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DUALIZING DISTANCE-HEREDITARY GRAPHS

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Abstract

Distance-hereditary graphs can be characterized by every cycle of length at least 5 having crossing chords. This makes distance-hereditary graphs susceptible to dualizing, using the common extension of geometric face/vertex planar graph duality to cycle/cutset duality as in abstract matroidal duality. The resulting "DH* graphs" are characterized and then analyzed in terms of connectivity. These results are used in a special case of plane-embedded graphs to justify viewing DH* graphs as the duals of distance-hereditary graphs.

Keywords: distance-hereditary graph, dual graph, graph duality.

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1. Distance-Hereditary and DH* Graphs

Unless otherwise noted, all graphs are simple (meaning no multiple edges or loops) and finite, with notation and terminology following [3]. A *chord* of a cycle C is an edge ab that has $a, b \in V(C)$ and $ab \notin E(C)$. Two chords ab and cd of C are *crossing chords* of C if their endpoints come in the order a, c, b, d around C. The notation "$\geq k$-cycle" abbreviates "cycle of length at least k."

Distance-hereditary graphs G were defined by Howorka in [4] by every connected induced subgraph H of G and every $x, y \in V(H)$ satisfying $\text{dist}_H(x, y) = \text{dist}_G(x, y)$; see [2, 3] for additional characterizations. At first glance, this graph class looks like a poor candidate for traditional graph duality, but another of Howorka's original characterizations, in Proposition 1, suggests a simple way to dualize distance-hereditary graphs. The resulting concept will be introduced (below), characterized (in Section 2), and motivated (in Section 4) in this paper.

Proposition 1 [4]**.** A graph is a distance-hereditary graph if and only if every ≥5-cycle has crossing chords.

A *minimal edge cutset* D of a graph G is an inclusion-minimal $D \subset E(G)$ such that deleting D would produce a subgraph $G - D$ with $V(G) = V(G - D)$ that consists of two *components* (maximal connected subgraphs); for convenience, we will simply call such sets D the *min-cutsets* of G. Call a min-cutset of cardinality k a *k-edge min-cutset* and a min-cutset of cardinality at least k a *≥k-edge min-cutset.*

As in [7], a *cut-chord* of a min-cutset D of G is an edge $e \in E(G) \setminus D$ whose deletion would disconnect one of the components of $G-D$. Say that a min-cutset D of G *separates* two cut-chords e_1 and e_2 of D (or that e_1 and e_2 are *separated by* D) if e_1 and e_2 are in different components of $G - D$.

Define a *DH* graph* to be a graph in which every ≥5-edge min-cutset separates two cut-chords. The graph G_1^1 in Figure 1 is a DH* graph, since its only ≥5-edge min-cutsets are (up to isomorphism) $\{1, 4, 7, 9, 11\}$, $\{1, 4, 7, 10, 13\}$, $\{1, 4, 7, 9, 12, 13\}$, and $\{1, 4, 7, 10, 11, 12\}$ (each of which separates the cut-chords 6 and 8) along with $\{1, 2, 5, 7, 8\}$, $\{1, 2, 5, 7, 9, 11\}$, $\{1, 2, 5, 7, 10, 13\}$, $\{1, 2, 5, 7, 9, 12, 13\}$, and $\{1, 2, 5, 7, 10, 11, 12\}$ (each of which separates the cut-chords 3 and 6). But the graph G_2^1 is not a DH* graph; for instance, its min-cutset $D = \{1, 4, 7, 9, 11\}$ has only the two cut-chords 6 and 8, which are not separated by D in this graph).

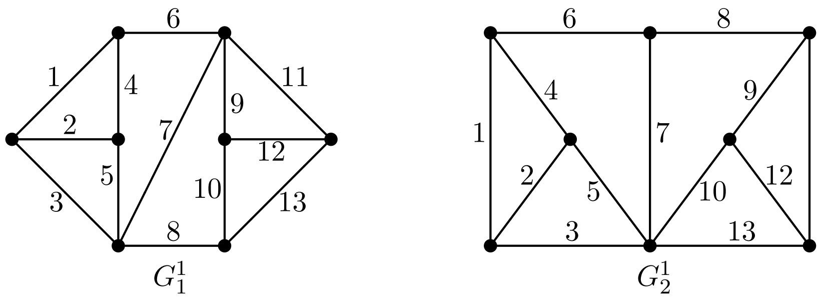

Figure 1. A DH* graph G_1^1 and a non-DH* graph G_2^1.

Section 2 will characterize the DH* graphs, and then Section 3 will exhibit all the 3-connected graphs that are DH* graphs and describe the structure of DH* graphs that are not 3-connected. Using those results, Section 4 will circle back to discuss how geometric duality of plane-embedded graphs motivates the definition of DH* graphs as a dual to distance-hereditary graphs, including how cut-chords of min-cutsets can be viewed as the duals of chords of cycles as in [7]. It is important to emphasize that, in all instances of graph-theoretical duality, different characterizations of any graph class can dualize to several nonequivalent dual classes. Thus each graph class—chordal graphs in [7] and distance-hereditary graphs here—can have more than one "dual class," each with a different fundamental structure for which different theoretical questions (eventually) arise. (See [5] for a more formal discussion.) Graph-theoretical duality becomes better behaved only when restricted to planar graphs (so as to benefit from geometric duality), and even then profits from making additional restrictions (as is done in Section 4 below, paralleling the same procedure used in [7]).

2. Characterizing DH* Graphs

Define a *relevant graph* to be a 2-connected, 3-edge-connected graph of order at least 3; thus, 3-connected graphs are always relevant graphs. This concept is motivated in [6, 7] by relevant, plane-embedded graphs always having well-defined, simple, relevant, plane-embedded dual graphs.

Lemma 2. *Every vertex of a relevant DH* graph has degree* 3 *or* 4.

Proof. Suppose v is a vertex of a relevant DH* graph G, and let $D = \{vx \in E(G) : x \in N_G(v)\}$. Since relevant graphs are 3-edge-connected, $\deg_G(v) \geq 3$. Since relevant graphs are 2-connected, $\{v\}$ induces an edgeless component of $G - D$ with $G - v$ the other component of $G - D$, and so D is a min-cutset of G that cannot separate cut-chords. Therefore, the DH* graph G has $|D| = \deg_G(v) \leq 4$. ■

For every induced subgraph H of a graph G, let G/H denote the multigraph that results from contracting all the edges of H down to one new vertex that is denoted v_H (allowing parallel edges, but deleting any loops thereby formed). For example, if $G \cong K_5$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and H is the induced subgraph with $V(H) = \{v_3, v_4, v_5\}$, then G/H has vertex set $\{v_1, v_2, v_H\}$ with one (simple) edge v_1v_2, three parallel edges between v_1 and v_H, and three parallel edges between v_2 and v_H. Let $\Delta(G)$ and $\Delta(G/H)$ denote the maximum degree of vertices in, respectively, the simple graph G and the multigraph G/H.

Theorem 3. *A relevant graph G with $\Delta(G) \leq 4$ is a DH* graph if and only if $\Delta(G/H) \leq 4$ for all* 2*-edge-connected induced subgraphs H of G for which the multigraph G/H is* 2*-connected.*

Proof. To show necessity, suppose G is a relevant DH* graph with $\Delta(G) \leq 4$. Suppose G has a 2-edge-connected induced subgraph H for which G/H is 2-connected, and let $D = \{xy \in E(G) : x \in V(H) \text{ and } y \notin V(H)\}$ and $D_H = \{yv_H : y \in N_{G/H}(v_H)\}$. Since G/H is 2-connected and H is connected, $G - H$ is connected, and D and D_H are min-cutsets of, respectively, G and G/H. Since H is 2-edge-connected, D has no cut-chords in the component H of $G - D$, and so D does not separate cut-chords. Thus G being a DH* graph requires that

$\deg_{G/H}(v_H) = |D_H| = |D| \leq 4$, while each $w \in V(G) \setminus \{v_H\}$ has $\deg_{G/H}(w) = \deg_G(w) \leq \Delta(G) \leq 4$. Therefore, $\Delta(G/H) \leq 4$ in G/H.

To show sufficiency, suppose G is a relevant graph with $\Delta(G) \leq 4$, but G is not a DH* graph (arguing by contraposition). Thus some $^{\geq}$5-edge min-cutset D of G does not separate cut-chords, so some component H of $G - D$ contains no cut-chord of D, and so H is 2-edge-connected. Thus $G/H - v_H \cong G - H$ is connected (since D is a min-cutset of G), while $G/H - w$ is connected for all $w \in V(G/H) \setminus \{v_H\} = V(G) \setminus V(H)$ (since G is 2-connected). Therefore, G/H is 2-connected, but $\deg_{G/H}(v_H) = |D| \geq 5$ implies $\Delta(G/H) \not\leq 4$. ■

3. The Role of 3-Connectivity for DH* Graphs

This section details the effect of 3-connectedness of a relevant graph on its being a DH* graph. Recall that relevant graphs are always 2-connected (and 3-edge-connected), and that 3-connected graphs are always relevant graphs.

Theorem 4. *Figure* 2 *shows all the* 3*-connected DH* graphs.*

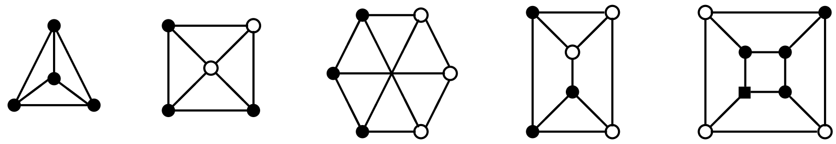

Figure 2. Five DH* graphs (the vertex colors are explained below).

Proof. We first show that the five 3-connected graphs in Figure 2 are indeed DH* graphs. Since G_1^2 has only six edges and each vertex has degree 3, there are no $^{\geq}$5-edge min-cutsets, and so G_1^2 is automatically a *DH** graph. In the other four graphs G_i^2, the five edges with one black and one white endpoint form a 5-edge min-cutset D_i with each component of $G - D_i$ containing at least one cut-edge of D_i (and these are the only 5-edge possibilities up to isomorphism). Graph G_5^2 also has 6-edge min-cutsets, one of which is obtained by changing the one black "square" vertex into a white vertex. No matter how such 6-edge min-cutsets D are chosen, the six edges in $E(G_5^2) \setminus D$ will form two components of $G - D$ that are trees, and with each containing a cut-edge of D. Therefore, all the five graphs in Figure 2 are DH* graphs.

Let G be an arbitrary 3-connected DH* graph, so G is a relevant graph and each $u \in V(G)$ has $\deg_G(u) \in \{3, 4\}$ by Lemma 2.

First suppose that G has adjacent degree-4 vertices v and w. Since G is 3-connected, there is a minimum-length chordless cycle C that has $v, w, x \in V(C)$

where $x \notin \{v, w\}$. But now G has at least five edges with one endpoint in C and the other not in C (two incident to each of v and w, and one incident to x), which would contradict Theorem 3 with $H = C$.

Now suppose that $\deg_G(v) = 4$ with $G \ncong G_2^2$ in Figure 2 where each $w \in N_G(v)$ has $\deg_G(w) = 3$. Since $G \ncong K_5$ and G is a relevant DH* graph, v has nonadjacent degree-3 neighbors x and y and a minimum-length chordless cycle C with $vx, vy \in E(C)$ that has some $z \in V(C) \setminus \{v, x, y\}$. But now G has at least five edges with one endpoint in C and the other not in C (two incident to v, and one incident to each of x, y, z), which would again contradict Theorem 3 with $H = C$.

Therefore, we can assume that G is a *cubic graph* (meaning that every vertex has degree 3), and so $|V(G)| \geq 4$ is even. By [1], a graph is both 3-connected and cubic if and only if it can be constructed from K_4 by repeated applications of the following operation.

> Given two (possibly adjacent) edges a_1b_1 and a_2b_2, subdivide each a_ib_i with a new vertex x_i and then insert a new edge x_1x_2,

forming a new graph that has two more vertices and three more edges than the original graph. (The other two operations described in [1] allow one or both x_i to be in $\{a_i, b_i\}$, which would prevent the new graph from being cubic.)

Note that applying this construction to adjacent edges of G_1^2 in Figure 2 produces the graph G_4^2, while applying it to nonadjacent edges of G_1^2 produces G_3^2. Similarly, applying the construction from [1] to G_2^2 would never produce a cubic graph.

The two graphs on the left in Figure 3 show the only graphs (up to isomorphism) that result from applying the construction from [1] to G_3^2; specifically, they result from letting a_1b_1 and a_2b_2 be, respectively, adjacent and nonadjacent edges of G_3^2. In each such graph G, the five edges that have both black and white vertices form a min-cutset D for which the black vertices induce a 2-connected (5-cycle) component of $G - D$, and so a component that contains no cut-chord of D. Therefore, such graphs G are not DH* graphs.

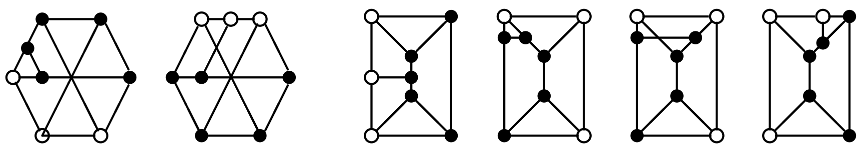

Figure 3. The non-DH* graphs constructed as in [1] from G_3^2 and G_4^2.

Note that applying the construction from [1] to two edges of G_4^2 that are in different triangles produces G_5^2. The four graphs on the right in Figure 3 show the remaining graphs (up to isomorphism) that result from applying the construction from [1] to G_4^2; specifically, they result from letting a_1b_1 and a_2b_2 be (from left to right) edges that are not in a triangle, edges a_1b_2 not in a triangle and a_2b_2 in a triangle with $a_1 = a_2$, edges a_1b_2 not in a triangle and a_2b_2 in a triangle with $a_1 \neq a_2$, and both edges in the same triangle. Just as in the G_3^2 discussion above, the five edges that have both black and white endpoints form a min-cutset that shows that graphs that are constructed this way are not DH* graphs.

The four graphs in Figure 4 show all the graphs G (up to isomorphism) that can result from applying the construction from [1] to G_5^2; specifically, they result from letting edges a_1b_1 and a_2b_2 be adjacent (the leftmost graph) or nonadjacent. Just as in the G_3^2 and G_4^2 discussions above, the five edges that have both black and white endpoints form a min-cutset that shows that graphs that are constructed this way are not DH* graphs.

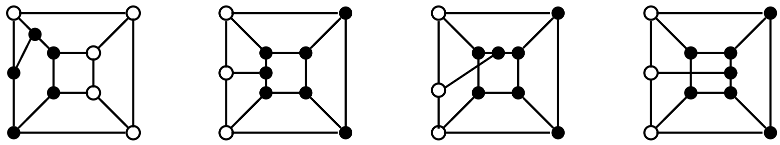

Figure 4. The non-DH* graphs constructed as in [1] from G_5^2.

The preceding four paragraphs show that applying the construction from [1] to the graphs in Figure 3 cannot produce additional 3-connected, cubic DH* graphs. Also, observe that G_1^2 is the only 3-connected DH* graph that has order 4.

Finally, to show that no additional 3-connected, cubic DH* graphs can exist, suppose that G'' is a 3-connected, cubic DH* graph that is constructed as in [1] from a 3-connected, cubic graph G' by replacing the edges $a_1b_1, a_2b_2 \in E(G')$ with a_1x_1, b_1x_1, a_2x_2, b_2x_2, $x_1x_2 \in E(G'')$ (toward showing that G' was also a 3-connected DH* graph).

Suppose D' is an arbitrary ${\geq}5$-edge min-cutset of G' (toward finding a new ${\geq}5$-edge min-cutset D'' of the DH* graph G'' that has separated cut-chords that correspond to separated cut-chords of D' in G'). We can assume that $D' \cap \{a_1b_1, a_2b_2\} \neq \emptyset$ (otherwise $D'' = D'$ will have the same separated cut-chords in G'' as D'' has in G').

Case 1. When $a_ib_i \in D'$ and $a_{3-i}b_{3-i} \notin D'$. One of the four edges $a_ix_i, b_ix_i \in E(G'')$ can replace a_ib_i to form the new min-cutset D'' of G''. The separated cut-chords of D'' in G'' will correspond to the separated cut-chords of D' in G' so long as, when $e \in \{a_{3-i}x_{3-i}, b_{3-i}x_{3-i}\}$ is a cut-chord of D'', the cut-chord $a_{3-i}b_{3-i}$ of D' corresponds to e.

Case 2. When both $a_1b_1, a_2b_2 \in D'$. Use the edges $a_1x_1, x_1x_2, b_2x_2 \in E(G'')$ to replace the pair a_1b_1, a_2b_2 to form the new min-cutset D'' of G''. Edges b_1x_1

and a_2x_2 will be separated cut-chords of D'' in G'' since x_1 and x_2 will be degree-1 vertices of $G'' - D''$.

Therefore, there are no 3-connected, cubic, DH* graphs beyond the four cubic graphs in Figure 2, and so there are no 3-connected DH* graphs beyond the five graphs G_i^2 shown there. ■

In Theorem 5, $\{s_1, s_2\} \subset V(G)$ is an order-2 *minimal separator* of a connected graph G if, for some $x, y \in V(G)$ from different components of $G - \{s_1, s_2\}$, each s_i is in an x-to-y path of $G - s_{3-i}$. Relevant graphs that are not 3-connected necessarily have an order-2 minimal separator.

Theorem 5. *If a relevant DH* graph G is not 3-connected, then G has a minimal separator $\{s, t\}$ for which $G - \{s, t\}$ has a component whose vertex set combines with $\{s, t\}$ to induce one of the subgraphs shown in Figure 5.*

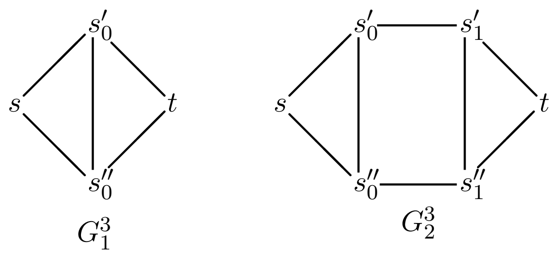

Figure 5. The two subgraphs of G mentioned in Theorem 5, where each $\deg_G(s_i') = \deg_G(s_i'') = 3$.

Proof. Suppose G is a relevant DH* graph that is not 3-connected, $\{s, t\}$ is a minimal separator of G, and H' is a component of $G - \{s, t\}$ such that $V(H') \cup \{s, t\}$ induces a 2-connected subgraph H of G. Further assume that such s, t, and H' are chosen so that $|V(H')|$ is as small as possible. Thus each $v \in V(H')$ has $N_G(v) \subseteq V(H)$, and each vertex of G has degree 3 or 4 by Lemma 2.

Since G is 3-edge connected, s and t cannot be endpoints of two edges of G that form a min-cutset of G. Thus, not both $\deg_G(s) = \deg_G(t) = 3$, and so we can assume that $\deg_G(s) = 4$ with $N_G(s) = \{p, q, s_0', s_0''\}$ where $p, q \notin V(H')$ and $s_0', s_0'' \in V(H')$. The assumed minimality of $|V(H')|$ ensures that t also has two neighbors in $V(H')$, and so $st \notin E(G)$ (otherwise, taking $p = t$ and $t' \in N_G(t) \setminus V(H)$ would make $\{qs, tt'\}$ a 2-edge min-cutset of G).

If s_0' is not adjacent to s_0'', then s, s_0', and s_0'' are vertices of a chordless $\geq$4-cycle C_1 of H for which G has at least five edges between vertices in C_1 and vertices not in C_1 (the edges ps and qs and one edge incident with each vertex in $V(C_1) \setminus \{s\}$); but then $\Delta(G/C_1) \geq 5$ (contradicting Theorem 3). Therefore, $s_0's_0'' \in E(H)$.

If $\deg_H(s_0') = 4$ or $\deg_H(s_0'') = 4$, then $\{s, s_0', s_0''\}$ induces a triangle C_1' that has at least five edges between vertices in C_1' and vertices not in C_1' (the edges ps and qs, two edges incident with s_0' or s_0'' and one edge incident with the other), and so for which $\Delta(G/C_1') \geq 5$ (contradicting Theorem 3). Therefore, $\deg_H(s_0') = \deg_H(s_0'') = 3$, say with $s_0's_1', s_0''s_1'' \in E(H) \setminus E(C_1')$.

Repeat the argument used in the preceding two paragraphs to introduce vertices $s_1', s_1'', \ldots, s_i', s_i''$ successively, where each $\{s, s_0', s_0'', \ldots, s_i', s_i''\}$ induces a $(2i+3)$-cycle H_i of H with exactly $i \geq 1$ chords $s_0's_0'', \ldots, s_{i-1}'s_{i-1}''$, stopping when finally $s_i' = s_i''$. Thus $\{s, s_i'\}$ is a minimal separator of G, and so $s_i' = s_i'' = t$ by the assumed minimality of $|V(H')|$.

If $i = 1$, then the 2-connected subgraph $H \cong G_1^3$, and if $i = 2$, then $H \cong G_2^3$. If $i \geq 3$, then $V(G) \setminus \{s_0', \ldots, s_i'\}$ would induce a 2-edge-connected subgraph H_i' of G with at least five edges between vertices in H_i' and vertices not in H_i' (namely, ss_0', $s_i't$, and $s_j's_j''$ for $1 \leq j \leq i$), and so for which $\Delta(G/H_i') \geq 5$ (contradicting Theorem 3).

Therefore, $i = 1$ or $i = 2$, and H is G_1^3 or G_2^3 as in Figure 5. ■

Noting the intrinsic role of Theorem 3 in the preceding proof, it is worth mentioning that Theorem 5 can in turn be used to simplify the application of Theorem 3 to graphs that are not 3-connected as follows: The choice of the 2-edge-connected induced subgraphs H in Theorem 3 can be limited to avoid H that contain degree-3 vertices such as s_i' and s_i'' in Figure 5.

4. The Planar Motivation for DH* Graphs

A plane embedding of a relevant planar graph G is transformed into its *geometric dual* graph G^* as described in this paragraph (with a detailed example in the next paragraph). Vertices of G, along with their incident edges, become the faces of G^* that are bordered by the corresponding edges, while the faces of G similarly become the vertices of G^*. Thus, vertices and faces are regarded as duals of each other. Since edges of G thereby correspond to edges of G^*, edges are regarded as self-dual, with each edge of either simultaneously joining two adjacent vertices and separating two adjacent faces. The plane embedding of G thus produces a plane embedding of G^*, with G also becoming the dual graph of G^* based on that embedding—in other words, with $G = (G^*)^*$.

Figure 6 illustrates this process of dualizing relevant plane-embedded graphs, showing the dual graph $(G_1^1)^*$ for the embedding of the graph G_1^1 in Figure 1. The vertices of $(G_1^1)^*$ are labeled with the edge sets that form the boundaries of the seven faces of G_1^1 (including the "exterior" hexagonal face), with the edges of $(G_1^1)^*$ labeled to match the corresponding edges of G_1^1. The vertices of G_1^1 (viewed as sets of incident edges) similarly correspond to (the edge sets of) the eight faces of $(G_1^1)^*$.

Each cycle C of G, as a set of edges, becomes a min-cutset D^* of G^* (with

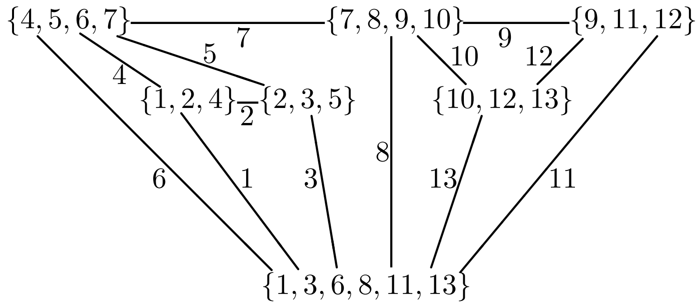

Figure 6. The geometric dual $(G_1^1)^*$ of G_1^1 as embedded in Figure 1.

the faces "inside" the geometric curve corresponding to C in the embedding of G becoming vertices of one of the components of $G^* - D^*$, and the faces "outside" C becoming vertices of the other component); similarly, each min-cutset D of G becomes a cycle C^* of G^*. Thus, cycles and min-cutsets are regarded as duals of each other. This concrete geometric duality generalizes to abstract matroid duality, interchanging cycles with min-cutsets (both viewed as sets of edges). Many elementary graph theory textbooks describe both geometric duality and matroidal duality, perhaps none more accessibly than Wilson's elementary text [10].

For instance, the cycle with edge set $\{1, 3, 6, 7\}$ of G_1^1 in Figure 1 (which happens not to be a face) corresponds to the min-cutset $\{1, 3, 6, 7\}$ in the dual graph $(G_i^1)^*$ in Figure 6 (which is not a set of edges incident to a vertex).

The chords of a cycle C can be characterized as the edges $e \notin E(C)$ for which $E(C)$ can be partitioned into the edge sets of two paths P_1 and P_2 such that both $E(P_i) \cup \{e\}$ are edge sets of cycles. (Notice that cycles can have crossing chords in a plane-embedding, with one of the chords inside of the geometric curve corresponding to C and the other outside of that curve.) Similarly, the cut-chords of a min-cutset D can be characterized as the edges $e \notin D$ for which D can be partitioned into two subsets D_1 and D_2 such that both $D_i \cup \{e\}$ are min-cutsets. Thus chords of cycles are regarded as the duals of cut-chords of min-cutsets. For instance, the min-cutset $\{1, 4, 7, 8\}$ of G_i^1 in Figure 1 has cut-chord 6, corresponding to a cycle $\left(G_i^1\right)^*$ in Figure 6 that has chord 6.

Two graphs are *cycle-isomorphic* if there is a bijection between their edge sets for which the cycles of each graph maps to the cycles of the other. As in [11], define a graph G to be *cycle-determined* if $G \cong G'$ for all graphs G' that are cycle-isomorphic to G. It is important that every 3-connected graph is cycle-determined; see [8, 9].

The two plane-embedded, relevant graphs G_1^1 and G_2^1 in Figure 1 are cycle-isomorphic, but $G_1^1 \ncong G_2^1$ shows that they are not cycle-determined. Also note that, no matter how G_1^1 and G_2^1 are embedded in the plane, $\left(G_1^1\right)^*$ and $\left(G_2^1\right)^*$ will not be distance-hereditary graphs—for instance, using Proposition 1, the edge set $\{1,4,7,9,11\}$ of each $\left(G_i^1\right)^*$ will correspond to a cycle with chords 6 and 8, but without crossing chords. This shows the need to require cycle-determined graphs in Theorem 6 (which largely motivates the "DH* graph" terminology).

Theorem 6. *A cycle-determined, plane-embedded, relevant graph is a DH* graph if and only if its geometric dual is a distance-hereditary graph.*

Proof. Suppose G is a cycle-determined, plane-embedded, relevant graph with geometric dual G^* (which also makes G^* a cycle-determined, plane-embedded, relevant graph).

To show necessity, suppose G is a DH* graph for which G^* is not a distance-hereditary graph (arguing by contradiction); further assume that, among all such graphs, G is chosen to minimize $|E(G)|$. By Theorem 4, G is not 3-connected, since G_1^2 and G_2^2 are both self-dual, $\left(G_4^2\right)^* \cong K_{1,1,1,2}$ (K_5 with one edge deleted), and $\left(G_5^2\right)^* \cong K_{2,2,2}$ (the octahedron graph), where each planar graph $\left(G_i^2\right)^*$ is a distance-hereditary graph by definition (all induced paths between nonadjacent vertices in each have the same length, namely 2).

Hence we can assume that the relevant (and so 2-connected) DH* graph G is not 3-connected. By [8], the cycle-determined graph G has a *generalized circuit representation*, defined as $k \geq 2$ subgraphs $G_1, \ldots, G_k$ of G that satisfy the following three conditions.

- Each G_i is 2-connected with $E(G_i) \neq \emptyset$, and with $|V(G_i)| \geq 3$ when $k = 2$.
- Sets $E(G_1), \ldots, E(G_k)$ partition $E(G)$, and each $V(G_i) \cap \bigcup_{j \neq i} V(G_j)$ consists of two distinguished vertices of G_i.
- Replacing each subgraph G_i by an edge between its distinguished vertices produces a cycle.

By Lemma 2, Theorem 5, and G being 3-edge-connected, some G_i has $G_i \cong G_1^3$ or $G_i \cong G_2^3$ as in Figure 5, say with its vertices labeled as shown there, with s and t as its distinguished vertices, and with exactly two edges $ps, qs \notin E(G) \setminus E(G_i)$ that have s as an endpoint.

Figure 7 shows the induced subgraph of G^* that corresponds to $G_i \cong G_2^3$ augmented with the edges $ps, qs \notin E(G_i)$, where the vertices of G^* are now labeled with the vertex sets of the faces in G. (The top and bottom vertices in Figure 7 correspond to the inside and outside faces of the plane-embedded generalized circuit representation of G from [8].) For the simpler $G_i \cong G_1^3$ case, simplify Figure 7 by deleting the vertex $\{s_0', s_0'', s_1', s_1''\}$ and inserting one new edge between vertices $\{s, s_0', s_0''\}$ and $\{s_1' s_1'', t\}$, and then replacing all occurrences of s_1' and s_1'' by s_0' and s_0'', respectively and labeling the newly inserted edge as $s_0' s_0''$.

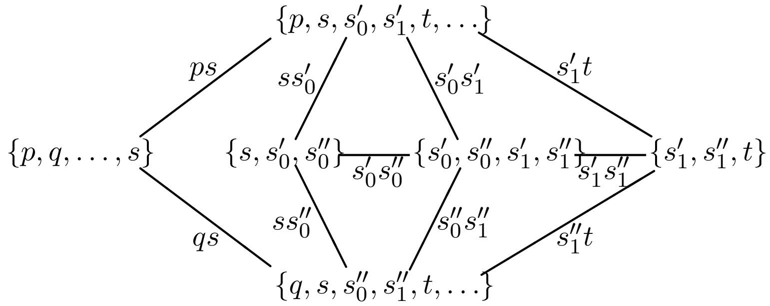

Figure 7. An induced subgraph of G^* that results from G_2^3 in Figure 5.

Since G^* is not a distance-hereditary graph, G^* has a ≥5-cycle C^* with no crossing chords by Proposition 1; further assume that, among all such cycles, C^* is chosen to minimize $|E(C^*) \cap E(G_i^*)|$. Let D be the ≥5-edge min-cutset of G whose edges correspond to the edges of C^*. Let $S_i^* \subset E(G^*)$ be the set of all edges of C^* that correspond to edges of G_i^*, so S_i^* forms a subpath of C^* (and $ps, qs \notin S_i^*$). Let S_i be the set of corresponding edges of G_i, so $S_i \subset D$ is a min-cutset of G_i that has s and t in different components of $G - D$.

If $S_i^* = \emptyset$, then C^* is also a ≥5-cycle without crossing chords of $(G/G_i)^*$, which would contradict the assumed minimality of $|E(G)| = |E(G^*)|$ in the original choice of G. Thus $S_i^* \neq \emptyset$.

If $|S_i^*| \in \{1, 2\}$, the only choices for the subpath S_i^* of G_i^* are $\{ss'_0, ss''_0\}$ or $\{s'_0t, s''_0t\}$ if $i = 1$, and $\{ss'_0, ss''_0\}$, $\{s'_0s'_1, s''_0s''_1\}$, or $\{s'_1t, s''_1t\}$ if $i = 2$. For each of these choices, the min-cutset S_i of G_i does not separate cut-chords in G_i. Replacing S_i with $\{ps, qs\}$ in D creates another min-cutset of G that separates the same cut-chords in G as D, which would contradict the assumed minimality of $|E(C^*) \cap E(G_i^*)| = |S_i^*| = |S_i|$. Thus $|S_i^*| \notin \{1, 2\}$.

Therefore, $|S_i^*| \geq 3$, and so the path S_i^* of G_i^* (and so the min-cutset D of G) contains one or both of the edges $s'_0s''_0$ and $s'_1s''_1$. In each case, D separates cut-chords of G (namely, ss'_0, s''_0t or ss''_0, s'_0t when $i = 1$, and $ss'_0, s''_0s''_1$ or $ss''_0, s'_0s'_1$ or $s'_0s'_1, s''_1t$, or $s''_0s''_1, s'_1t$ when $i = 2$). These cut-chords correspond to crossing chords of C^* in G^*, which would contradict choosing C^* to have no crossing chords.

To show sufficiency, suppose G^* is a distance-hereditary graph and D is a ≥5-edge min-cutset of G. Let C^* be the ≥5-cycle of G^* whose edges correspond to the edges of D. By Proposition 1, C^* has crossing chords e_1 and e_2, say with e_1 inside the geometric curve corresponding to C^* in the embedding of G^* and e_2 outside that curve. Thus e_1 is the boundary of two faces inside that curve, while e_2 is the boundary of two faces outside of that curve. Hence, the cut-chords of D that e_1 and e_2 correspond to are separated by D. Therefore, every ≥5-edge min-cutset of G separates cut-chords, and so G is a DH* graph. ■

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