NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES\textsuperscript{1}

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Abstract

For a given graph $G = (V(G), E(G))$, a proper total coloring $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is neighbor sum distinguishing if $f(u) \neq f(v)$ for each edge $uv \in E(G)$, where $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. The smallest integer $k$ in such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak first introduced this coloring and conjectured that $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$ for any graph with maximum degree $\Delta(G)$. In this paper, by using the discharging method, we prove that for any planar graph $G$ without 5-cycles, $\chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 10\}$. The bound $\Delta(G) + 2$ is sharp. Furthermore, we get the exact value of $\chi''_{\Sigma}(G)$ if $\Delta(G) \geq 9$.

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph $G$, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If $G$ is a planar graph embedded in the plane, we use $F(G)$ to denote its face set. A vertex $v$ is a $t$-vertex, $t^-$-vertex, $t^+$-vertex if $d_G(v) = t$, $d_G(v) \leq t$, $d_G(v) \geq t$ in $G$, respectively. A t-face is defined similarly. An $t$-face $v_1v_2\cdots v_l$ is a ($b_1,b_2,\ldots,b_l$)-face, where $v_i$ is a $b_i$-vertex, for $i = 1,2,\ldots,l$. Let $d^t_G(v)$ denote the number of $t$-vertices adjacent to $v$ in $G$. Let $n^d_G(v)$ denote the number of $d$-faces incident with $v$ in $G$. A configuration $F$ is reducible to $G$, if it cannot be a configuration of $G$.

Given a graph $G$, set $n_i(G) = |\{v \in V(G) : d_G(v) = i\}|$ for $i = 1,2,\ldots,\Delta(G)$. A graph $G'$ is smaller than $G$ if one of the following holds:

1. $|E(G')| < |E(G)|$,
2. $|E(G')| = |E(G)|$ and $(n_t(G'),n_{t-1}(G'),\ldots,n_1(G'))$ precedes $(n_t(G),n_{t-1}(G),\ldots,n_1(G))$ with respect to the standard lexicographic order, where $t = \max \{\Delta(G),\Delta(G')\}$.

A graph is minimum for a property if no smaller graph satisfies it.

Given a graph $G$ and a positive integer $k$, a proper total $k$-coloring of $G$ is a mapping $\phi: V(G) \cup E(G) \to \{1,2,\ldots,k\}$ such that $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements $x,y \in V(G) \cup E(G)$. Let $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. If $f(u) \neq f(v)$ for each edge $uv \in E(G)$, then $\phi$ is a neighbor sum distinguishing total $k$-coloring, or $k$-tnsd-coloring for simplicity. The smallest number $k$ is the neighbor sum distinguishing total chromatic number of $G$, denoted by $\chi^t_\Sigma(G)$. For $k$-tnsd-coloring, Pilśniak and Woźniak gave the following conjecture.

Conjecture 1 [11]. For any graph $G$, $\chi^t_\Sigma(G) \leq \Delta(G) + 3$.

Pilśniak and Woźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong et al. [3] showed that Conjecture 1 holds for some sparse graphs. Yao et al. [21, 22] considered tnsd-coloring of degenerate graphs. Li et al. [9] proved that Conjecture 1 holds for $K_4$-minor free graphs. Song et al. [15] determined $\chi^t_\Sigma(G)$ for $K_4$-minor free graph $G$ with $\Delta(G) \geq 5$. For planar graph, it was proved that this conjecture holds with $\Delta(G) \geq 13$ by Li et al. [7] and $\Delta(G) \geq 11$ by Qu et al. [12]. For planar graph, it was proved that $\chi^t_\Sigma(G) \leq \Delta(G) + 2$ holds with $\Delta(G) \geq 14$ by Cheng et al. [2], $\Delta(G) \geq 12$ by Song et al. [14] and $\Delta(G) \geq 11$ by Yang et al. [20]. The bound $\Delta(G) + 2$ is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].
Recently, Ge et al. [6] got the following result.

**Theorem 2** [6]. Let \( G \) be a planar graph without 5-cycles. Then
\[
\chi''_{\Sigma}(G) \leq \max \{ \Delta(G) + 3, 10 \}.
\]

In this paper, we prove the following results.

**Theorem 3.** Let \( G \) be a planar graph without 5-cycles. Then
\[
\chi''_{\Sigma}(G) \leq \max \{ \Delta(G) + 2, 10 \}.
\]

**Theorem 4.** Let \( G \) be a planar graph without 5-cycles and without adjacent \( \Delta(G) \)-vertices. Then \( \chi''_{\Sigma}(G) \leq \max \{ \Delta(G) + 1, 10 \} \).

Clearly, \( \chi''_{\Sigma}(G) \geq \Delta(G) + 1 \) for any graph \( G \). If \( G \) has adjacent \( \Delta(G) \)-vertices, then \( \chi''_{\Sigma}(G) \geq \Delta(G) + 2 \). Thus we get the following corollary.

**Corollary 5.** Let \( G \) be a planar graph without 5-cycles and \( \Delta(G) \geq 9 \). If \( G \) has no adjacent \( \Delta(G) \)-vertices, then \( \chi''_{\Sigma}(G) = \Delta(G) + 1 \), otherwise \( \chi''_{\Sigma}(G) = \Delta(G) + 2 \).

## 2. The Proof of Theorem 3

We will prove it by contradiction. Let \( G \) be a minimum counterexample to Theorem 3 which is embedded in the plane. Set \( k = \max \{ \Delta(G) + 2, 10 \} \). By the choice of \( G \), any planar graph \( G' \) without 5-cycles which is smaller than \( G \) has a \( k \)-tnsd-coloring \( \phi' \). In the following, we will choose some \( G' \) and extend the coloring \( \phi' \) of \( G' \) to a desired coloring \( \phi \) of \( G \) to get a contradiction. Unless otherwise stated, for any \( x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G')) \), set \( \phi(x) = \phi'(x) \).

In the following proof, we will omit the coloring of all \( 3^- \)-vertices. Since they have at most 9 forbidden colors and \( k \geq 10 \), they can be colored easily.

In Figure 1, we draw a vertex \( x \) in black if it has no other neighbors than the ones already depicted, and a vertex \( x \) in white if it might have more neighbors than the ones shown in the figure.

**Claim 1.** These configurations of \( F_1, F_2, F_3 \) and \( F_4 \) in Figure 1 are reducible.

**Proof.** (1) Suppose to the contrary that \( G \) contains configuration \( F_1 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( u_i, v_i \) for \( i = 1, 2 \) (see \( F_1' \) in Figure 1). Thus \( G' \) is a planar graph without 5-cycles which is smaller than \( G \). Hence \( G' \) admits a \( k \)-tnsd-coloring \( \phi' \). We can stick \( u_i, v_i \) together properly for \( i = 1, 2 \) (if necessary, exchange the colors of \( uu_1 \) and \( uu_2 \)), and then recolor \( u_i, v_i \), thus we can obtain a \( k \)-tnsd-coloring \( \phi \) of \( G \), a contradiction.

(2) Suppose to the contrary that \( G \) contains configuration \( F_2 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( u_i, v_i \) for \( i = 1, 2 \) (see \( F_2' \) in Figure 1) without producing 5-cycles. Thus \( G' \) has a \( k \)-tnsd-coloring \( \phi' \).
Suppose that \( G \) contains configuration \( F_1 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( v_{i1}, v_{i2} \) for \( i = 1, 2 \) (see Figure 1) without producing 5-cycles. Thus \( G' \) has a \( k \)-tnsd-coloring \( \phi' \).

(i) If \( \phi'(wu_1) \neq \phi'(uw_2) \) or \( \phi'(wu_1) = \phi'(uw_2) \in \{ \phi'(vv_1), \phi'(vv_2) \} \), then we can stick \( u_i, v_i \) together for \( i = 1, 2 \) (if necessary, exchange the colors of \( vv_{i1} \) and \( vv_{i2} \)).

(ii) If \( \phi'(wu_1) = \phi'(uw_2) \in \{ \phi'(vv_1), \phi'(vv_2) \} \), without loss of generality, suppose that \( \phi'(wu_2) = \phi'(vv_1) \). Exchange the colors of \( vv_{i1} \) and \( uv \). Therefore, we can stick \( u_i, v_i \) together for \( i = 1, 2 \). Thus, by recoloring, we can obtain a \( k \)-tnsd-coloring \( \phi \) of \( G \), a contradiction.

3. Suppose to the contrary that \( G \) contains configuration \( F_3 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( v_{i1}, v_{i2} \) for \( i = 1, 3 \) (see Figure 1) without producing 5-cycles. Thus \( G' \) has a \( k \)-tnsd-coloring \( \phi' \).

(i) If \( \phi'(wu_{i2}) \neq \phi'(wu_{i3}) \) or \( \phi'(wu_{i2}) = \phi'(wu_{i3}) \in \{ \phi'(vv_{i1}), \phi'(vv_{i2}) \} \), then we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 3 \) (if necessary, exchange the colors of \( vv_{i1} \) and \( vv_{i3} \)).

(ii) If \( \phi'(wu_{i2}) = \phi'(wu_{i3}) \in \{ \phi'(vv_{i1}), \phi'(vv_{i3}) \} \), without loss of generality, suppose that \( \phi'(wu_{i2}) = \phi'(vv_{i1}) \). Then we exchange the colors of \( wu_{i2} \) and \( uv \). Therefore, we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 3 \). Thus, by recoloring, we can obtain a \( k \)-tnsd-coloring \( \phi \) of \( G' \), a contradiction.

4. Suppose to the contrary that \( G \) contains configuration \( F_1 \). We obtain a smaller graph \( G' \) by splitting \( v_i \) into \( v_{i1}, v_{i2} \) for \( i = 1, 4 \) (see Figure 1) without producing 5-cycles. Thus \( G' \) admits a \( k \)-tnsd-coloring \( \phi' \).

(i) If \( \phi'(uw_{i2}) \neq \phi'(zw_{i2}) \) or \( \phi'(uw_{i2}) = \phi'(zw_{i2}) \in \{ \phi'(vv_{i1}), \phi'(vv_{i4}) \} \), then we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 4 \) (if necessary, exchange the colors of \( vv_{i1} \) and \( vv_{i4} \)).
(ii) If $\phi'(uw_{12}) = \phi'(zv_{42}) \in \{\phi'(vw_{11}), \phi'(vw_{41})\}$, without loss of generality, suppose that $\phi'(uw_{12}) = \phi'(zv_{42}) = \phi'(vw_{11})$. Since $\phi'(uw_{2}) \neq \phi'(uw_{3})$, suppose that $\phi'(uw_{2}) \neq \phi'(uw_{12})$. We exchange the colors of $uw_{12}$ and $uw_{2}$. Therefore, we can stick $v_{i1}, v_{i2}$ together for $i = 1, 4$. Thus, by recoloring, we can obtain a $k$-tsd-coloring $\phi$ of $G$, a contradiction. 

It is easy to see that the following claim given in [16] also holds with the graph $G$ in our proof.

**Claim 2** [16]. In the graph $G$, the following results holds.

1. Each $t^{-}$-vertex is not adjacent to any $(T-t^{-})$-vertex, where $t = 4, 5$.
2. For each vertex $v \in V(G)$, if $d_{G}^{1}(v) \geq 1$, then $d_{G}^{2}(v) = 0$; if $d_{G}^{1}(v) \geq 2$, then $d_{G}^{3}(v) = 0$.
3. If $d_{G}(v) = 5$, then $d_{G}^{2}(v) \leq 1$.
4. If $d_{G}(v) = 6$, then $d_{G}^{2}(v) \leq 2$. Furthermore, if $d_{G}^{2}(v) \geq 1$, then $d_{G}^{3}(v) \leq 1$.
5. If $d_{G}(v) = 7$, then $d_{G}^{2}(v) \leq 2$. Furthermore, if $d_{G}^{2}(v) \geq 1$, then $d_{G}^{3}(v) \leq 2$.
6. If $d_{G}(v) = l \ (i \geq 8)$, then $d_{G}^{1}(v) < \left[ \frac{l}{2} \right]$.
7. If $d_{G}(v) = l \ (i \geq 8)$ and $d_{G}^{2}(v) \geq 1$, then $d_{G}^{2}(v) + d_{G}^{3}(v) \leq l - 1$.
8. Each 3-face in $G$ is a $(2, 6^{+}, 6^{+})$-face, a $(3, 5^{+}, 5^{+})$-face or a $(4^{+}, 4^{+}, 5^{+})$-face.

**Claim 3.** Each 4-face in $G$ is a $(2, 6^{+}, 3^{+}, 6^{+})$-face, a $(3, 6^{+}, 3, 6^{+})$-face, a $(3, 5^{+}, 4^{+}, 5^{+})$-face or a $(4^{+}, 4^{+}, 4^{+})$-face.

**Proof.** Let $T = v_{1}v_{2}v_{3}v_{4}v_{1}$ be a 4-face of $G$, and assume that $d_{G}(v_{1}) \leq d_{G}(v_{4})$, where $i = 2, 3, 4$. If $d_{G}(v_{1}) = 2$, by Claim 2(1), $d_{G}(v_{2}) \geq 6, d_{G}(v_{4}) \geq 6$. By Claim 1, $F_{1}$ is reducible, thus $T$ is a $(2, 6^{+}, 3^{+}, 6^{+})$-face. If $d_{G}(v_{1}) = d_{G}(v_{4}) = 3$, by Claim 2(1) and Claim 2(3), $d_{G}(v_{2}) \geq 6$ and $d_{G}(v_{4}) \geq 6$, thus $T$ is a $(3, 6^{+}, 3, 6^{+})$-face. If $d_{G}(v_{1}) = 3$ and $d_{G}(v_{4}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 5$ and $d_{G}(v_{4}) \geq 5$, thus $T$ is a $(3, 5^{+}, 4^{+}, 5^{+})$-face. If $d_{G}(v_{1}) \geq 4$ and $d_{G}(v_{3}) \geq 4$, by Claim 2(1), $d_{G}(v_{2}) \geq 4$ and $d_{G}(v_{4}) \geq 4$, thus $T$ is a $(4^{+}, 4^{+}, 4^{+})$-face.

Let $H$ be the graph obtained from $G$ by removing all 1-vertices. By Claims 1–3, we have the following facts.

**Fact 1.** For the graph $H$, we have $\delta(H) \geq 2$; $d_{H}(v) = d_{G}(v)$, for $2 \leq d_{G}(v) \leq 5$. If $d_{G}(v) \geq 6$, then $d_{H}(v) \geq 5$.

**Fact 2.**

1. In the graph $H$, each $3^{-}$-vertex is not adjacent to any $4^{-}$-vertex.
2. If $d_{H}(v) = 5$, then $d_{H}^{2}(v) = 0$ and $d_{H}^{3}(v) \leq 1$.
3. If $d_{H}(v) = 6$, then $d_{H}^{2}(v) \leq 1$; furthermore, if $d_{H}(v) = 1$, then $d_{H}^{2}(v) = 0$; if $d_{H}^{2}(v) = 0$, then $d_{H}^{3}(v) \leq 2$. 
(4) If \( d_H(v) = 7 \), then \( d_2^2 H(v) \leq 2 \); furthermore, if \( d_2^2 H(v) = 2 \), then \( d_3^3 H(v) = 0 \); if \( d_2^2 H(v) = 1 \), then \( d_3^3 H(v) \leq 1 \).

(5) If \( d_H(v) = l \ (l \geq 8) \), then \( d_2^2 H(v) \leq l - 1 \).

**Fact 3.**

(1) Each 3-face in \( H \) is a \((2, 6^+, 6^+)-face, a (3, 5^+, 6^+)-face or a (4^+, 4^+, 5^+)-face.

(2) Each 4-face in \( H \) is a \((2, 6^+, 3^+, 6^+)-face, a (3, 6^+, 3, 6^+)-face, a (3, 5^+, 4^+, 5^+)-face or a (4^+, 4^+, 4^+, 4^+)-face.

A \((2, 6^+, 6^+)-face or a (3, 5^+, 6^+)-face is called a \textit{bad} 3-face. A \((4^+, 5^+, 5^+)-face is called a \textit{normal} 3-face. A \((2, 6^+, 3^+, 6^+)-face or a (3, 6^+, 3, 6^+)-face, a (3, 5^+, 4^+, 5^+)-face is called a \textit{bad} 4-face, and other 4-face is a \textit{normal} 4-face. We use \( n_i'(v) \), \( n_i''(v) \) to denote the number of bad \( i \)-faces and the number of normal \( i \)-faces incident with \( v \) in \( H \), respectively, \( i = 3, 4 \).

Since \( G \) has no 5-cycles, we have the following fact.

**Fact 4.** These configurations are reducible to \( H \):

(1) a 5-face,

(2) a 3-face adjacent to two 3-faces,

(3) a 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

**Fact 5.** If \( d_H(v) = l \) and \( n_3^1 H(v) > 0 \), then \( n_3^3 H(v) + n_4^4 H(v) \leq l - 2 \).

By Euler’s formula, we have

\[
\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.
\]

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: \( w(v) = 2d_H(v) - 6 \) for each \( v \in V(H) \); \( w(f) = d_H(f) - 6 \) for each \( f \in F(H) \). Next, we will design some discharging rules. Let \( w' \) be the new charge after the discharging process. It suffices to show that \( w'(x) \geq 0 \) for each \( x \in V(H) \cup F(H) \), which leads to a contradiction.

In the following, a \( k \)-face means a \( k \)-face in \( H \), the discharging rules are defined as follows.

**R1** Every 2-vertex \( v \) in \( H \) takes 1 from each neighbor.

**R2** Every 4-vertex \( v \) in \( H \) gives 1 to each incident 3-face, gives \( \frac{1}{2} \) to each incident 4-face.

**R3** Every 5\(^+\)-vertex \( v \) in \( H \) gives \( \frac{3}{2} \) to each incident bad 3-face, gives 1 to each incident normal 3-face.
R4 Every 5+-vertex \( v \) in \( H \) gives 1 to each incident bad 4-face, gives \( \frac{3}{2} \) to each incident normal 4-face.

We will verify the new charge of each \( x \in V(H) \cup F(H) \). In the following, we use \( d(v) \), \( d_i(v) \), \( n_i(v) \) and \( d(f) \) to denote \( d_H(v) \), \( d'_H(v) \), \( n'_H(v) \) and \( d_H(f) \), respectively. We first consider the new charge of each \( f \in F(H) \).

- \( d(f) = 3 \). If \( f \) is a bad 3-face, by R3, \( w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0 \); otherwise, by R2 and R3, \( w'(f) = 3 - 6 + 1 \cdot 3 = 0 \).

- \( d(f) = 4 \). If \( f \) is a bad 4-face, by R4, \( w'(f) = 4 - 6 + 1 \cdot 2 = 0 \). If \( f \) is a \((2, 6^+, 4^+, 6^+)\)-face or a \((3, 5^+, 4^+, 5^+)\)-face, by R2 and R4, \( w'(f) \geq 4 - 6 + \frac{3}{2} \cdot 2 + \frac{1}{2} = 0 \). If \( f \) is a \((4^+, 4^+, 4^+, 4^+)\)-face, by R2 and R4, \( w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0 \).

- \( d(f) = t \) (\( t \geq 6 \)). \( w'(f) = w(f) = t - 6 \geq 0 \).

Next we will consider the new charge of each \( v \in V(H) \).

- \( d(v) = 2 \). By R1, \( w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0 \).

- \( d(v) = 3 \). No rule applies to \( v \), \( w'(v) = 2 \cdot 3 - 6 = 0 \).

- \( d(v) = 4 \). By Fact 2(1), \( d_2(v) = d_3(v) = 0 \). If \( n_3(v) = 0 \), by R2, \( w'(v) = 2 \cdot 4 - 6 - \frac{1}{2} \cdot n_4(v) \geq 2 - \frac{1}{2} \cdot 4 = 0 \). If \( n_3(v) > 0 \), by Fact 5, \( n_3(v) + n_4(v) \leq 2 \).

- \( d(v) = 5 \). By Fact 2(2), \( d_2(v) = 0 \), \( d_3(v) \leq 1 \), so we have \( n'_3(v) \leq 2 \) and \( n'_4(v) = 0 \). If \( n_3(v) = 0 \), by R4, \( w'(v) = 2 \cdot 5 - 6 - \frac{3}{4} \cdot n'_4(v) \geq 4 - \frac{3}{4} \cdot 5 = \frac{1}{4} \geq 0 \).

- \( d(v) = 6 \). By Fact 2(3), \( d_2(v) \leq 1 \).

- \( d(v) = 7 \). By Fact 2(4), \( d_2(v) \leq 2 \).

\[ n'_3(v) = \frac{2}{3} \cdot n'_4(v) \]

- \( d(v) = l \) (\( l \geq 8 \)), by Fact 2(5), \( d_2(v) \leq l - 1 \).
(a) $d_2(v) = l - 1$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3(v) = 0$ and $n_4(v) \leq 2$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 1) - 1 \cdot 2 = l - 7 > 0$.

(b) $d_2(v) = l - 2$.

(b1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 4$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 2) - 4 = l - 8 \geq 0$.

(b2) $n_3(v) > 0$. By Claim 1, $F_1$ and $F_2$ are reducible, and by Fact 4, we have $n_3(v) = 1$ and $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 2) - \frac{3}{2} = l - \frac{11}{2} > 0$.

(c) $d_2(v) = l - 3$.

(c1) $n_3(v) = 0$. By Claim 1, $F_2$ is reducible, so we have $n_4(v) \leq 6$.

If $n_4(v) = 6$, by Claim 1, $F_3$ is reducible, so we have $n_3'(v) = 0$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - \frac{3}{4} \cdot n_3''(v) = 2l - 6 - (l - 3) - \frac{3}{4} \cdot 6 = l - \frac{15}{2} > 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - 1 \cdot 5 = l - 8 \geq 0$.

(c2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so we have $n_4(v) \leq 2$. By Claim 1, $F_1$ is reducible, and by Fact 4, we have $n_4(v) \leq 2$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - \frac{3}{2} \cdot 2 - 2 = l - 8 \geq 0$.

(d) $d_2(v) = l - 4$.

(d1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 8$.

If $n_4(v) = 6$, by Claim 1, $F_3$ is reducible, so we have $n_3'(v) \leq 8 - i$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_3'(v) \geq 2l - 6 - (l - 4) - 1 \cdot (8 - i) - \frac{3}{4} \cdot (i - (8 - i)) = l - 4 - \frac{i}{2} \geq 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot 6 = l - 8 \geq 0$.

(d2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so each 2-neighbor of $v$ is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have $n_3(v) \leq 2$.

$n_3(v) = i (i = 1, 2)$. By Claim 1, $F_1$ and $F_2$ are reducible, and note that each 3-face is not adjacent to a 4-face, we have $n_4(v) \leq 6 - 2i$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - \frac{3}{2} \cdot i - 1 \cdot (6 - 2i) = l - 8 + \frac{i}{2} > 0$.

(e) $d_2(v) = l - 5$.

(e1) $n_3(v) = 0$. If $n_4(v) \leq l - 1$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot (l - 1) = 0$. Now suppose that $n_4(v) = l$. By Claim 1, $F_1$ is reducible, so we have $d_2(v) \leq \lfloor \frac{1}{2} \rfloor$. Noting that $d_2(v) = l - 5$, we have $8 \leq l \leq 10$. By Claim 1, $F_1$, $F_3$ and $F_4$ are reducible, so we have $n_3'(v) \leq 4$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_3''(v) \geq 2l - 6 - (l - 5) - 1 \cdot (6 - 2 \cdot 4) = \frac{1}{4} - 2 \geq 0$.

(e2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, and by Fact 4, we have $n_3(v) \leq 3$.
\(n_3(v) = 3\). By Claim 1, \(F_1\) is reducible, and by Fact 4, we have \(n_4(v) = 0\). By \(R_1\) and \(R_3\), \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot 3 = l - \frac{11}{2} > 0\).

\(n_3(v) = i\) (\(i = 1, 2\)). By Claim 1, \(F_1\) is reducible, and by Fact 4, we have \(n_4(v) \leq 8 - 2i\). By Claim 1, \(F_3\) is reducible. So if \(n_4(v) = 8 - 2i\), we have \(n'_4(v) = 0\). By \(R_1\), \(R_3\) and \(R_4\), \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - \frac{3}{4} \cdot n'_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - \frac{3}{4} \cdot (8 - 2i) = l - 7 > 0\). If \(n_4(v) \leq 7 - 2i\), by \(R_1\), \(R_3\) and \(R_4\), \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - \frac{3}{2} \cdot i - 1 \cdot (7 - 2i) = l + \frac{1}{2} - 8 > 0\).

\((f)\) \(d_2(v) \leq l - 6\). Set \(t = \left\lfloor \frac{2(l - d_2(v) - 1)}{3} \right\rfloor\). By Claim 1, \(F_2\) is reducible, and by Fact 4, we have \(n_3(v) \leq t, n_4(v) \leq l\) and if \(n_3(v) > 0\), then \(n_3(v) + n_4(v) \leq l - 2\).

\((f_1)\) \(n_3(v) = 0\), by \(R_1\) and \(R_4\), \(w'(v) \geq 2l - 6 - d_2(v) - l \geq 2l - 6 - (l - 6) - l = 0\).

\((f_2)\) \(n_3(v) > 0\), by \(R_1\), \(R_3\) and \(R_4\), \(w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - n_4(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - (l - 2 - n_3(v)) \geq l - 4 - d_2(v) - \frac{1}{2} \cdot t = l - 4 - d_2(v) - \frac{1}{2} \cdot \left\lfloor \frac{2(l - d_2(v) - 1)}{3} \right\rfloor \geq 0\).

Now we get that for each \(x \in V(H) \cup F(H)\), \(w'(x) \geq 0\), which is a contradiction. This completes the proof of Theorem 3.

### 3. The Proof of Theorem 4

The proof of Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let \(G\) be a minimum counterexample to Theorem 4 which is embedded in the plane. Set \(k = \max\{\Delta(G) + 1, 10\}\). By the choice of \(G\), any planar graph \(G'\) without 5-cycles and without adjacent \(\Delta(G)\)-vertices which is smaller than \(G\) has a \(k\)-tusd-coloring \(\phi'\). Similarly, we will choose some \(G'\) and extend the coloring \(\phi'\) of \(G'\) to a desired coloring \(\phi\) of \(G\) to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

### References


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