DUALIZING DISTANCE-HEREDITARY GRAPHS

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Abstract

Distance-hereditary graphs can be characterized by every cycle of length at least 5 having crossing chords. This makes distance-hereditary graphs susceptible to dualizing, using the common extension of geometric face/vertex planar graph duality to cycle/cutset duality as in abstract matroidal duality. The resulting “DH* graphs” are characterized and then analyzed in terms of connectivity. These results are used in a special case of plane-embedded graphs to justify viewing DH* graphs as the duals of distance-hereditary graphs.

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1. DISTANCE-HEREDITARY AND DH* GRAPHS

Unless otherwise noted, all graphs are simple (meaning no multiple edges or loops) and finite, with notation and terminology following [3]. A chord of a cycle $C$ is an edge $ab$ that has $a, b \in V(C)$ and $ab \notin E(C)$. Two chords $ab$ and $cd$ of $C$ are crossing chords of $C$ if their endpoints come in the order $a, c, b, d$ around $C$. The notation “≥k-cycle” abbreviates “cycle of length at least $k$.”

Distance-hereditary graphs $G$ were defined by Howorka in [4] by every connected induced subgraph $H$ of $G$ and every $x, y \in V(H)$ satisfying $\text{dist}_H(x, y) = \text{dist}_G(x, y)$; see [2, 3] for additional characterizations. At first glance, this graph class looks like a poor candidate for traditional graph duality, but another of Howorka’s original characterizations, in Proposition 1, suggests a simple way to dualize distance-hereditary graphs. The resulting concept will be introduced (below), characterized (in Section 2), and motivated (in Section 4) in this paper.
Proposition 1 [4]. A graph is a distance-hereditary graph if and only if every \( \geq 5 \)-cycle has crossing chords.

A minimal edge cutset \( D \) of a graph \( G \) is an inclusion-minimal \( D \subset E(G) \) such that deleting \( D \) would produce a subgraph \( G - D \) with \( V(G) = V(G - D) \) that consists of two components (maximal connected subgraphs); for convenience, we will simply call such sets \( D \) the min-cutsets of \( G \). Call a min-cutset of cardinality \( k \) a \( k \)-edge min-cutset and a min-cutset of cardinality at least \( k \) a \( \geq k \)-edge min-cutset.

As in [7], a cut-chord of a min-cutset \( D \) of \( G \) is an edge \( e \in E(G) \setminus D \) whose deletion would disconnect one of the components of \( G - D \). Say that a min-cutset \( D \) of \( G \) separates two cut-chords \( e_1 \) and \( e_2 \) of \( D \) (or that \( e_1 \) and \( e_2 \) are separated by \( D \)) if \( e_1 \) and \( e_2 \) are in different components of \( G - D \).

Define a DH* graph to be a graph in which every \( \geq 5 \)-edge min-cutset separates two cut-chords. The graph \( G_1 \) in Figure 1 is a DH* graph, since its only \( \geq 5 \)-edge min-cutsets are (up to isomorphism) \( \{1, 4, 7, 9, 11\} \), \( \{1, 4, 7, 10, 13\} \), \( \{1, 4, 7, 9, 12, 13\} \), and \( \{1, 4, 7, 10, 11, 12\} \) (each of which separates the cut-chords 6 and 8) along with \( \{1, 2, 5, 7, 8\} \), \( \{1, 2, 5, 7, 9, 11\} \), \( \{1, 2, 5, 7, 10, 13\} \), \( \{1, 2, 5, 7, 9, 12, 13\} \), and \( \{1, 2, 5, 7, 10, 11, 12\} \) (each of which separates the cut-chords 3 and 6). But the graph \( G_2 \) is not a DH* graph; for instance, its min-cutset \( D = \{1, 4, 7, 9, 11\} \) has only the two cut-chords 6 and 8, which are not separated by \( D \) in this graph.

![Figure 1. A DH* graph \( G_1 \) and a non-DH* graph \( G_2 \).](image-url)
Define a relevant graph to be a 2-connected, 3-edge-connected graph of order at least 3; thus, 3-connected graphs are always relevant graphs. This concept is motivated in [6, 7] by relevant, plane-embedded graphs always having well-defined, simple, relevant, plane-embedded dual graphs.

**Lemma 2.** Every vertex of a relevant DH* graph has degree 3 or 4.

**Proof.** Suppose \( v \) is a vertex of a relevant DH* graph \( G \), and let \( D = \{ vx \in E(G) : x \in N_G(v) \} \). Since relevant graphs are 3-edge-connected, \( \deg_G(v) \geq 3 \). Since relevant graphs are 2-connected, \( \{ v \} \) induces an edgeless component of \( G - D \) with \( G - v \) the other component of \( G - D \), and so \( D \) is a min-cutset of \( G \) that cannot separate cut-chords. Therefore, the DH* graph \( G \) has \( |D| = \deg_G(v) \leq 4 \).

For every induced subgraph \( H \) of a graph \( G \), let \( G/H \) denote the multigraph that results from contracting all the edges of \( H \) down to one new vertex that is denoted \( v_H \) (allowing parallel edges, but deleting any loops thereby formed). For example, if \( G \cong K_5 \) with \( V(G) = \{ v_1, v_2, v_3, v_4, v_5 \} \) and \( H \) is the induced subgraph with \( V(H) = \{ v_3, v_4, v_5 \} \), then \( G/H \) has vertex set \( \{ v_1, v_2, v_H \} \) with one (simple) edge \( v_1v_2 \), three parallel edges between \( v_1 \) and \( v_H \), and three parallel edges between \( v_2 \) and \( v_H \). Let \( \Delta(G) \) and \( \Delta(G/H) \) denote the maximum degree of vertices in, respectively, the simple graph \( G \) and the multigraph \( G/H \).

**Theorem 3.** A relevant graph \( G \) with \( \Delta(G) \leq 4 \) is a DH* graph if and only if \( \Delta(G/H) \leq 4 \) for all 2-edge-connected induced subgraphs \( H \) of \( G \) for which the multigraph \( G/H \) is 2-connected.

**Proof.** To show necessity, suppose \( G \) is a relevant DH* graph with \( \Delta(G) \leq 4 \). Suppose \( G \) has a 2-edge-connected induced subgraph \( H \) for which \( G/H \) is 2-connected, and let \( D = \{ xy \in E(G) : x \in V(H) \text{ and } y \notin V(H) \} \) and \( D_H = \{ yv_H : y \in N_G(H(v_H)) \} \). Since \( G/H \) is 2-connected and \( H \) is connected, \( G - H \) is connected, and \( D \) and \( D_H \) are min-cutsets of, respectively, \( G \) and \( G/H \). Since \( H \) is 2-edge-connected, \( D \) has no cut-chords in the component \( H \) of \( G - D \), and so \( D \) does not separate cut-chords. Thus \( G \) being a DH* graph requires that...
deg_{G/H}(v_H) = |D_{H}| = |D| \leq 4, while each w \in V(G) \setminus \{v_H\} has deg_{G/H}(w) = deg_G(w) \leq \Delta(G) \leq 4. Therefore, \Delta(G/H) \leq 4 in G/H.

To show sufficiency, suppose G is a relevant graph with \Delta(G) \leq 4, but G is not a DH* graph (arguing by contraposition). Thus some \geq 5-edge min-cutset D of G does not separate cut-chords, so some component H of G - D contains no cut-chord of D, and so H is 2-edge-connected. Thus G/H - v_H \cong G - H is connected (since D is a min-cutset of G), while G/H - w is connected for all w \in V(G/H) \setminus \{v_H\} = V(G) \setminus V(H) (since G is 2-connected). Therefore, G/H is 2-connected, but deg_{G/H}(v_H) = |D| \geq 5 implies \Delta(G/H) \neq 4.

3. The Role of 3-Connectivity for DH* Graphs

This section details the effect of 3-connectedness of a relevant graph on its being a DH* graph. Recall that relevant graphs are always 2-connected (and 3-edge-connected), and that 3-connected graphs are always relevant graphs.

Theorem 4. Figure 2 shows all the 3-connected DH* graphs.

![Five DH* graphs](image)

Figure 2. Five DH* graphs (the vertex colors are explained below).

Proof. We first show that the five 3-connected graphs in Figure 2 are indeed DH* graphs. Since G_1^2 has only six edges and each vertex has degree 3, there are no \geq 5-edge min-cutsets, and so G_1^2 is automatically a DH* graph. In the other four graphs G_i^2, the five edges with one black and one white endpoint form a 5-edge min-cutset D_i with each component of G - D_i containing at least one cut-edge of D_i (and these are the only 5-edge possibilities up to isomorphism). Graph G_3^2 also has 6-edge min-cutsets, one of which is obtained by changing the one black “square” vertex into a white vertex. No matter how such 6-edge min-cutsets D are chosen, the six edges in \text{E}(G_3^2) \setminus D will form two components of G - D that are trees, and with each containing a cut-edge of D. Therefore, all the five graphs in Figure 2 are DH* graphs.

Let G be an arbitrary 3-connected DH* graph, so G is a relevant graph and each u \in V(G) has deg_G(u) \in \{3, 4\} by Lemma 2.

First suppose that G has adjacent degree-4 vertices v and w. Since G is 3-connected, there is a minimum-length chordless cycle C that has v, w, x \in V(C)
where $x \not\in \{v, w\}$. But now $G$ has at least five edges with one endpoint in $C$ and the other not in $C$ (two incident to each of $v$ and $w$, and one incident to $x$), which would contradict Theorem 3 with $H = C$.

Now suppose that $\deg_G(v) = 4$ with $G \not\cong G_2^2$ in Figure 2 where each $w \in N_G(v)$ has $\deg_G(w) = 3$. Since $G \not\cong K_5$ and $G$ is a relevant DH* graph, $v$ has nonadjacent degree-3 neighbors $x$ and $y$ and a minimum-length chordless cycle $C$ with $vx, vy \in E(C)$ that has some $z \in V(C) \setminus \{v, x, y\}$. But now $G$ has at least five edges with one endpoint in $C$ and the other not in $C$ (two incident to $v$, and one incident to each of $x, y, z$), which would again contradict Theorem 3 with $H = C$.

Therefore, we can assume that $G$ is a cubic graph (meaning that every vertex has degree 3), and so $|V(G)| \geq 4$ is even. By [1], a graph is both 3-connected and cubic if and only if it can be constructed from $K_4$ by repeated applications of the following operation.

Given two (possibly adjacent) edges $a_1b_1$ and $a_2b_2$, subdivide each $a_ib_i$ with a new vertex $x_i$ and then insert a new edge $x_1x_2$, forming a new graph that has two more vertices and three more edges than the original graph. (The other two operations described in [1] allow one or both $x_i$ to be in $\{a_i, b_i\}$, which would prevent the new graph from being cubic.)

Note that applying this construction to adjacent edges of $G_4^1$ in Figure 2 produces the graph $G_4^2$, while applying it to nonadjacent edges of $G_4^1$ produces $G_2^3$. Similarly, applying the construction from [1] to $G_2^2$ would never produce a cubic graph.

The two graphs on the left in Figure 3 show the only graphs (up to isomorphism) that result from applying the construction from [1] to $G_4^2$; specifically, they result from letting $a_1b_1$ and $a_2b_2$ be, respectively, adjacent and nonadjacent edges of $G_4^2$. In each such graph $G$, the five edges that have both black and white vertices form a min-cutset $D$ for which the black vertices induce a 2-connected (5-cycle) component of $G - D$, and so a component that contains no cut-chord of $D$. Therefore, such graphs $G$ are not DH* graphs.

![Figure 3. The non-DH* graphs constructed as in [1] from $G_4^2$ and $G_2^2$.](image)

Note that applying the construction from [1] to two edges of $G_4^2$ that are in different triangles produces $G_4^3$. The four graphs on the right in Figure 3 show the remaining graphs (up to isomorphism) that result from applying the construction...
from [1] to $G_4^2$: specifically, they result from letting $a_1b_1$ and $a_2b_2$ be (from left to right) edges that are not in a triangle, edges $a_1b_2$ not in a triangle and $a_2b_2$ in a triangle with $a_1 = a_2$, edges $a_1b_2$ not in a triangle and $a_2b_2$ in a triangle with $a_1 \neq a_2$, and both edges in the same triangle. Just as in the $G_3^2$ discussion above, the five edges that have both black and white endpoints form a min-cutset that shows that graphs that are constructed this way are not DH* graphs.

The four graphs in Figure 4 show all the graphs $G$ (up to isomorphism) that can result from applying the construction from [1] to $G_4^2$: specifically, they result from letting edges $a_1b_1$ and $a_2b_2$ be adjacent (the leftmost graph) or nonadjacent. Just as in the $G_3^2$ and $G_4^2$ discussions above, the five edges that have both black and white endpoints form a min-cutset that shows that graphs that are constructed this way are not DH* graphs.

![Figure 4. The non-DH* graphs constructed as in [1] from $G_4^2$.](image-url)

The preceding four paragraphs show that applying the construction from [1] to the graphs in Figure 3 cannot produce additional 3-connected, cubic DH* graphs. Also, observe that $G_4^2$ is the only 3-connected DH* graph that has order 4.

Finally, to show that no additional 3-connected, cubic DH* graphs can exist, suppose that $G''$ is a 3-connected, cubic DH* graph that is constructed as in [1] from a 3-connected, cubic graph $G'$ by replacing the edges $a_1b_1, a_2b_2 \in E(G')$ with $a_1x_1, b_1x_1, a_2x_2, b_2x_2, x_1x_2 \in E(G'')$ (toward showing that $G''$ was also a 3-connected DH* graph).

Suppose $D'$ is an arbitrary $\geq 5$-edge min-cutset of $G'$ (toward finding a new $\geq 5$-edge min-cutset $D''$ of the DH* graph $G''$ that has separated cut-chords that correspond to separated cut-chords of $D'$ in $G'$). We can assume that $D' \cap \{a_1b_1, a_2b_2\} \neq \emptyset$ (otherwise $D'' = D'$ will have the same separated cut-chords in $G''$ as $D''$ has in $G'$).

**Case 1.** When $a_1b_1 \in D'$ and $a_3-i b_3-i \notin D'$. One of the four edges $a_1x_1, b_1x_1 \in E(G'')$ can replace $a_1b_1$ to form the new min-cutset $D''$ of $G''$. The separated cut-chords of $D''$ in $G''$ will correspond to the separated cut-chords of $D'$ in $G'$ so long as, when $e \in \{a_3-i x_3-i, b_3-i x_3-i\}$ is a cut-chord of $D''$, the cut-chord $a_3-i b_3-i$ of $D'$ corresponds to $e$.

**Case 2.** When both $a_1b_1, a_2b_2 \in D'$. Use the edges $a_1x_1, x_1x_2, b_2x_2 \in E(G'')$ to replace the pair $a_1b_1, a_2b_2$ to form the new min-cutset $D''$ of $G''$. Edges $b_1x_1$
and $a_2x_2$ will be separated cut-chords of $D''$ in $G''$ since $x_1$ and $x_2$ will be degree-1 vertices of $G'' - D''$.

Therefore, there are no 3-connected, cubic, DH* graphs beyond the four cubic graphs in Figure 2, and so there are no 3-connected DH* graphs beyond the five graphs $G^2_7$ shown there.

In Theorem 5, \{s_1, s_2\} $\subset V(G)$ is an order-2 minimal separator of a connected graph $G$ if, for some $x, y \in V(G)$ from different components of $G - \{s_1, s_2\}$, each $s_i$ is in an $x$-to-$y$ path of $G - s_{3-i}$. Relevant graphs that are not 3-connected necessarily have an order-2 minimal separator.

**Theorem 5.** If a relevant DH* graph $G$ is not 3-connected, then $G$ has a minimal separator $\{s, t\}$ for which $G - \{s, t\}$ has a component whose vertex set combines with $\{s, t\}$ to induce one of the subgraphs shown in Figure 5.

![Figure 5](image-url)

**Figure 5.** The two subgraphs of $G$ mentioned in Theorem 5, where each $\deg_G(s_i') = \deg_G(s_i'') = 3$.

**Proof.** Suppose $G$ is a relevant DH* graph that is not 3-connected, $\{s, t\}$ is a minimal separator of $G$, and $H'$ is a component of $G - \{s, t\}$ such that $V(H') \cup \{s, t\}$ induces a 2-connected subgraph $H$ of $G$. Further assume that such $s$, $t$, and $H'$ are chosen so that $|V(H')|$ is as small as possible. Thus each $v \in V(H')$ has $N_G(v) \subseteq V(H)$, and each vertex of $G$ has degree 3 or 4 by Lemma 2.

Since $G$ is 3-edge connected, $s$ and $t$ cannot be endpoints of two edges of $G$ that form a min-cutset of $G$. Thus, not both $\deg_G(s) = \deg_G(t) = 3$, and so we can assume that $\deg_G(s) = 4$ with $N_G(s) = \{p, q, s_0', s_0''\}$ where $p, q \notin V(H')$ and $s_0', s_0'' \in V(H')$. The assumed minimality of $|V(H')|$ ensures that $t$ also has two neighbors in $V(H')$, and so $st \notin E(G)$ (otherwise, taking $p = t$ and $t' \notin N_G(t) \setminus V(H)$ would make $\{qs, tt'\}$ a 2-edge min-cutset of $G$).

If $s_0'$ is not adjacent to $s_0''$, then $s$, $s_0'$, and $s_0''$ are vertices of a chordless 4-cycle $C_1$ of $H$ for which $G$ has at least five edges between vertices in $C_1$ and vertices not in $C_1$ (the edges $ps$ and $qs$ and one edge incident with each vertex in $V(C_1) \setminus \{s\}$); but then $\Delta(G/C_1) \geq 5$ (contradicting Theorem 3). Therefore, $s_0's_0'' \in E(H)$.

If $\deg_H(s_0') = 4$ or $\deg_H(s_0'') = 4$, then $\{s, s_0', s_0''\}$ induces a triangle $C_1'$ that has at least five edges between vertices in $C_1'$ and vertices not in $C_1'$ (the
edges ps and qs, two edges incident with $s'_0$ or $s''_0$ and one edge incident with the other), and so for which $\Delta(G/C^1_i) \geq 5$ (contradicting Theorem 3). Therefore, $\deg_H(s'_0) = \deg_H(s''_0) = 3$, say with $s'_0s'_1, s''_0s''_1 \in E(H) \setminus E(C^1_i)$.

Repeat the argument used in the preceding two paragraphs to introduce vertices $s'_1, s''_1, \ldots, s'_t, s''_t$ successively, where each $\{s, s'_0, s''_0, \ldots, s'_t, s''_t\}$ induces a $(2i+3)$-cycle $H_i$ of $H$ with exactly $i \geq 1$ chords $s'_0s''_0, \ldots, s'_{i-1}s''_{i-1}$, stopping when finally $s'_i = s''_i$. Thus $\{s, s'_t\}$ is a minimal separator of $G$, and so $s'_t = s''_t = t$ by the assumed minimality of $|V(H')|$. If $i = 1$, then the $2$-connected subgraph $H \cong G^3_1$, and if $i = 2$, then $H \cong G^3_2$. If $i \geq 3$, then $V(G) \setminus \{s'_0, \ldots, s'_i\}$ would induce a $2$-edge-connected subgraph $H'_i$ of $G$ with at least five edges between vertices in $H'_i$ and vertices not in $H'_i$ (namely, $ss'_0, s'_it$, and $s'_js''_j$ for $1 \leq j \leq i$), and so for which $\Delta(G/H'_i) \geq 5$ (contradicting Theorem 3).

Therefore, $i = 1$ or $i = 2$, and $H$ is $G^3_1$ or $G^3_2$ as in Figure 5.

Noting the intrinsic role of Theorem 3 in the preceding proof, it is worth mentioning that Theorem 5 can in turn be used to simplify the application of Theorem 3 to graphs that are not 3-connected as follows: The choice of the $2$-edge-connected induced subgraphs $H$ in Theorem 3 can be limited to avoid $H$ that contain degree-3 vertices such as $s'_t$ and $s''_t$ in Figure 5.

4. The Planar Motivation for DH* Graphs

A plane embedding of a relevant planar graph $G$ is transformed into its geometric dual graph $G^*$ as described in this paragraph (with a detailed example in the next paragraph). Vertices of $G$, along with their incident edges, become the faces of $G^*$ that are bordered by the corresponding edges, while the faces of $G$ similarly become the vertices of $G^*$. Thus, vertices and faces are regarded as duals of each other. Since edges of $G$ thereby correspond to edges of $G^*$, edges are regarded as self-dual, with each edge of either simultaneously joining two adjacent vertices and separating two adjacent faces. The plane embedding of $G$ thus produces a plane embedding of $G^*$, with $G$ also becoming the dual graph of $G^*$ based on that embedding—in other words, with $G = (G^*)^*$.

Figure 6 illustrates this process of dualizing relevant plane-embedded graphs, showing the dual graph $(G^1_1)^*$ for the embedding of the graph $G^1_1$ in Figure 1. The vertices of $(G^1_1)^*$ are labeled with the edge sets that form the boundaries of the seven faces of $G^1_1$ (including the “exterior” hexagonal face), with the edges of $(G^1_1)^*$ labeled to match the corresponding edges of $G^1_1$. The vertices of $G^1_1$ (viewed as sets of incident edges) similarly correspond to (the edge sets of) the eight faces of $(G^1_1)^*$.

Each cycle $C$ of $G$, as a set of edges, becomes a min-cutset $D^*$ of $G^*$ (with
the faces “inside” the geometric curve corresponding to \( C \) in the embedding of \( G \) becoming vertices of one of the components of \( G^* - D^* \), and the faces “outside” \( C \) becoming vertices of the other component); similarly, each min-cutset \( D \) of \( G \) becomes a cycle \( C^* \) of \( G^* \). Thus, cycles and min-cutsets are regarded as duals of each other. This concrete geometric duality generalizes to abstract matroid duality, interchanging cycles with min-cutsets (both viewed as sets of edges). Many elementary graph theory textbooks describe both geometric duality and matroidal duality, perhaps none more accessibly than Wilson’s elementary text [10].

For instance, the cycle with edge set \( \{1, 3, 6, 7\} \) of \( G_1^1 \) in Figure 1 (which happens not to be a face) corresponds to the min-cutset \( \{1, 3, 6, 7\} \) in the dual graph \( (G_1^1)^* \) in Figure 6 (which is not a set of edges incident to a vertex).

The chords of a cycle \( C \) can be characterized as the edges \( e \not\in E(C) \) for which \( E(C) \) can be partitioned into the edge sets of two paths \( P_1 \) and \( P_2 \) such that both \( E(P_i) \cup \{e\} \) are edge sets of cycles. (Notice that cycles can have crossing chords in a plane-embedding, with one of the chords inside of the geometric curve corresponding to \( C \) and the other outside of that curve.) Similarly, the cut-chords of a min-cutset \( D \) can be characterized as the edges \( e \not\in D \) for which \( D \) can be partitioned into two subsets \( D_1 \) and \( D_2 \) such that both \( D_i \cup \{e\} \) are min-cutsets. Thus chords of cycles are regarded as the duals of cut-chords of min-cutsets. For instance, the min-cutset \( \{1, 4, 7, 8\} \) of \( G_1^1 \) in Figure 1 has cut-chord 6, corresponding to a cycle \( (G_1^1)^* \) in Figure 6 that has chord 6.

Two graphs are \textit{cycle-isomorphic} if there is a bijection between their edge sets for which the cycles of each graph maps to the cycles of the other. As in [11], define a graph \( G \) to be \textit{cycle-determined} if \( G \cong G' \) for all graphs \( G' \) that are cycle-isomorphic to \( G \). It is important that every 3-connected graph is cycle-determined; see [8, 9].

The two plane-embedded, relevant graphs \( G_1^1 \) and \( G_2^1 \) in Figure 1 are cycle-isomorphic, but \( G_1^1 \not\cong G_2^1 \) shows that they are not cycle-determined. Also note that, no matter how \( G_1^1 \) and \( G_2^1 \) are embedded in the plane, \( (G_1^1)^* \) and \( (G_2^1)^* \) will
not be distance-hereditary graphs—for instance, using Proposition 1, the edge set \( \{1,4,7,9,11\} \) of each \((G_i^1)^*\) will correspond to a cycle with chords 6 and 8, but without crossing chords. This shows the need to require cycle-determined graphs in Theorem 6 (which largely motivates the “DH* graph” terminology).

**Theorem 6.** A cycle-determined, plane-embedded, relevant graph is a DH* graph if and only if its geometric dual is a distance-hereditary graph.

**Proof.** Suppose \( G \) is a cycle-determined, plane-embedded, relevant graph with geometric dual \( G^* \) (which also makes \( G^* \) a cycle-determined, plane-embedded, relevant graph).

To show necessity, suppose \( G \) is a DH* graph for which \( G^* \) is not a distance-hereditary graph (arguing by contradiction); further assume that, among all such graphs, \( G \) is chosen to minimize \( |E(G)| \). By Theorem 4, \( G \) is not 3-connected, since \( G_3^1 \) and \( G_3^2 \) are both self-dual, \((G_3^2)^* \equiv K_{1,1,1,2} \) (\( K_5 \) with one edge deleted), and \((G_3^2)^* \equiv K_{2,2,2} \) (the octahedron graph), where each planar graph \((G_3^2)^* \) is a distance-hereditary graph by definition (all induced paths between nonadjacent vertices in each have the same length, namely 2).

Hence we can assume that the relevant (and so 2-connected) DH* graph \( G \) is not 3-connected. By [8], the cycle-determined graph \( G \) has a **generalized circuit representation**, defined as \( k \geq 2 \) subgraphs \( G_1, \ldots, G_k \) of \( G \) that satisfy the following three conditions.

- Each \( G_i \) is 2-connected with \( E(G_i) \neq \emptyset \), and with \( |V(G_i)| \geq 3 \) when \( k = 2 \).
- Sets \( E(G_1), \ldots, E(G_k) \) partition \( E(G) \), and each \( V(G_i) \cap \bigcup_{j \neq i} V(G_j) \) consists of two distinguished vertices of \( G_i \).
- Replacing each subgraph \( G_i \) by an edge between its distinguished vertices produces a cycle.

By Lemma 2, Theorem 5, and \( G \) being 3-edge-connected, some \( G_i \) has \( G_i \cong G_3^1 \) or \( G_i \cong G_3^2 \) as in Figure 5, say with its vertices labeled as shown there, with \( s \) and \( t \) as its distinguished vertices, and with exactly two edges \( ps, qs \notin E(G) \setminus E(G_i) \) that have \( s \) as an endpoint.

Figure 7 shows the induced subgraph of \( G^* \) that corresponds to \( G_i \cong G_3^1 \) or \( G_i \cong G_3^2 \) augmented with the edges \( ps,qs \notin E(G_i) \), where the vertices of \( G^* \) are now labeled with the vertex sets of the faces in \( G \). (The top and bottom vertices in Figure 7 correspond to the inside and outside faces of the plane-embedded generalized circuit representation of \( G \) from [8].) For the simpler \( G_i \cong G_3^1 \) case, simplify Figure 7 by deleting the vertex \( \{s_0', s_0'', s_1', s_1''\} \) and inserting one new edge between vertices \( \{s,s_0', s_0''\} \) and \( \{s_1', s_1'', t\} \), and then replacing all occurrences of \( s_1 \) by \( s_0' \) and \( s_0'' \), respectively and labeling the newly inserted edge as \( s_0's_0'' \).
Since \( G^* \) is not a distance-hereditary graph, \( G^* \) has a \( ≥5 \)-cycle \( C^* \) with no crossing chords by Proposition 1; further assume that, among all such cycles, \( C^* \) is chosen to minimize \( |E(C^*) \cap E(G_i^*)| \). Let \( D \) be the \( ≥5 \)-edge min-cutset of \( G \) whose edges correspond to the edges of \( C^* \). Let \( S_i^* \subset E(G^*) \) be the set of all edges of \( C^* \) that correspond to edges of \( G_i^* \), so \( S_i^* \) forms a subpath of \( C^* \) (and \( ps, qs \not\in S_i^* \)). Let \( S_i \) be the set of corresponding edges of \( G_i \), so \( S_i \subset D \) is a min-cutset of \( G_i \) that has \( s \) and \( t \) in different components of \( G - D \).

If \( S_i^* = \emptyset \), then \( C^* \) is also a \( ≥5 \)-cycle without crossing chords of \((G/G_i)^*\), which would contradict the assumed minimality of \( |E(G)| = |E(G^*)| \) in the original choice of \( G \). Thus \( S_i^* \neq \emptyset \).

If \( |S_i^*| \in \{1,2\} \), the only choices for the subpath \( S_i^* \) of \( G_i^* \) are \( \{ss_0, ss_0''\} \) or \( \{s_0t, s_0''t\} \) if \( i = 1 \), and \( \{ss_0'', ss_0''\}, \{s_0''s_1', s_0''s_1''\} \), or \( \{s_1't, s_1''t\} \) if \( i = 2 \). For each of these choices, the min-cutset \( S_i \) of \( G_i \) does not separate cut-chords in \( G_i \). Replacing \( S_i \) with \( \{ps, qs\} \) in \( D \) creates another min-cutset of \( G \) that separates the same cut-chords in \( G \) as \( D \), which would contradict the assumed minimality of \( |E(C^*) \cap E(G_i^*)| = |S_i^*| = |S_i| \). Thus \( |S_i^*| \notin \{1,2\} \).

Therefore, \( |S_i^*| \geq 3 \), and so the path \( S_i^* \) of \( G_i^* \) (and so the min-cutset \( D \) of \( G \)) contains one or both of the edges \( s_0's_0'' \) and \( s_1's_1'' \). In each case, \( D \) separates cut-chords of \( G \) (namely, \( ss_0, ss_0''t \) or \( ss_0''s_0''t \) when \( i = 1 \), and \( ss_0'', ss_0's_1s_1'' \) or \( ss_0's_1's_1'' \) or \( s_0's_1', s_1't \), or \( s_0's_1'', s_1''t \) when \( i = 2 \)). These cut-chords correspond to crossing chords of \( C^* \) in \( G^* \), which would contradict choosing \( C^* \) to have no crossing chords.

To show sufficiency, suppose \( G^* \) is a distance-hereditary graph and \( D \) is a \( ≥5 \)-edge min-cutset of \( G \). Let \( C^* \) be the \( ≥5 \)-cycle of \( G^* \) whose edges correspond to the edges of \( D \). By Proposition 1, \( C^* \) has crossing chords \( e_1 \) and \( e_2 \), say with \( e_1 \) inside the geometric curve corresponding to \( C^* \) in the embedding of \( G^* \) and \( e_2 \) outside that curve. Thus \( e_1 \) is the boundary of two faces inside that curve, while \( e_2 \) is the boundary of two faces outside of that curve. Hence, the cut-chords of \( D \) that \( e_1 \) and \( e_2 \) correspond to are separated by \( D \). Therefore, every \( ≥5 \)-edge min-cutset of \( G \) separates cut-chords, and so \( G \) is a DH* graph.

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