6 - 6 - 1 \cdot n_4(v) \geq 6 - 1 \cdot 6 = 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n_4(v) \leq 4 \). By R3 and R4, \( w'(v) \geq 2 \cdot 6 - 6 - \frac{3}{2} \cdot n'_3(v) - 1 \cdot n_4(v) \geq 6 - \frac{3}{2} \cdot 4 = 0 \).

- \( d(v) = 7 \). By Fact 2(4), \( d_2(v) \leq 2 \).
  
  (a) \( d_2(v) = 2 \). By Fact 2(4), \( d_3(v) = 0 \). By Claim 1, \( F_1 \) and \( F_2 \) are reducible, so we have \( n'_3(v) = n'_4(v) = 0 \). If \( n_3(v) = 0 \), by R1 and R4, \( w'(v) = 2 \cdot 7 - 6 - d_2(v) - \frac{3}{4} \cdot n'_4(v) \geq 8 - 2 - \frac{3}{4} \cdot 7 = \frac{3}{2} > 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n_4(v) \leq 5 \). Noting that \( n'_3(v) = n'_4(v) = 0 \). By R1, R3 and R4, \( w'(v) = 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n'_4(v) - \frac{3}{4} \cdot n'_3(v) \geq 8 - 1 - 1 \cdot 5 = 0 \).

  (b) \( d_2(v) \leq 1 \). If \( n_3(v) = 0 \), by R1 and R4, \( w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_4(v) \geq 8 - 1 - 1 \cdot 7 = 0 \). If \( n_3(v) > 0 \), by Fact 4 and Fact 5, \( n_3(v) \leq 4 \) and \( n_3(v) + n_4(v) \leq 5 \). By R1, R3 and R4, \( w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 8 - 1 - \frac{3}{2} \cdot 4 - 1 = 0 \).

- \( d(v) = l (l \geq 8) \), by Fact 2(5), \( d_2(v) \leq l - 1 \).
(a) $d_2(v) = l - 1$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3(v) = 0$ and $n_4(v) \leq 2$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 1) - 1 \cdot 2 = l - 7 > 0$.

(b) $d_2(v) = l - 2$.

(b1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 4$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 2) - 4 = l - 8 \geq 0$.

(b2) $n_3(v) > 0$. By Claim 1, $F_1$ and $F_2$ are reducible, and by Fact 4, we have $n_3(v) = 1$ and $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 2) - \frac{3}{2} = l - \frac{11}{2} > 0$.

(c) $d_2(v) = l - 3$.

(c1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 6$.

If $n_4(v) = 6$, by Claim 1, $F_3$ is reducible, so we have $n'_4(v) = 0$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - \frac{3}{4} \cdot n''_3(v) = 2l - 6 - (l - 3) - \frac{3}{4} \cdot 6 = l - \frac{15}{2} > 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - 1 \cdot 5 = l - 8 \geq 0$.

(c2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so we have $n_3(v) \leq 2$. By Claim 1, $F_1$ is reducible, and by Fact 4, we have $n_4(v) \leq 2$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - \frac{3}{2} \cdot 2 - 2 = l - 8 \geq 0$.

(d) $d_2(v) = l - 4$.

(d1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 8$.

If $n_4(v) = 6$, By Claim 1, $F_3$ is reducible, so we have $n'_4(v) \leq 8 - i$.

By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n'_4(v) - \frac{3}{4} \cdot n''_3(v) \geq 2l - 6 - (l - 4) - 1 \cdot (8 - i) - \frac{3}{4} \cdot (i - (8 - i)) = l - 4 - \frac{i}{2} \geq 0$.

$n_4(v) \leq 6$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot 6 = l - 8 \geq 0$.

(d2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so each 2-neighbor of $v$ is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have $n_3(v) \leq 2$.

$n_3(v) = i$ ($i = 1, 2$). By Claim 1, $F_1$ and $F_2$ are reducible, and note that each 3-face is not adjacent to a 4-face, we have $n_4(v) \leq 6 - 2i$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - \frac{3}{2} \cdot i - 1 \cdot (6 - 2i) = l - 8 + \frac{i}{2} > 0$.

(e) $d_2(v) = l - 5$.

(e1) $n_3(v) = 0$. If $n_4(v) \leq l - 1$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot (l - 1) = 0$. Now suppose that $n_4(v) = l$. By Claim 1, $F_1$ is reducible, so we have $d_2(v) \leq \left\lfloor \frac{l}{2} \right\rfloor$. Noting that $d_2(v) = l - 5$, we have $8 \leq l \leq 10$. By Claim 1, $F_1$, $F_3$ and $F_4$ are reducible, so we have $n'_4(v) \leq 4$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n'_4(v) - \frac{3}{4} \cdot n''_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot 4 - \frac{3}{4} \cdot (l - 4) = \frac{l}{4} - 2 \geq 0$.

(e2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, and by Fact 4, we have $n_3(v) \leq 3$. 
n_3(v) = 3. By Claim 1, F_1 is reducible, and by Fact 4, we have n_4(v) = 0. By R1 and R3, w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot 3 = l - \frac{11}{2} > 0.

n_3(v) = i (i = 1, 2). By Claim 1, F_1 is reducible, and by Fact 4, we have n_4(v) \leq 8 - 2i. By Claim 1, F_3 is reducible. So if n_4(v) = 8 - 2i, we have n'_4(v) = 0. By R1, R3 and R4, w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - \frac{3}{4} \cdot n'_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - \frac{3}{4} \cdot (8 - 2i) = l - 7 > 0. If n_4(v) \leq 7 - 2i, by R1, R3 and R4, w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - 1 \cdot (7 - 2i) = l + \frac{2}{2} - 8 > 0.

(f) d_2(v) \leq l - 6. Set t = \left\lceil \frac{2l - d_2(v) - 1}{3} \right\rceil. By Claim 1, F_2 is reducible, and by Fact 4, we have n_3(v) \leq t, n_4(v) \leq t and if n_3(v) + n_4(v) \leq l - 2.

(F1) n_3(v) = 0, by R1 and R4, w'(v) \geq 2l - 6 - d_2(v) - l \geq 2l - 6 - (l - 6) - l = 0.

(F2) n_3(v) > 0, by R1, R3 and R4, w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - n_4(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - (l - 2 - n_3(v)) \geq l - 4 - d_2(v) - \frac{1}{2} \cdot (2l - d_2(v) - \frac{1}{3} \left\lceil \frac{2l - d_2(v) - 1}{3} \right\rceil) \geq 0.

Now we get that for each x \in V(H) \cup F(H), w'(x) \geq 0, which is a contradiction. This completes the proof of Theorem 3.

3. The Proof of Theorem 4

The proof of Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let G be a minimum counterexample to Theorem 4 which is embedded in the plane. Set k = \max\{\Delta(G) + 1, 10\}. By the choice of G, any planar graph G' without 5-cycles and without adjacent \Delta(G)-vertices which is smaller than G has a k-tusd-coloring \phi'. Similarly, we will choose some G' and extend the coloring \phi' of G' to a desired coloring \phi of G to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

References


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