NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES\(^1\)

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Abstract

For a given graph \( G = (V(G), E(G)) \), a proper total coloring \( \phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \) is neighbor sum distinguishing if \( f(u) \neq f(v) \) for each edge \( uv \in E(G) \), where \( f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v) \), \( v \in V(G) \). The smallest integer \( k \) in such a coloring of \( G \) is the neighbor sum distinguishing total chromatic number, denoted by \( \chi'^\Sigma(G) \). Pilśniak and Woźniak first introduced this coloring and conjectured that \( \chi'^\Sigma(G) \leq \Delta(G) + 3 \) for any graph with maximum degree \( \Delta(G) \). In this paper, by using the discharging method, we prove that for any planar graph \( G \) without 5-cycles, \( \chi'^\Sigma(G) \leq \max\{\Delta(G) + 2, 10\} \). The bound \( \Delta(G) + 2 \) is sharp. Furthermore, we get the exact value of \( \chi'^\Sigma(G) \) if \( \Delta(G) \geq 9 \).

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

2010 Mathematics Subject Classification: 05C15.

\(^1\)This work was supported by NSFC (No.11671232), HNSF(No.A2015202301) and HUSTP (No.ZD2015106, QN2017044).
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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph \( G \), we denote its vertex set, edge set and maximum degree by \( V(G) \), \( E(G) \) and \( \Delta(G) \), respectively. If \( G \) is a planar graph embedded in the plane, we use \( F(G) \) to denote its face set. A vertex \( v \) is a \( t \)-vertex, \( t^- \)-vertex, \( t^+ \)-vertex if \( d_G(v) = t \), \( d_G(v) \leq t \), \( d_G(v) \geq t \) in \( G \), respectively. A \( t \)-face is defined similarly. An \( l \)-face \( v_1v_2 \cdots v_l \) is a \( (b_1, b_2, \ldots, b_l) \)-face, where \( v_i \) is a \( b_i \)-vertex, for \( i = 1, 2, \ldots, l \). Let \( d_G^l(v) \) denote the number of \( l \)-vertices adjacent to \( v \) in \( G \). Let \( n^d_G(v) \) denote the number of \( d \)-faces incident with \( v \) in \( G \). A configuration \( F \) is reducible to \( G \), if it cannot be a configuration of \( G \).

Given a graph \( G \), set \( n_i(G) = |\{v \in V(G) : d_G(v) = i\}| \) for \( i = 1, 2, \ldots, \Delta(G) \). A graph \( G' \) is smaller than \( G \) if one of the following holds:

1. \(|E(G')| < |E(G)|\),
2. \(|E(G')| = |E(G)| \) and \( (n_t(G'), n_{t-1}(G'), \ldots, n_1(G')) \) precedes \( (n_t(G), n_{t-1}(G), \ldots, n_1(G)) \) with respect to the standard lexicographic order, where \( t = \max\{\Delta(G), \Delta(G')\} \).

A graph is minimum for a property if no smaller graph satisfies it.

Given a graph \( G \) and a positive integer \( k \), a proper total \( k \)-coloring of \( G \) is a mapping \( \phi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) such that \( \phi(x) \neq \phi(y) \) for each pair of adjacent or incident elements \( x, y \in V(G) \cup E(G) \). Let \( f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v), v \in V(G) \). If \( f(u) \neq f(v) \) for each edge \( uv \in E(G) \), then \( \phi \) is a neighbor sum distinguishing total \( k \)-coloring, or \( k \)-tnsd-coloring for simplicity. The smallest number \( k \) is the neighbor sum distinguishing total chromatic number of \( G \), denoted by \( \chi^n(G) \). For \( k \)-tnsd-coloring, Pilśniak and Woźniak gave the following conjecture.

Conjecture 1 [11]. For any graph \( G \), \( \chi^n(G) \leq \Delta(G) + 3 \).

Pilśniak and Woźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong et al. [3] showed that Conjecture 1 holds for some sparse graphs. Yao et al. [21, 22] considered tnsd-coloring of degenerate graphs. Li et al. [9] proved that Conjecture 1 holds for \( K_4 \)-minor free graphs. Song et al. [15] determined \( \chi^n(G) \) for \( K_4 \)-minor free graph \( G \) with \( \Delta(G) \geq 5 \). For planar graph, it was proved that this conjecture holds with \( \Delta(G) \geq 13 \) by Li et al. [7] and \( \Delta(G) \geq 11 \) by Qu et al. [12]. For planar graph, it was proved that \( \chi^n(G) \leq \Delta(G) + 2 \) holds with \( \Delta(G) \geq 14 \) by Cheng et al. [2], \( \Delta(G) \geq 12 \) by Song et al. [14] and \( \Delta(G) \geq 11 \) by Yang et al. [20]. The bound \( \Delta(G) + 2 \) is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].
Recently, Ge et al. [6] got the following result.

**Theorem 2** [6]. Let $G$ be a planar graph without 5-cycles. Then

$$\chi_{\Sigma}^n(G) \leq \max\{\Delta(G) + 3, 10\}.$$ 

In this paper, we prove the following results.

**Theorem 3.** Let $G$ be a planar graph without 5-cycles. Then

$$\chi_{\Sigma}^n(G) \leq \max\{\Delta(G) + 2, 10\}.$$ 

**Theorem 4.** Let $G$ be a planar graph without 5-cycles and without adjacent $\Delta(G)$-vertices. Then $\chi_{\Sigma}^n(G) \leq \max\{\Delta(G) + 1, 10\}.$

Clearly, $\chi_{\Sigma}^n(G) \geq \Delta(G) + 1$ for any graph $G$. If $G$ has adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) \geq \Delta(G) + 2$. Thus we get the following corollary.

**Corollary 5.** Let $G$ be a planar graph without 5-cycles and $\Delta(G) \geq 9$. If $G$ has no adjacent $\Delta(G)$-vertices, then $\chi_{\Sigma}^n(G) = \Delta(G) + 1$, otherwise $\chi_{\Sigma}^n(G) = \Delta(G) + 2$.

2. **The Proof of Theorem 3**

We will prove it by contradiction. Let $G$ be a minimum counterexample to Theorem 3 which is embedded in the plane. Set $k = \max\{\Delta(G) + 2, 10\}$. By the choice of $G$, any planar graph $G'$ without 5-cycles which is smaller than $G$ has a $k$-tnsd-coloring $\phi'$. In the following, we will choose some $G'$ and extend the coloring $\phi'$ of $G'$ to a desired coloring $\phi$ of $G$ to get a contradiction. Unless otherwise stated, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G')), \phi(x) = \phi'(x)$.

In the following proof, we will omit the coloring of all 3$^-$-vertices. Since they have at most 9 forbidden colors and $k \geq 10$, they can be colored easily.

In Figure 1, we draw a vertex $x$ in black if it has no other neighbors than the ones already depicted, and a vertex $x$ in white if it might have more neighbors than the ones shown in the figure.

**Claim 1.** These configurations of $F_1$, $F_2$, $F_3$ and $F_4$ in Figure 1 are reducible.

**Proof.** (1) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F'_1$ in Figure 1). Thus $G'$ is a planar graph without 5-cycles which is smaller than $G$. Hence $G'$ admits a $k$-tnsd-coloring $\phi'$. We can stick $u_i$, $v_i$ together properly for $i = 1, 2$ (if necessary, exchange the colors of $uu_1$ and $uu_2$), and then recolor $u_i$, $v_i$, thus we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(2) Suppose to the contrary that $G$ contains configuration $F_2$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F'_2$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$.
(i) If \( \phi'(wu_1) \neq \phi'(uu_2) \) or \( \phi'(wu_1) = \phi'(uu_2) \not\in \{ \phi'(vv_1), \phi'(vv_2) \} \), then we can stick \( u_i, v_i \) together for \( i = 1, 2 \) (if necessary, exchange the colors of \( vv_1 \) and \( vv_2 \)).

(ii) If \( \phi'(wu_1) = \phi'(uu_2) \in \{ \phi'(vv_1), \phi'(vv_2) \} \), without loss of generality, suppose that \( \phi'(uu_2) = \phi'(vv_1) \). Exchange the colors of \( vv_1 \) and \( uu_2 \). Therefore, we can stick \( u_i, v_i \) together for \( i = 1, 2 \). Thus, by recoloring, we can obtain a \( k \)-tnsd-coloring \( \phi \) of \( G \), a contradiction.

Figure 1. Illustration of Claim 1.

(3) Suppose to the contrary that \( G \) contains configuration \( F_3 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( v_{i1}, v_{i2} \) for \( i = 1, 3 \) (see \( F'_3 \) in Figure 1) without producing 5-cycles. Thus \( G' \) has a \( k \)-tnsd-coloring \( \phi' \).

(i) If \( \phi'(wu_{12}) \neq \phi'(wu_{32}) \) or \( \phi'(wu_{12}) = \phi'(wu_{32}) \not\in \{ \phi'(vv_{11}), \phi'(vv_{31}) \} \), then we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 3 \) (if necessary, exchange the colors of \( vv_{11} \) and \( vv_{31} \)).

(ii) If \( \phi'(wu_{12}) = \phi'(wu_{32}) \in \{ \phi'(vv_{11}), \phi'(vv_{31}) \} \), without loss of generality, suppose that \( \phi'(wu_{12}) = \phi'(vv_{11}) \). Then we exchange the colors of \( uu_{12} \) and \( uu_{2} \). Therefore, we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 3 \). Thus, by recoloring, we can obtain a \( k \)-tnsd-coloring \( \phi \) of \( G \), a contradiction.

(4) Suppose to the contrary that \( G \) contains configuration \( F_1 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( v_{i1}, v_{i2} \) for \( i = 1, 4 \) (see \( F'_4 \) in Figure 1) without producing 5-cycles. Thus \( G' \) admits a \( k \)-tnsd-coloring \( \phi' \).

(i) If \( \phi'(wu_{12}) \neq \phi'(zw_{12}) \) or \( \phi'(wu_{12}) = \phi'(zw_{12}) \not\in \{ \phi'(vv_{11}), \phi'(vv_{41}) \} \), then we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 4 \) (if necessary, exchange the colors of \( vv_{11} \) and \( vv_{41} \)).
(ii) If $\phi'(uv_{12}) = \phi'(zv_{42}) \in \{ \phi'(vv_{11}), \phi'(vv_{41}) \}$, without loss of generality, suppose that $\phi'(uv_{12}) = \phi'(zv_{42}) = \phi'(vv_{11})$. Since $\phi'(uw_{2}) \neq \phi'(uv_{12})$, suppose that $\phi'(uw_{2}) = \phi'(uv_{12})$. We exchange the colors of $uw_{12}$ and $uw_{2}$. Therefore, we can stick $v_{i1}, v_{i2}$ together for $i = 1, 4$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

It is easy to see that the following claim given in [16] also holds with the graph $G$ in our proof.

**Claim 2** [16]. In the graph $G$, the following results holds.

(1) Each $t^-$-vertex is not adjacent to any $(7-t^-)$-vertex, where $t = 4, 5$.
(2) For each vertex $v \in V(G)$, if $d^{1}_{G}(v) \geq 1$, then $d^{2}_{G}(v) = 0$; if $d^{1}_{G}(v) \geq 2$, then $d^{3}_{G}(v) = 0$.
(3) If $d_{G}(v) = 5$, then $d^{2}_{G}(v) \leq 1$.
(4) If $d_{G}(v) = 6$, then $d^{2}_{G}(v) \leq 2$. Furthermore, if $d^{2}_{G}(v) \geq 1$, then $d^{3}_{G}(v) \leq 1$.
(5) If $d_{G}(v) = 7$, then $d^{2}_{G}(v) \leq 2$. Furthermore, if $d^{2}_{G}(v) \geq 1$, then $d^{3}_{G}(v) \leq 2$.
(6) If $d_{G}(v) = l (l \geq 8)$, then $d^{1}_{G}(v) \leq \left[ \frac{l}{2} \right]$.
(7) If $d_{G}(v) = l (l \geq 8)$ and $d^{2}_{G}(v) \geq 1$, then $d^{2}_{G}(v) + d^{3}_{G}(v) \leq l - 1$.
(8) Each 3-face in $G$ is a $(2, 6^+, 6^+)$-face, a $(3, 5^+, 5^+)$-face or a $(4^+, 4^+, 5^+)$-face.

**Claim 3.** Each 4-face in $G$ is a $(2, 6^+, 3^+, 6^+)$-face, a $(3, 6^+, 3^+, 6^+)$-face, a $(3, 5^+, 4^+, 5^+)$-face or a $(4^+, 4^+, 4^+, 4^+)$-face.

**Proof.** Let $T = v_{1}v_{2}v_{3}v_{4}v_{1}$ be a 4-face of $G$, and assume that $d_{G}(v_{1}) \leq d_{G}(v_{4})$, where $i = 2, 3, 4$. If $d_{G}(v_{1}) = 2$, by Claim 2(1), $d_{G}(v_{2}) \geq 6, d_{G}(v_{4}) \geq 6$. By Claim 1, $F_{1}$ is reducible, thus $T$ is a $(2, 6^+, 3^+, 6^+)$-face. If $d_{G}(v_{1}) = d_{G}(v_{4}) = 3$, by Claim 2(1) and Claim 2(3), $d_{G}(v_{2}) \geq 6$ and $d_{G}(v_{3}) \geq 6$, thus $T$ is a $(3, 6^+, 3^+, 6^+)$-face. If $d_{G}(v_{1}) = 3$ and $d_{G}(v_{3}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 5$ and $d_{G}(v_{4}) \geq 5$, thus $T$ is a $(3, 5^+, 4^+, 5^+)$-face. If $d_{G}(v_{1}) \geq 4$ and $d_{G}(v_{3}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 4$ and $d_{G}(v_{4}) \geq 4$, thus $T$ is a $(4^+, 4^+, 4^+, 4^+)$-face.

Let $H$ be the graph obtained from $G$ by removing all 1-vertices. By Claims 1–3, we have the following facts.

**Fact 1.** For the graph $H$, we have $\delta(H) \geq 2, d_{H}(v) = d_{G}(v)$, for $2 \leq d_{G}(v) \leq 5$. If $d_{G}(v) \geq 6$, then $d_{H}(v) \geq 5$.

**Fact 2.**

(1) In the graph $H$, each 3^-vertex is not adjacent to any 4^-vertex.
(2) If $d_{H}(v) = 5$, then $d^{2}_{H}(v) = 0$ and $d^{3}_{H}(v) \leq 1$.
(3) If $d_{H}(v) = 6$, then $d^{2}_{H}(v) \leq 1$; furthermore, if $d_{H}(v) = 1$, then $d^{3}_{H}(v) = 0$; if
$\quad d^{2}_{H}(v) = 0$, then $d^{3}_{H}(v) \leq 2$. 

\[ \]
(4) If $d_H(v) = 7$, then $d_H^2(v) \leq 2$; furthermore, if $d_H^2(v) = 2$, then $d_H^3(v) = 0$; if $d_H^2(v) = 1$, then $d_H^3(v) \leq 1$.

(5) If $d_H(v) = l$ ($l \geq 8$), then $d_H^2(v) \leq l - 1$.

**Fact 3.**

1. Each 3-face in $H$ is a $(2, 6+, 6+)$-face, a $(3, 5+, 5+)$-face or a $(4+, 4+, 5+)$-face.
2. Each 4-face in $H$ is a $(2, 6+, 3+, 6+)$-face, a $(3, 6+, 3+, 6+)$-face, a $(3, 5+, 4+, 5+)$-face or a $(4+, 4+, 4+, 4+)$-face.

A $(2, 6+, 6+)$-face or a $(3, 5+, 5+)$-face is called a **bad** 3-face. A $(4+, 5+, 5+)$-face is called a **normal** 3-face. A $(2, 6+, 3+, 6+)$-face or a $(3, 6+, 3+, 6+)$-face is called a **bad** 4-face, and other 4-face is a **normal** 4-face. We use $n'_i(v)$, $n''_i(v)$ to denote the number of bad $i$-faces and the number of normal $i$-faces incident with $v$ in $H$, respectively, $i = 3, 4$.

Since $G$ has no 5-cycles, we have the following fact.

**Fact 4.** These configurations are reducible to $H$:

1. A 5-face,
2. A 3-face adjacent to two 3-faces,
3. A 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

**Fact 5.** If $d_H(v) = l$ and $n_H^3(v) > 0$, then $n_H^3(v) + n_H^4(v) \leq l - 2$.

By Euler’s formula, we have

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.$$ 

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: $w(v) = 2d_H(v) - 6$ for each $v \in V(H)$; $w(f) = d_H(f) - 6$ for each $f \in F(H)$. Next, we will design some discharging rules. Let $w'$ be the new charge after the discharging process. It suffices to show that $w'(x) \geq 0$ for each $x \in V(H) \cup F(H)$, which leads to a contradiction.

In the following, a $k$-face means a $k$-face in $H$, the discharging rules are defined as follows.

**R1** Every 2-vertex $v$ in $H$ takes 1 from each neighbor.

**R2** Every 4-vertex $v$ in $H$ gives 1 to each incident 3-face, gives $\frac{1}{2}$ to each incident 4-face.

**R3** Every 5+-vertex $v$ in $H$ gives $\frac{3}{2}$ to each incident bad 3-face, gives 1 to each incident normal 3-face.
**R4** Every $5^+$-vertex $v$ in $H$ gives 1 to each incident bad 4-face, gives $\frac{3}{2}$ to each incident normal 4-face.

We will verify the new charge of each $x \in V(H) \cup F(H)$. In the following, we use $d(v)$, $d_i(v)$, $n_i(v)$ and $d(f)$ to denote $d_{H}(v)$, $d_i^H(v)$, $n_i^H(v)$ and $d_{H}(f)$, respectively. We first consider the new charge of each $f \in F(H)$.

- $d(f) = 3$. If $f$ is a bad 3-face, by R3, $w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0$; otherwise, by R2 and R3, $w'(f) = 3 - 6 + 1 \cdot 3 = 0$.

- $d(f) = 4$. If $f$ is a bad 4-face, by R4, $w'(f) = 4 - 6 + 1 \cdot 2 = 0$. If $f$ is a $(2,6^+,4^+,6^+)$-face or a $(3,5^+,4^+,5^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{3}{2} \cdot 2 + \frac{1}{2} = 0$. If $f$ is a $(4^+,4^+,4^+,4^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0$.

- $d(f) = t$ ($t \geq 6$). $w'(f) = w(f) = t - 6 \geq 0$.

Next we will consider the new charge of each $v \in V(H)$.

- $d(v) = 2$. By R1, $w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0$.

- $d(v) = 3$. No rule applies to $v$, $w'(v) = 2 \cdot 3 - 6 = 0$.

- $d(v) = 4$. By Fact 2(1), $d_2(v) = d_3(v) = 0$. If $n_3(v) = 0$, by R2, $w'(v) = 2 \cdot 4 - 6 - \frac{3}{2} \cdot n_4(v) \geq 2 - \frac{1}{2} \cdot 4 = 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 2$. By R2, $w'(v) = 2 \cdot 4 - 6 - 1 \cdot n_3(v) - \frac{1}{2} \cdot n_4(v) \geq 2 - 1 \cdot 2 = 0$.

- $d(v) = 5$. By Fact 2(2), $d_2(v) = d_3(v) = 0$, $d_3(v) \leq 1$, so we have $n_3'(v) \leq 2$ and $n_4'(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{4} \cdot n_4'(v) \geq 4 - \frac{3}{4} \cdot 5 = \frac{1}{4} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4'(v) \leq 3$. By R3 and R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_4'(v) \geq 4 - \frac{3}{2} \cdot 2 - 1 = 0$.

- $d(v) = 6$. By Fact 2(3), $d_2(v) \leq 1$.

  (a) $d_2(v) = 1$. By Fact 2(3), $d_3(v) = 0$, so we have $n_3'(v) \leq 1$ and $n_4'(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{4} \cdot n_4'(v) \geq 6 - 6 - \frac{3}{2} \cdot 6 = \frac{1}{2} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4'(v) \leq 4$. By R1, R3 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_4'(v) \geq 6 - 1 \cdot \frac{3}{2} \cdot 1 - 1 \cdot 3 = \frac{1}{2} > 0$.

  (b) $d_2(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) \geq 2 \cdot 6 - 6 - 1 \cdot n_4'(v) \geq 6 - 1 \cdot 0 = 6$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4'(v) \leq 4$. By R3 and R4, $w'(v) \geq 2 \cdot 6 - 6 - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_4'(v) \geq 6 - \frac{3}{2} \cdot 0 = 6$.

- $d(v) = 7$. By Fact 2(4), $d_2(v) \leq 2$.

  (a) $d_2(v) = 2$. By Fact 2(4), $d_3(v) = 0$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3'(v) = n_4'(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - \frac{3}{4} \cdot n_4'(v) \geq 8 - 2 - \frac{3}{2} \cdot 7 = \frac{3}{2} > 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 5$. Noting that $n_3'(v) = n_4'(v) = 0$, By R1, R3 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_3'(v) - \frac{3}{4} \cdot n_4'(v) \geq 8 - 2 - 1 \cdot 5 > 0$.

  (b) $d_2(v) \leq 1$. If $n_3(v) = 0$, by R1 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_4(v) \geq 8 - 1 \cdot 7 = 0$. If $n_3(v) > 0$, by Fact 4 and Fact 5, $n_3(v) \leq 4$ and $n_3(v) + n_4(v) \leq 5$. By R1, R3 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_3'(v) - 1 \cdot n_4'(v) \geq 8 - 1 \cdot \frac{3}{2} \cdot 4 - 1 = 0$.

- $d(v) = l$ ($l \geq 8$), by Fact 2(5), $d_2(v) \leq l - 1$. 

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**Neighbor Sum Distinguishing Total Chromatic Number**

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(a) $d_2(v) = l - 1$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3(v) = 0$ and $n_4(v) \leq 2$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 1) - 1 \cdot 2 = l - 7 > 0$.

(b) $d_2(v) = l - 2$.

(b1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 4$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 2) - 4 = l - 8 \geq 0$.

(b2) $n_3(v) > 0$. By Claim 1, $F_1$ and $F_2$ are reducible, and by Fact 4, we have $n_3(v) = 1$ and $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 2) - \frac{3}{2} = l - \frac{11}{2} > 0$.

(c) $d_2(v) = l - 3$.

(c1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 6$. If $n_4(v) = 6$, by Claim 1, $F_3$ is reducible, so we have $n'_4(v) = 0$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - \frac{3}{4} \cdot n'_4(v) = 2l - 6 - (l - 3) - \frac{3}{4} \cdot 6 = l - \frac{15}{2} > 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - 1 \cdot 5 = l - 8 \geq 0$.

(c2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so we have $n_3(v) \leq 2$. By Claim 1, $F_1$ is reducible, and by Fact 4, we have $n_4(v) \leq 2$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - \frac{3}{2} \cdot 2 - 2 = l - 8 \geq 0$.

(d) $d_2(v) = l - 4$.

(d1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 8$. If $n_4(v) = 8$, by Claim 1, $F_3$ is reducible, so we have $n'_4(v) \leq 8 - i$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n'_4(v) - 3 \cdot i \cdot n'_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot (8 - i) - \frac{3}{4} \cdot (i - (8 - i)) = l - 4 - \frac{i}{2} \geq 0$.

$n_4(v) \leq 6$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot 6 = l - 8 \geq 0$.

(d2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so each 2-neighbor of $v$ is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have $n_3(v) \leq 2$.

If $n_4(v) = i \ (i = 1, 2)$. By Claim 1, $F_1$ and $F_2$ are reducible, and note that each 3-face is not adjacent to a 4-face, we have $n_4(v) \leq 6 - 2i$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - \frac{3}{2} \cdot i - 1 \cdot (6 - 2i) = l - 8 + \frac{i}{2} > 0$.

(e) $d_2(v) = l - 5$.

(e1) $n_3(v) = 0$. If $n_4(v) \leq l - 1$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot (l - 1) = 0$. Now suppose that $n_4(v) = l$. By Claim 1, $F_1$ is reducible, so we have $d_2(v) \leq \left\lfloor \frac{l}{2} \right\rfloor$. Noting that $d_2(v) = l - 5$, we have $8 \leq l \leq 10$. By Claim 1, $F_1$, $F_3$ and $F_4$ are reducible, so we have $n'_4(v) \leq 4$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n'_4(v) - \frac{3}{4} \cdot n'_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot 4 - \frac{3}{4} \cdot (l - 4) = \frac{l}{4} - 2 \geq 0$.

(e2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, and by Fact 4, we have $n_3(v) \leq 3$.  

n_3(v) = 3. By Claim 1, F_1 is reducible, and by Fact 4, we have n_4(v) = 0. By R1 and R3, w'(v) ≥ 2ℓ - d_2(v) - \frac{3}{2} \cdot n_3(v) ≥ 2ℓ - 6 - (l - 5) - \frac{3}{2} \cdot 3 = l - \frac{11}{2} > 0.

n_3(v) = i (i = 1, 2). By Claim 1, F_1 is reducible, and by Fact 4, we have n_4(v) ≤ 8 - 2i. By Claim 1, F_3 is reducible. So if n_4(v) = 8 - 2i, we have n'_4(v) = 0. By R1, R3 and R4, w'(v) ≥ 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - \frac{3}{2} \cdot n'_4(v) ≥ 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - \frac{3}{2} \cdot (8 - 2i) = l - 7 > 0. If n_4(v) ≤ 7 - 2i, by R1, R3 and R4, w'(v) ≥ 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) ≥ 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - 1 \cdot (7 - 2i) = l + \frac{2}{2} - 8 > 0.

(f) d_2(v) ≤ l - 6. Set t = \left[\frac{2l - d_2(v) - 1}{4}\right]. By Claim 1, F_2 is reducible, and by Fact 4, we have n_3(v) ≤ t, n_4(v) ≤ l and if n_3(v) > 0, then n_3(v) + n_4(v) ≤ l - 2.

(f1) n_3(v) = 0, by R1 and R4, w'(v) ≥ 2l - 6 - d_2(v) - l ≥ 2l - 6 - (l - 6) - l = 0.

(f2) n_3(v) > 0, by R1, R3 and R4, w'(v) ≥ 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - n_4(v) ≥ 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - (l - 2 - n_3(v)) ≥ l - 4 - d_2(v) - \frac{1}{2} \cdot t = l - 4 - d_2(v) - \frac{1}{2} \cdot \left[\frac{2l - d_2(v) - 1}{3}\right] ≥ 0.

Now we get that for each x ∈ V(H) ∪ F(H), \( w'(x) ≥ 0 \), which is a contradiction. This completes the proof of Theorem 3.

3. The Proof of Theorem 4

The proof of Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let G be a minimum counterexample to Theorem 4 which is embedded in the plane. Set k = \max\{Δ(G) + 1, 10\}. By the choice of G, any planar graph G' without 5-cycles and without adjacent Δ(G)-vertices which is smaller than G has a k-tusd-coloring \( φ' \). Similarly, we will choose some G' and extend the coloring \( φ' \) of G' to a desired coloring \( φ \) of G to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

References


doi:10.1016/j.dam.2015.03.013


doi:10.1007/s10114-014-2454-7


Received 5 December 2017
Revised 8 February 2018
Accepted 7 March 2018