THE EDIT DISTANCE FUNCTION OF SOME GRAPHS

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Abstract

The edit distance function of a hereditary property $\mathcal{H}$ is the asymptotically largest edit distance between a graph of density $p \in [0,1]$ and $\mathcal{H}$. Denote by $P_n$ and $C_n$ the path graph of order $n$ and the cycle graph of order $n$, respectively. Let $C_{2n}^*$ be the cycle graph $C_{2n}$ with a diagonal, and $\tilde{C}_n$ be the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(\tilde{C}_n) = E(C_n) \cup \{v_0v_2\}$. Marchant and Thomason determined the edit distance function of $C_{2n}^*$. Peck studied the edit distance function of $C_n$, while Berikkyzy \textit{et al.} studied the edit distance of powers of cycles. In this paper, by using the methods of Peck and Martin, we determine the edit distance function of $C_{2n}^*$, $\tilde{C}_n$ and $P_n$, respectively.

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1. Introduction

The edit distance in graphs was introduced by Axenovich, Kézdy and Martin [5] and by Alon and Stav [4] independently. The edit distance problem considered here is “How many edges need to be added or deleted (edited) in a graph $G$
so that it will have a certain property?” The presence or absence of edges in a certain graph corresponds to pairs of genes which activate or deactivate one another in evolutionary biology. In evolutionary theory, the gene reconstruction avoiding forbidden induced subgraphs is studied \[9\], which is equivalent to the edit distance problem. The edit distance problem is also important to the algorithmic aspects of property testing \[1–4\].

The *edit distance* between a graph \(G\) and a property \(\mathcal{H}\) is

\[
\text{dist}(G, \mathcal{H}) = \min \left\{ \frac{|E(G) \triangle E(G')|}{n^2} : \forall n, |V(G) = V(G')|, G' \in \mathcal{H} \right\}.
\]

The *edit distance function* of a property \(\mathcal{H}\), denoted \(ed_{\mathcal{H}}(p)\), measures the maximum distance of a graph with density \(p\) from \(\mathcal{H}\). Formally,

\[
ed_{\mathcal{H}}(p) = \lim_{n \to \infty} \max \left\{ \text{dist}(G, \mathcal{H}) : |V(G)| = n, |E(G)| = \left\lceil p \left(\frac{n}{2}\right) \right\rceil \right\}.
\]

if this limit exists.

A *hereditary property* is a family of graphs that is closed under the taking of induced subgraphs. For a given graph \(H\), the property of having no \(H\) as an induced subgraph is called a *principal hereditary property*, denoted by \(\text{Forb}(H)\). Clearly, \(\text{Forb}(H)\) is a hereditary property for any graph \(H\). In fact, for every hereditary property \(\mathcal{H}\) there exists a family of graphs \(\mathcal{F}(\mathcal{H})\) such that \(\mathcal{H} = \bigcap_{H \in \mathcal{F}(\mathcal{H})} \text{Forb}(H)\). A hereditary property is said to be *nontrivial* if there is an infinite sequence of graphs that is in the property. The properties for which we study the edit distance are usually hereditary property.

Balogh and Martin \[6\] showed that the limit in (1) exists and the edit distance function has a number of interesting properties.

**Proposition 1** \[11\]. Let \(\mathcal{H}\) be a nontrivial hereditary property. For \(p \in [0, 1]\),

(a) \(ed_{\mathcal{H}}(p)\) is continuous.

(b) \(ed_{\mathcal{H}}(p)\) is concave down.

In \[4\], Alon and Stav proved that for every hereditary property \(\mathcal{H}\), there exists a \(p^* = p^*(\mathcal{H}) \in [0, 1]\) such that the maximum distance of a graph \(G\) on \(n\) vertices from \(\mathcal{H}\) is asymptotically the same as that of the Erdős-Rényi random graph \(G(n, p^*)\). Namely,

\[
\max \left\{ \text{dist}(G, \mathcal{H}) : |V(G)| = n \right\} = \mathbb{E}[\text{dist}(G(n, p^*), \mathcal{H})] + o(1).
\]

We denote the limit in (2) by \(d^*_{\mathcal{H}}\).

The edit distance functions of some kinds of graphs have been investigated in recent years, including complete graphs \[13\] and split graphs \[12\]. Actually, complete bipartite graphs are also studied. Marchant and Thomason \[10\] studied
The edit distance functions of $K_{2,2}$ and $K_{3,3}$, respectively. Balogh and Martin [6] established the value of $p_{\text{Forb}(K_{3,3})}^*$ and $d_{\text{Forb}(K_{3,3})}^*$. Martin and McKay studied the edit distance function of $K_{2,t}$ in [14]. Recently, Berikkyzy et al. [7] settled the edit distance function for many powers of cycles.

Denote by $P_n$ and $C_n$ the path graph of order $n$ and the cycle graph of order $n$, respectively. Let $C_{2n}^*$ be the cycle graph $C_{2n}$ with a diagonal, and $\tilde{C}_n$ be the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and $E(\tilde{C}_n) = E(C_n) \cup \{v_0v_2\}$.

In [10], Marchant and Thomason studied the edit distance function of the graph $C_{8}^*$. Motivated by this result, we study the edit distance function of the graph $C_{8}^*$ and prove the following result.

**Theorem 2.** Let $\mathcal{H} = \text{Forb}(C_{8}^*)$.

$$\text{ed}_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1+p}, \frac{1-p}{3} \right\}, \text{ for } p \in [0,1].$$


**Theorem 3 [15].** Let $\mathcal{H} = \text{Forb}(C_n)$.

(a) If $n$ is odd, then $\text{ed}_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1+\left(\frac{n}{3}\right)^2p}, \frac{1-p}{\left(\frac{n}{3}\right)^2} \right\}, \text{ for } p \in [0,1].$

(b) If $n$ is even, then $\text{ed}_{\mathcal{H}}(p) = \min \left\{ \frac{p(1-p)}{1+\left(\frac{n}{3}\right)^2p}, \frac{1-p}{\left(\frac{n}{3}\right)^2} \right\}, \text{ for } p \in \left[\frac{n}{3}, 1\right].$

Motivated by this result, we study the edit distance function of $\tilde{C}_n$ and $P_n$.

**Theorem 4.** Let $\mathcal{H} = \text{Forb}(\tilde{C}_n)$ and $n \geq 9$.

$$\text{ed}_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1+\left(\frac{n-1}{3}\right)(2p)}, \frac{1-p}{\frac{n-1}{3}} \right\}, \text{ for } p \in [0,1].$$

**Theorem 5.** Let $\mathcal{H} = \text{Forb}(P_n)$ and $n \geq 3$.

$$\text{ed}_{\mathcal{H}}(p) = \min \left\{ \frac{p(1-p)}{1+\left(\frac{n-1}{3}\right)(2p)}, \frac{1-p}{\frac{n}{2}} \right\}, \text{ for } p \in \left[\frac{n-1}{3}, 1\right].$$

Our paper is organized as follows. Some definitions and tools are explained in Section 2. We prove Theorems 2, 4 and 5 in Sections 3, 4 and 5, respectively.
2. Definitions and Tools

All graphs considered in this paper are simple. The standard graph theory notation not defined here will conform to that in [8]. The edit distance notation not defined here will conform to that in [11].

In order to estimate the edit distance function, Alon and Stav [4] defined a colored regularity graph (CRG) $K$ as follows. Let $K$ be a simple complete graph, together with a partition of the vertices into white and black, and a partition of the edges into white, gray, and black. Denote by $VW(K)$ and $VB(K)$ the set of white vertices and the set of black vertices, respectively. Then $V(K) = VW(K) \cup VB(K)$. Denote by $EW(K)$, $EG(K)$ and $EB(K)$ the set of white edges, the set of gray edges and the set of black edges, respectively. Then we have $E(K) = EW(K) \cup EG(K) \cup EB(K)$. A CRG $K'$ is said to be a sub-CRG of $K$ if $K'$ can be obtained by deleting vertices of $K$ and is a proper sub-CRG if $K' \neq K$.

We say that a graph $H$ embeds in $K$ (writing $H \rightarrow K$), if there is a function $\varphi : V(H) \rightarrow V(K)$ so that if $h_1 h_2 \in E(H)$, then either $\varphi(h_1) = \varphi(h_2) \in VB(K)$ or $\varphi(h_1) \varphi(h_2) \in EB(K) \cup EG(K)$ and if $h_1 h_2 \notin E(H)$, then either $\varphi(h_1) = \varphi(h_2) \in VW(K)$ or $\varphi(h_1) \varphi(h_2) \in EW(K) \cup EG(K)$. For a hereditary property $\mathcal{H}$, we denote by $\mathcal{H}(\mathcal{H})$ the subset of CRGs $K$ such that any graph $H \in \mathcal{F}(\mathcal{H})$ does not embed in $K$. That is, $\mathcal{H}(\mathcal{H}) = \{ K : H \not\rightarrow K, \forall H \in \mathcal{F}(\mathcal{H}) \}$.

For a hereditary property $\mathcal{H}$, we can use the $g$ function of each CRG $K$ to compute the edit distance function, where $g$ function is defined by

$$
g_K(p) = \min \left\{ x^T M_K(p) x : x^T 1 = 1, x \geq 0 \right\},
$$

and

$$
[M_K(p)]_{ij} = \begin{cases} 
p & \text{if } v_i v_j \in EW(K) \text{ or } v_i = v_j \in VW(K), \\
1-p & \text{if } v_i v_j \in EB(K) \text{ or } v_i = v_j \in VB(K), \\
0 & \text{if } v_i v_j \in EG(K).
\end{cases}
$$

Marchant and Thomason in [10] proved that for every $p \in [0, 1]$, there is a CRG $K \in \mathcal{H}(\mathcal{H})$ such that $ed_{\mathcal{H}}(p) = g_K(p)$. That is

**Proposition 6** [10]. Let $\mathcal{H}$ be a nontrivial hereditary property. For $p \in [0, 1]$,

$$
ed_{\mathcal{H}}(p) = \min \{ g_K(p) : K \in \mathcal{H}(\mathcal{H}) \}.
$$

In [10], the authors also proved that in order to find such CRGs, we only need to look at all $p$-core CRGs. A CRG $K$ is $p$-core if, for any proper sub-CRG $K'$ of $K$, we have $g_{K'}(p) > g_K(p)$.

The gray-edge CRG $K(r, s)$ is the CRG $K$ with $r$ white vertices, $s$ black vertices and all edges gray. The clique spectrum of $\mathcal{H}$ is the set $\Gamma(\mathcal{H}) := \{ (r, s) : H \not\rightarrow K(r, s), \forall H \in \mathcal{F}(\mathcal{H}) \}$. Clearly, we obtain
Proposition 7 [11]. Let $\mathcal{H}$ be a nontrivial hereditary property and $\Gamma(H)$ denote the clique spectrum of $\mathcal{H}$. If we define

$$
\gamma_{\mathcal{H}}(p) := \min_{(r,s) \in \Gamma(\mathcal{H})} g_{K(r,s)}(p) = \min_{(r,s) \in \Gamma(\mathcal{H})} \frac{p(1-p)}{r(1-p) + sp},
$$

then $ed_{\mathcal{H}}(p) \leq \gamma_{\mathcal{H}}(p)$.

Let $K$ be a $p$-core CRG, $v \in V(K)$, and let $x$ be an optimal weight vector in the quadratic program (3) that defines $g_K(p)$. The weight of $v$, denoted by $x(v)$, is the entry corresponding to $v$ of the vector $x$. We denote the gray neighborhood of $v$ by $N_G(v) = \{v' \in V(K) : vv' \in EG(K)\}$. The weighted gray degree of vertex $v \in V(K)$ is $d_G(v) = \Sigma_{v' \in N_G(v)} x(v')$ and the number of vertices adjacent to $v$ via gray edges is denoted by $deg_G(v)$, i.e., $deg_G(v) = |N_G(v)|$. We use similar notation for the white and black cases. Now we get $d_G(v) + d_W(v) + d_B(v) = 1$ for each $v \in V(K)$.

The weighted gray codegree of vertices $v$ and $v'$, denoted by $d_G(v,v')$, is the sum of the weights of the common gray neighbors of $v$ and $v'$. Denote the number of common gray neighbors of vertices $v$ and $v'$ by $deg_G(v,v')$.

Marchant and Thomason [10] gave the following characterization of all $p$-core CRGs.

Proposition 8 [10]. Let $K$ be a $p$-core CRG.

(a) If $p \leq 1/2$, then there are no black edges, and the white edges are only incident to black vertices.

(b) If $p \geq 1/2$, then there are no white edges, and the black edges are only incident to white vertices.

Martin [13] gave a formula for $d_G(v)$ for all $v \in V(K)$ and a bound on the weight of each $v$.

Proposition 9 [13]. Let $p \in (0,1)$ and $K$ be a $p$-core CRG with optimum weight vector $x$.

(a) If $p \leq 1/2$, then $x(u) = g_K(p)/p$ for all $v \in VW(K)$ and

$$
x(u) \leq g_K(p)/(1-p), \quad d_G(u) = \frac{p - g_K(p)}{p} + \frac{1 - 2p}{p} x(u), \text{ for each } u \in VB(K).
$$

(b) If $p \geq 1/2$, then $x(u) = g_K(p)/(1-p)$ for all $u \in VB(K)$ and

$$
x(v) \leq g_K(p)/p, \quad d_G(v) = \frac{1 - p - g_K(p)}{1-p} + \frac{2p - 1}{1-p} x(v), \text{ for each } v \in VW(K).
$$

The following results will be used in this paper.
Proposition 10 [13]. Let $p \in (0, 1/2)$ and $K$ be a $p$-core CRG with black vertices and white or gray edges.
(a) If $K$ has no gray 3-cycle, then $g_K(p) > p/2$.
(b) If $K$ has a gray 3-cycle, but no gray $C_4^+$ (that is, four vertices that induce 5 gray edges), then $g_K(p) \geq \min \{2p/3, (1-p)/3\}$.

Proposition 11 [7]. Let $F$ be a connected graph. If some path of maximum length forms a cycle, then $F$ is Hamiltonian.

Proposition 12 [7]. Let $F$ be a graph on $n$ vertices with no cycle of length longer than $\left\lceil \frac{n}{3} \right\rceil - 1$, with every vertex having degree at least $\left\lceil \frac{n-1}{3} \right\rceil \geq 2$ and with every pair of vertices having at least one common neighbor. Furthermore, let $F$ have the property that no maximum length path forms a cycle.

Let $v_1 \cdots v_k$ be a path of maximum length in $F$. Then $v_1$ and $v_k$ have exactly one common neighbor $v_c$ on this path. Furthermore, $N(v_1) \subseteq \{v_2, \ldots, v_c\}$ and $N(v_k) \subseteq \{v_c, \ldots, v_{k-1}\}$.

3. Proof of Theorem 2

In this section, we consider the edit distance function for the hereditary property that forbids $C_{2n}^*$ where $n$ is even and prove that $ed_{\text{Forb}(C_{2n}^*)}(p) = \gamma_{\text{Forb}(C_{2n}^*)}(p)$ for all $p \in [0, 1]$.

First, we obtain the value of $\gamma_{\text{Forb}(C_{2n}^*)}(p)$ for $p \in [0, 1]$ and restrict $ed_{\text{Forb}(C_{2n}^*)}(p)$ to $p \in [0, 1/2]$ and CRGs $K$ with only black vertices. Finally, we determine the edit distance $ed_{\text{Forb}(C_{2n}^*)}(p) = \gamma_{\text{Forb}(C_{2n}^*)}(p)$ and then prove Theorem 2.

Lemma 13. Let $\mathcal{H} = \text{Forb}(C_{2n}^*)$, $p \in [0, 1]$ and $n \geq 4$ be even.

$$\gamma_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1 + \left(\left\lceil \frac{2n-1}{3} \right\rceil - 2\right)p}, \frac{1-p}{n-1} \right\}.$$ 

Furthermore, if there is a $p$-core CRG $K \in K(\text{Forb}(C_{2n}^*))$ such that $g_K(p) < \gamma_{\text{Forb}(C_{2n}^*)}(p)$ for any $p \in [0, 1]$, then $p < 1/2$ and $K$ has all black vertices.

Proof. If $n$ is even, the extreme points of the clique spectrum of $\text{Forb}(C_{2n}^*)$ are $(2, 0), (1, \left\lceil \frac{2n-1}{3} \right\rceil - 1)$ and $(0, n-1)$. Then $\gamma_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1 + \left(\left\lceil \frac{2n-1}{3} \right\rceil - 2\right)p}, \frac{1-p}{n-1} \right\}$.

Since $ed_{\mathcal{H}}(1/2) = \gamma_{\mathcal{H}}(1/2)$ for any hereditary property and $\gamma_{\mathcal{H}}(1) = 0$, we may use continuity and concavity to conclude that $ed_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p) = \frac{1-p}{n-1}$ for $p \in [1/2, 1]$. Now we suppose $p \in [0, 1/2)$ and $K$ is a $p$-core CRG such that $g_K(p) < \gamma_{\mathcal{H}}(p)$. 

If \( K \) has only white vertices, then \(|V(K)| \leq 2\) and \( g_K(p) \geq \frac{p}{2} \geq \gamma_{\mathcal{H}}(p) \) since \( C_{2n}^* \rightarrow K(3,0) \). If \( K \) has both white and black vertices, then it has 1 white vertex \( \omega \) since \( C_{2n}^* \rightarrow K(2,1) \). Furthermore, it can have at most \( \lceil \frac{2n-1}{3} \rceil - 1 \) black vertices.

To see this, denote the vertices of \( C_{2n}^* \) by \( \{0, \ldots, 2n-1\} \) where \( i \sim i+1 \) for \( 0 \leq i \leq 2n-2, 2n-1 \sim 0 \) and \( 0 \sim n \). If \( n \) is not divisible by 3, then let \( S \) consist of the members of \( \{0,1,\ldots,2n-1\} \) that are divisible by 3. The graph \( C_{2n}^* - S \) has \( \lceil \frac{2n-1}{3} \rceil \) connected components, each of which are cliques of size 1 or 2. If \( n \) is divisible by 3, then let \( S = \{i : i \in \{0,1,\ldots,2n-1\}, i-1 \text{ is divisible by } 3\} \). The graph \( C_{2n}^* - S \) has \( \lceil \frac{2n-1}{3} \rceil \) connected components, each of which are cliques of size 2 except three edges \( n-1 \sim n, n \sim 0 \) and \( 0 \sim 2n-1 \).

If \( d_G(v_i) = x(\omega) \) for any \( v_i \in VB(K) \), then by Proposition 9(a), we have \( \frac{g_K(p)}{p} = \frac{\gamma_{\mathcal{H}}(p)}{\frac{2p^2}{3}} + \frac{1-2p}{p} > \frac{g_K(p)}{p} > \frac{\gamma_{\mathcal{H}}(p)}{p} \). Rearranging the terms, we obtain \( g_K(p) > \frac{p}{2} \geq \gamma_{\mathcal{H}}(p) \), a contradiction. So, there are two black vertices \( v_1, v_2 \) in \( K \) such that \( v_1v_2 \in EG(K) \). Let \( v_1 \) receive \( n-1 \sim n \) and \( v_2 \) receive \( 0 \sim 2n-1 \). Then \( C_{2n}^* \rightarrow K \). Thus, regardless of whether the edges are white or gray, there are at most \( \lceil \frac{2n-1}{3} \rceil - 1 \) black vertices in \( K \) and \( g_K(p) \geq \frac{p(1-p)}{1+(\frac{2n-1}{3}-2)p} \geq \gamma_{\mathcal{H}}(p) \).

So, if \( p \in [0,1/2] \) and \( g_K(p) = ed_{Forb(C_{2n}^*)}(p) \), then \( K \) is either \( K(2,0) \), \( K(1,\lceil \frac{2n-1}{3} \rceil - 1) \), \( K(0,n-1) \) or \( K \) has all black vertices (and white or gray edges).

**Proof of Theorem 2.** Now, we calculate \( ed_{\mathcal{H}}(p) \) where \( \mathcal{H} = Forb(C_8^*) \). By Lemma 13, we know \( \gamma_{\mathcal{H}}(p) = \min \left\{ \frac{p}{2}, \frac{1-p}{3}, \frac{p(1-p)}{1+p} \right\} \) and only need to consider the \( p \)-core CRGs \( K \) with only black vertices for some \( p \in [0,1/2] \).

If \( K \) has only black vertices, then \( K \) has no gray \( C_4^+ \) otherwise \( C_8^* \rightarrow K \). By Proposition 10, we know either \( g_K(p) > \gamma_{\mathcal{H}}(p) \) or \( g_K(p) \geq \min \left\{ \frac{2p}{3}, (1-p)/3 \right\} > \gamma_{\mathcal{H}}(p) \). By straightforward calculations, this contradicts to \( g_K(p) < \gamma_{\mathcal{H}}(p) \) for all \( p \in [0,1/2] \).

4. Proof of Theorem 4

In this section, we consider the edit distance function for hereditary property that forbids \( C_n \). Let \( \mathcal{H} = Forb(C_n) \). First, we obtain the value of \( \gamma_{\mathcal{H}}(p) \) for \( p \in [0,1] \). Then we suppose there is a \( p \)-core CRG \( K \in \mathcal{H}(Forb(C_n)) \) such that \( g_K(p) < \gamma_{\mathcal{H}}(p) \) and establish some characterizations of such a \( p \)-core CRG \( K \). Finally, we obtain a contradiction to such a CRG existing in \( \mathcal{H}(Forb(C_n)) \) for our desired range of \( p \) values, establishing \( \gamma_{\mathcal{H}}(p) \leq ed_{\mathcal{H}}(p) \). 
Lemma 14. Let $H = \text{Forb}(\tilde{C}_n)$, and $n \geq 6$. Then

$$\gamma_H(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1+\left(\left\lfloor \frac{n-1}{3} \right\rfloor - 2\right)p}, \frac{1-p}{\left\lfloor \frac{n-3}{2} \right\rfloor} \right\}, \quad \text{for } p \in [0,1].$$

Furthermore, if there is a p-core CRG $K \in \mathcal{H}(\mathcal{H})$ such that $g_K(p) < \gamma_H(p)$ for any $p \in [0,1]$, then $p < \frac{1}{2}$ and $K$ has all black vertices.

Proof. The extreme points of the clique spectrum of $\text{Forb}(\tilde{C}_n)$ are $(2,0)$, $(1,\left\lfloor \frac{n-1}{3} \right\rfloor - 1)$ and $(0,\left\lfloor \frac{n-3}{2} \right\rfloor)$, which establishes the value of $\gamma_H(p)$.

Since $ed_H(1/2) = \gamma_H(1/2)$ for any hereditary property and $\gamma_H(1) = 0$, we may use continuity and concavity to conclude that $ed_H(p) = \frac{1-p}{\left\lfloor \frac{n-3}{2} \right\rfloor}$ for $p \in [1/2,1]$.

Now, let $p \in [0,1/2)$ and $K$ be a p-core CRG such that $\tilde{C}_n \not\sim K$. If $K$ has at most two vertices, then $g_K(p) \geq \frac{p}{2}$ since $\tilde{C}_n \Rightarrow K(3,0)$. If $K$ has both white and black vertices, then it has at most one white vertex since $\tilde{C}_n \Rightarrow K(2,1)$. Furthermore, it can have at most $\left\lceil \frac{n-1}{3} \right\rceil - 1$ black vertices.

To see this, denote the vertices of $\tilde{C}_n$ by $\{0,1,\ldots,n-1\}$ where $i \sim i+1$ for $0 \leq i \leq n-2$, $n-1 \sim 0$ and $0 \sim 2$. Let $S$ consist of the members of $\{3,\ldots,n-1\}$ that are divisible by 3. If $n-1$ is not divisible by 3, then add 0 to $S$. The graph $\tilde{C}_n - S$ has $\left\lfloor \frac{n-1}{3} \right\rfloor$ connected components, each of which are cliques of size 1 or 2 or 3. Thus, regardless of whether the edges are white or gray, there are at most $\left\lceil \frac{n-1}{3} \right\rceil - 1$ black vertices in $K$ and $g_K(p) \geq \frac{p(1-p)}{1+\left(\left\lfloor \frac{n-1}{3} \right\rfloor - 2\right)p}$, with equality if and only if $K \cong K\left(1,\left\lfloor \frac{n-1}{3} \right\rfloor - 1\right)$.

Summarizing, if $p \in [0,1/2)$ and $g_K(p) = ed_H(p)$, then $K$ is either $K(2,0)$, $K\left(1,\left\lfloor \frac{n-1}{3} \right\rfloor - 1\right)$, $K\left(0,\left\lfloor \frac{n-3}{2} \right\rfloor\right)$, or $K$ has all black vertices (and white or gray edges).

We only need to consider the $K \in \mathcal{H}(\text{Forb}(\tilde{C}_n))$ with all black vertices such that $g_K(p) < \gamma_{\text{Forb}(\tilde{C}_n)}(p)$. Now, we establish some characterizations of such a p-core CRG $K$.

Proposition 15. Let $p \in [0,1/2)$ and $K$ be a p-core CRG such that $K$ has only black vertices and white and gray edges. If $\tilde{C}_n \not\sim K$ then $K$ has no gray cycle of length $l \in \left\{\left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, n-1\right\}$.

Proof. Suppose $K$ has some gray cycle of length $l \in \left\{\left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, n-1\right\}$. Partition the vertices of $\tilde{C}_n$ into $l$ parts so that one part is the triangle and each of the others parts is either a set of two consecutive vertices (an edge) or single vertex. Because of the structure of $\tilde{C}_n$ and the fact that $\left\lfloor \frac{n-1}{2} \right\rfloor \leq l \leq n-1$, it is always possible to do so. This partition witnesses an embedding of $\tilde{C}_n$ into
the l-cycle of $K$ because we can map consecutive parts to consecutive vertices on the l-cycle. Since non-consecutive parts do not have edges between them and Proposition 8(a) gives that the edges of $K$ are either white or gray, this map is an embedding that demonstrates $\tilde{C}_n \mapsto K$, a contradiction.

**Proposition 16.** Let $p \in \left[ \frac{1}{n-1}, \frac{1}{2} \right)$, and $K$ be a p-core CRG with all black vertices such that $g_K(p) < \gamma_{\text{Forb}(\tilde{C}_n)}(p)$. Then

(a) for every $v \in V(K)$, $\deg_G(v) \geq \lceil \frac{n-1}{3} \rceil$, and

(b) for every $v, w \in V(K)$, $\deg_G(v, w) \geq 1$.

**Proof.** (a) Let $v, w \in V(K)$. By using Proposition 9(a),

\[
\deg_G(v) \geq \left\lceil \frac{d_G(v)}{\max\{x(w)\}} \rightceil \geq \frac{p-g_K(p)}{p} + \frac{1-2p}{p} x(v) = \frac{(p-g_K(p))(1-p)}{pg_k(p)} = \frac{1-p}{p} - \frac{1-p}{p} = \lceil \frac{n-1}{3} \rceil - 1.
\]

(b) By the inclusion-exclusion principle, $d_G(v) + d_G(w) - d_G(v, w) \leq 1$, and by using Proposition 9(a), we have $d_G(v, w) \geq 2(1-p) \leq 2(1-p)$. Therefore,

\[
\deg_G(v, w) \geq \left\lceil \frac{d_G(v, w)}{\max\{x(u)\}} \right\rceil \geq \left\lceil \frac{p-g_K(p)}{p} - \frac{2(1-p)}{p} \right\rceil = \frac{1-p}{g_K(p)} - \frac{2(1-p)}{p} = \left\lceil \frac{n-1}{3} \right\rceil - \frac{1}{p}.
\]

Since $p \geq \frac{1}{n-1}$, we have $\deg_G(v, w) \geq 1$.

We consider the value of $ed_{\text{Forb}(\tilde{C}_n)}(p)$ from the perspective of the gray subgraphs of CRGs $K$. Let $F$ be a graph such that $V(F) = V(K)$ and $E(F) = EG(K)$, where $K \in \mathcal{H}(\text{Forb}(\tilde{C}_n))$ is a p-core CRG with all black vertices such that $g_K(p) < \gamma_{\text{Forb}(\tilde{C}_n)}(p)$. By Proposition 16, $F$ is a connected graph and each pair of vertices has at least one common neighbor.

**Proposition 17.** Let $n \geq 9$ and $F$ be a graph with no cycle with length in $\left\{ \frac{n-1}{2}, \ldots, n-1 \right\}$ and every pair of vertices having at least one common neighbor. Then $F$ has no cycle of with length greater than $\left\lceil \frac{n-1}{3} \right\rceil - 1$. 

Proof. Let $v_1 \cdots v_\ell v_1$ be a shortest cycle in $F$ among all those with length greater than $n - 1$. Consider the path $v_1 \cdots v_{\lceil \frac{n-1}{2} \rceil - 1}$ on the cycle $v_1 \cdots v_\ell v_1$.

Assume $v_i$ is a common neighbor of $v_1$ and $v_{\lceil \frac{n-1}{2} \rceil - 1}$, then either $v_1 v_i v_{i+1} \cdots v_\ell v_1$ or $v_1 v_i v_{i+1} \cdots v_{\lceil \frac{n-1}{2} \rceil - 1} v_\ell v_1$ has length less than $\ell$. Without loss of generality, we assume $v_1 v_i v_{i+1} \cdots v_\ell v_1$ has length less than $\ell$, which implies

$$\left\lfloor \frac{n-1}{2} \right\rfloor - 1 \geq \ell - i + 2 \geq \ell - \left( \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \right) + 2 \geq n - \left\lfloor \frac{n-1}{2} \right\rfloor + 4.$$ 

Thus,

$$2 \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n - 4 \geq 0,$$

a contradiction, since $2 \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n - 4 < 2 \left( \frac{n-1}{2} + 1 \right) - 1 - n - 4 < 0$.

Therefore, $F$ has no cycle of with length greater than $\left\lfloor \frac{n-1}{2} \right\rfloor - 1$.

Then, we consider the maximum-length path in the graph $F$. If this path forms a cycle, then Proposition 11 gives that $F$ must be Hamiltonian. By Proposition 17, $|V(K)| \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$ and $g_K(p) \geq \frac{1-p}{\left\lceil \frac{n-1}{2} \right\rceil - 1}$, a contradiction. Thus, no maximum-length path in $F$ forms a cycle. By Proposition 17, $F$ has no cycle of with length greater than $\left\lfloor \frac{n-1}{2} \right\rfloor - 1$. And, by Proposition 16, every vertex in $F$ has degree at least $\left\lceil \frac{n-1}{3} \right\rceil \geq 2$ and every pair of vertices has at least one common neighbor.

Let $v_1 \cdots v_\ell$ be a maximum-length path in $F$ such that the sum $x(v_1) + x(v_\ell)$ is largest among all such paths. Then by Proposition 12, we have $v_1$ and $v_\ell$ have a unique common neighbor $v_c$ and $N(v_1) \subseteq \{v_2, \ldots, v_c\}$. Let $v_1$ have $d$ neighbors in $F$. Since $v_1$ cannot have neighbors outside of this path, $d_G(v_1) \leq x(v_2) + \cdots + x(v_c)$. And if $v_i \in \{v_1, \ldots, v_c-1\}$ is a predecessor of a neighbor of $v_1$ in $F$, then it is an endpoint of a path containing the same $\ell$ vertices, namely $v_i v_{i-1} \cdots v_{i+1} v_{i+2} \cdots v_\ell v_1$. Since all $d$ predecessors of gray neighbors of $v_1$ (including $v_1$ itself) have weight at most $x(v_1)$. By Proposition 9, $\frac{\ell - g_K(p)}{p} + \frac{1-p}{p} x(v_1) = x(v_1) + d_G(v_1) \leq x(v_1) + \cdots + x(v_c) \leq d x(v_1) + (c-d) \frac{q}{1-p}$, which implies

$$g_K(p) \left( \frac{c-d}{c} + \frac{1}{p} \right) \geq 1 - x(v_1) \left( d - \frac{1-p}{p} \right).$$

By Propositions 15 and 16, we have $c \leq \left\lceil \frac{n-1}{2} \right\rceil - 1$ and $d > \left\lceil \frac{n-1}{3} \right\rceil - 1$. So when $p \geq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, by Proposition 9(a), we have $x(v) \leq g_K(p)/(1-p)$, hence

$$g_K(p) \geq \frac{1-p}{c} \geq \frac{1-p}{\left\lceil \frac{n-1}{2} \right\rceil - 1} \geq \gamma_{x}(p),$$

a contradiction. So $ed_{x}(p) = \gamma_{x}(p)$ for all $p \in \left[ \frac{1}{\lceil \frac{n-1}{2} \rceil}, \frac{1}{2} \right]$. 

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Finally, $\text{ed}_H(p) = \gamma_H(p) = \frac{p}{2}$ for $p = \frac{1}{\lfloor \frac{n-1}{3} \rfloor}$, and $\text{ed}_H(p) = \gamma_H(p) = \frac{p}{2}$ for $p = 0$. Then, since the function $\gamma_H(p)$ is linear over this interval and $\text{ed}_H(p)$ is continuous and concave down, we have $\text{ed}_H(p) = \gamma_H(p)$ for $p \in \left[0, \frac{1}{\lfloor \frac{n-1}{3} \rfloor}\right]$. Hence the two functions are equal for all $p \in [0,1]$.

5. **Proof of Theorem 5**

Similarly as Section 4, but it also involves some crucial differences. We first prove the following lemma.

**Lemma 18.** Let $\mathcal{H} = \text{Forb}(P_n)$ where $P_n$ denotes the path on $n \geq 3$ vertices.

$$\gamma_{\mathcal{H}}(p) = \min \left\{ \frac{p(1-p)}{1 + (\lfloor \frac{n-1}{3} \rfloor - 2)p}, \frac{1-p}{\lfloor \frac{n}{2} \rfloor - 1} \right\}, \text{ for } p \in [0,1].$$

Furthermore, if there is a $p$-core CRG $K \in \mathcal{H}(\mathcal{H})$ such that $g_K(p) < \gamma_{\mathcal{H}}(p)$ for any $p \in (0,1)$, then $p < \frac{1}{2}$ and $K$ has all black vertices.

**Proof.** The extreme points of the clique spectrum of $\text{Forb}(P_n)$ are $(1, \lfloor \frac{n-1}{3} \rfloor - 1)$ and $(0, \lfloor \frac{n}{2} \rfloor - 1)$, which establishes the value of $\gamma_{\mathcal{H}}(p)$.

Since $\text{ed}_{\mathcal{H}}(1/2) = \gamma_{\mathcal{H}}(1/2)$ for any hereditary property and $\gamma_{\mathcal{H}}(1) = 0$, we may use continuity and concavity to conclude that $\text{ed}_{\mathcal{H}}(p) = \frac{1-p}{\lfloor \frac{n}{2} \rfloor - 1}$ for $p \in [1/2,1]$.

Now, let $p \in [0,1/2)$ and $K$ be a $p$-core CRG such that $P_n \not\rightarrow K$. If $K$ has only white vertices, then $K \approx K(1,0)$ and $g_K(p) = p > \gamma_{\mathcal{H}}(p)$. If $K$ has both white and black vertices, then it has at most one white vertex since $P_n \rightarrow K(2,1)$. Furthermore, it can have at most $\lfloor \frac{n-1}{3} \rfloor - 1$ black vertices. To see this, denote the vertices of $P_n$ by $\{0,1,\ldots,n-1\}$ where $0 \sim 1 \sim 2 \sim \cdots \sim n-1$. Let $S$ consist of the members of $\{0,1,\ldots,n-1\}$ that are divisible by 3. The graph $P_n - S$ has $\lfloor \frac{n-1}{3} \rfloor$ connected components, each of which are cliques of size 1 or 2. Thus, regardless of whether the edges are white or gray, there are at most $\lfloor \frac{n-1}{3} \rfloor - 1$ black vertices in $K$ and $g_K(p) \geq \frac{p(1-p)}{1 + (\lfloor \frac{n-1}{3} \rfloor - 2)p}$, with equality if and only if $K \approx K\left(1, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right)$.

Summarizing, if $p \in [0,1/2)$ and $g_K(p) = \text{ed}_{\mathcal{H}}(p)$, then $K$ is either $K\left(1, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right)$, $K(0, \lfloor \frac{n}{2} \rfloor - 1)$ or $K$ has all black vertices (and white or gray edges).

When $n < 5$, $\gamma_{\mathcal{H}}(p) = \min\{p,1-p\}$. This observation plus continuity and concavity give that $\text{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p)$ for all $p \in [0,1]$. From now on, we assume $n \geq 5$. 

We only need to consider the $K \in \mathcal{K}(\text{Forb}(P_n))$ with all black vertices such that $g_K(p) < \gamma_{\text{Forb}(P_n)}(p)$. Now, we establish some characterizations of such a $p$-core CRG $K$.

**Proposition 19.** Let $p \in \left[\frac{1}{2}, 1\right)$, and $K$ be a $p$-core CRG with all black vertices such that $g_K(p) < \gamma_{\text{Forb}(P_n)}(p)$. Then
(a) for every $v \in V(K)$, $\deg_G(v) \geq \left\lceil \frac{n-1}{3} \right\rceil$, and
(b) for every $v, w \in V(K)$, $\deg_G(v, w) \geq 1$.

**Proof.** (a) Let $v, w \in V(K)$. By using Proposition 9(a),

$$\deg_G(v) \geq \left\lceil \frac{d_G(v)}{\max\{x(w)\}} \right\rceil \geq \frac{p-g_K(p)}{p} + \frac{1-2p}{p} x(v) \geq \frac{(p-g_K(p))(1-p)}{pg_K(p)} = \frac{1-p}{g_K(p)} - \frac{1}{p} \geq \frac{1-p}{\left\lceil \frac{n-1}{3} \right\rceil} - 1.$$

(b) By the inclusion-exclusion principle, $d_G(v) + d_G(w) - d_G(v, w) \leq 1$, and by using Proposition 9(a), we have $d_G(v, w) \geq 2p \frac{p-g_K(p)}{p} + \frac{1-2p}{p} (x(v) + x(w)) - 1 \geq \frac{p-g_K(p)}{p} \geq \frac{p-2g_K(p)}{p}$ and for all $u \in V(K)$, $x(u) \leq g_K(p)/(1-p)$. Therefore,

$$\deg_G(v, w) \geq \left\lceil \frac{d_G(v, w)}{\max\{x(u)\}} \right\rceil \geq \left\lceil \frac{p-2g_K(p)}{p} \right\rceil = \frac{1-p}{g_K(p)} - \frac{2(1-p)}{p} \geq \frac{1-p}{\left\lceil \frac{n-1}{3} \right\rceil} - \frac{1}{p}.$$

Since $p \geq \frac{1}{\left\lceil \frac{n-1}{3} \right\rceil}$, we have $\deg_G(v, w) \geq 1$. 

**Proposition 20.** Let $p \in [0, 1/2)$ and $K$ be a $p$-core CRG such that $K$ has only black vertices and white and gray edges. If $P_n \not\rightarrow K$ then $K$ has no gray path with length greater than $\left\lceil \frac{n}{2} \right\rceil - 1$.

**Proof.** Suppose $K$ has some gray path of length $l > \left\lceil \frac{n}{2} \right\rceil - 1$. Partition the vertices of $P_n$ into $l$ parts so that each of parts is either a set of two consecutive vertices (an edge) or single vertex. Because of the structure of $P_n$ and the fact that $l > \left\lceil \frac{n}{2} \right\rceil - 1$, it is always possible to do so. This partition witnesses an embedding of $P_n$ into $l$-path of $K$ because we can map consecutive parts to consecutive vertices on the $l$-path. Since non-consecutive parts do not have edges
between them and Proposition 8(a) gives that the edges of $K$ are either white or gray, this map is an embedding that demonstrates $P_n \rightarrow K$, a contradiction. ■

We consider the value of $ed_{Forb(P_n)}(p)$ from the perspective of the gray subgraphs of CRGs $K$. Let $F$ be a graph, $V(F) = V(K)$, $E(F) = EG(K)$ where $K \in \mathcal{K}(Forb(P_n))$ is a p-core CRG with all black vertices such that $g_K(p) < \gamma_{Forb(P_n)}(p)$. By Proposition 19, we obtain $F$ is a connected graph.

Suppose a maximum-length path forms a cycle in the graph $F$. Then Proposition 11 implies that $F$ must be Hamiltonian. By Proposition 20, $|V(K)| \leq \lceil \frac{n}{2} \rceil - 1$ and $g_K(p) \geq \frac{1-p}{1-p}$, a contradiction, and so we may assume that no maximum-length path in $F$ forms a cycle. By Proposition 20, $F$ has no path with length greater than $\lceil \frac{n}{2} \rceil - 1$, so $F$ has no cycle with length greater than $\lceil \frac{n}{2} \rceil - 1$. And, by Proposition 19, every vertex in $F$ has degree at least $\lceil \frac{n-1}{2} \rceil \geq 2$ and every pair of vertices has at least one common neighbor.

Let $v_1 \cdots v_\ell$ be such a maximum-length path in $K$ such that the sum $x(v_1) + x(v_\ell)$ is the largest among all such paths. By Proposition 12, $v_1$ and $v_\ell$ have a unique common neighbor $v_c$ and $N(v_1) \subseteq \{v_2, \ldots, v_c\}$. Let $v_1$ have $d$ neighbors in $F$. Since $v_1$ cannot have neighbors outside of this path, the sum of the weights of the neighbors of $v_1$ satisfies $d_G(v_1) \leq x(v_2) + \cdots + x(v_c)$. And if $v_i \in \{v_1, \ldots, v_{c-1}\}$ is a predecessor of a neighbor of $v_1$, then it is an endpoint of a path containing the same $\ell$ vertices, namely $v_1v_{i-1}\cdots v_{i+1}v_{i+2}\cdots v_c\cdots v_\ell$. Hence all $d$ predecessors of gray neighbors of $v_1$ (including $v_1$ itself) have weight at most $x(v_1)$. By Proposition 9, $\frac{\nu - g_K(p)}{p} + \frac{1-p}{p}x(v_1) = x(v_1) + d_G(v_1) \leq x(v_1) + \cdots + x(v_c) \leq dx(v_1) + (c-d)\frac{g_K(p)}{1-p}$, which implies

$$g_K(p) \left( \frac{c-d}{1-p} + \frac{1}{p} \right) \geq 1 - x(v_1) \left( d - \frac{1-p}{p} \right).$$

By Propositions 19 and 20, we have $c \leq \lceil \frac{n}{2} \rceil - 1$ and $d \geq \lceil \frac{n-1}{3} \rceil$. And, when $\lceil \frac{n-1}{3} \rceil - 1 \leq p \leq \frac{1}{2}$, by Proposition 9(a), we have $x(v) \leq g_K(p)/(1-p)$, hence

$$g_K(p) \geq \frac{1-p}{c} \geq \frac{1-p}{\lceil \frac{n}{2} \rceil - 1} \geq \gamma_{\mathcal{K}}(p),$$
a contradiction.

So we can get $ed_{\mathcal{K}}(p) = \gamma_{\mathcal{K}}(p)$ for $p \in \left[ \frac{1}{\lceil \frac{n}{2} \rceil - 1}, 1 \right]$. The proof is thus complete.

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