CHANGING AND UNCHANGING OF THE DOMINATION NUMBER OF A GRAPH: PATH ADDITION NUMBERS

Vladimir Samodivkin
Department of Mathematics
University of Architecture, Civil Engineering and Geodesy
Sofia 1164, Bulgaria

e-mail: vl.samodivkin@gmail.com

Abstract

Given a graph $G = (V, E)$ and two its distinct vertices $u$ and $v$, the $(u, v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k-2}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_{k-1}$. We prove that $\gamma(G) - 1 \leq \gamma(G_{u,v,k})$ for all $k \geq 1$, and $\gamma(G_{u,v,k}) > \gamma(G)$ when $k \geq 5$. We also provide necessary and sufficient conditions for the equality $\gamma(G_{u,v,k}) = \gamma(G)$ to be valid for each pair $u, v \in V(G)$. In addition, we establish sharp upper and lower bounds for the minimum, respectively maximum, $k$ in a graph $G$ over all pairs of vertices $u$ and $v$ in $G$ such that the $(u, v)$-$P_k$-addition graph of $G$ has a larger domination number than $G$, which we consider separately for adjacent and non-adjacent pairs of vertices.

Keywords: domination number, path addition.

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1. Introduction

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [8]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The complement $\overline{G}$ of $G$ is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. We write $K_n$ for the complete graph of order $n$, $K_{m,n}$ for the complete bipartite graph with partite sets of order $m$ and $n$, and $P_n$ for the path on $n$ vertices. Let $C_m$ denote the cycle of length $m$. For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the
degree of $x$ is $\text{deg}(x, G) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_G[A] = N_G(A) \cup A$. A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. Let $G$ be a graph and $uv$ be an edge of $G$. By subdividing the edge $uv$ we mean forming a graph $H$ from $G$ by adding a new vertex $w$ and replacing the edge $uv$ by $uw$ and $wv$. Formally, $V(H) = V(G) \cup \{ w \}$ and $E(H) = (E(G) \setminus \{ uv \}) \cup \{ uw, wv \}$. For a graph $G$, let $x \in S \subseteq V(G)$. A vertex $y \in V(G)$ is a $S$-private neighbor of $x$ if $N_G[y] \cap S = \{ x \}$. The set of all $S$-private neighbors of $x$ is denoted by $pn_G[x, S]$.

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a comprehensive introduction to the theory of domination in graphs we refer the reader to Haynes et al. [8]. A dominating set for a graph $G$ is a subset $D \subseteq V(G)$ of vertices such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. The concept of $\gamma$-bad/good vertices in graphs was introduced by Fricke et al. in [5]. A vertex $v$ of a graph $G$ is called

(i) [5] $\gamma$-good, if $v$ belongs to some $\gamma$-set of $G$, and

(ii) [5] $\gamma$-bad, if $v$ belongs to no $\gamma$-set of $G$.

A graph $G$ is said to be $\gamma$-excellent whenever all its vertices are $\gamma$-good [5]. Brigham et al. [3] defined a vertex $v$ of a graph $G$ to be $\gamma$-critical if $\gamma(G - v) < \gamma(G)$, and $G$ to be vertex domination-critical (from now on called vc-graph) if each vertex of $G$ is $\gamma$-critical. For a graph $G$ we define $V^- (G) = \{ x \in V(G) \mid \gamma(G - x) < \gamma(G) \}$.

It is often of interest to known how the value of a graph parameter $\mu$ is affected when a change is made in a graph, for instance vertex or edge removal, edge addition, edge subdivision and edge contraction. In this connection, here we consider this question in the case $\mu = \gamma$ when a path is added to a graph.

Path-addition is an operation that takes a graph and adds an internally vertex-disjoint path between two vertices together with a set of supplementary edges. This operation can be considered as a natural generalization of the edge addition. Formally, let $u$ and $v$ be distinct vertices of a graph $G$. The $(u, v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k-2}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_k$. When $k \geq 3$ we call $x_1, x_2, \ldots, x_{k-2}$ path-addition vertices. By $pa_\gamma(u, v)$ we denote the minimum number $k$ such that $\gamma(G) < \gamma(G_{u,v,k})$. For every graph $G$ with at least 2 vertices we define

$\triangleright$ the $e$-path addition ($\mathcal{E}$-path addition) number with respect to domination, de-
noted $epa_\gamma(G)$ ($\overline{epa}_\gamma(G)$, respectively), to be

\begin{itemize}
  \item $epa_\gamma(G) = \min \{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}$,
  \item $\overline{epa}_\gamma(G) = \min \{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}$, and
\end{itemize}

the upper e-path addition (upper $\overline{\gamma}$-path addition) number with respect to domination, denoted $E_{pa}_\gamma(G)$ ($E_{\overline{pa}}_\gamma(G)$, respectively), to be

\begin{itemize}
  \item $E_{pa}_\gamma(G) = \max \{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}$,
  \item $E_{\overline{pa}}_\gamma(G) = \max \{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}$.
\end{itemize}

If $G$ is complete, then we write $E_{\overline{pa}}_\gamma(G) = \overline{epa}_\gamma(G) = \infty$, and if $G$ is edgeless then $epa_\gamma(G) = E_{pa}\gamma(G) = \infty$. In what follows the subscript $\gamma$ will be omitted from the notation.

The remainder of this paper is organized as follows. In Section 2, we prove that $1 \leq epa(G) \leq 3$ and $2 \leq E_{pa}(G) \leq 5$, and we present necessary and sufficient conditions for $pa(u, v) = i$, $i = 1, 2, 3$, where $uv \in E(G)$. In Section 3, we show that $1 \leq \overline{epa}(G) \leq \overline{E_{pa}}(G) \leq 5$, and we give necessary and sufficient conditions for $\overline{epa}(G) = \overline{E_{pa}}(G) = j$, $1 \leq j \leq 5$. We conclude in Section 4 with open problems.

We end this section with some known results which will be useful in proving our main results.

**Lemma 1** [2]. If $G$ is a graph and $H$ is any graph obtained from $G$ by subdividing some edges of $G$, then $\gamma(H) \geq \gamma(G)$.

**Lemma 2.** Let $G$ be a graph and $v \in V(G)$.

(i) [5] If $v$ is $\gamma$-bad, then $\gamma(G - v) = \gamma(G)$.

(ii) [3] $v$ is $\gamma$-critical if and only if $\gamma(G - v) = \gamma(G) - 1$.

(iii) [5] If $v$ is $\gamma$-critical, then all its neighbors are $\gamma$-bad vertices of $G - v$.

(iv) [11] If $e \in E(G)$, then $\gamma(G) - 1 \leq \gamma(G + e) \leq \gamma(G)$.

In most cases, Lemma 2 will be used in the sequel without specific reference.

2. **The Adjacent Case**

The aim of this section is to prove that $1 \leq pa(u, v) \leq 3$ and to find necessary and sufficient conditions for $pa(u, v) = i$, $i = 1, 2, 3$, where $uv \in E(G)$.

**Observation 3.** If $u$ and $v$ are adjacent vertices of a graph $G$, then $\gamma(G) = \gamma(G_{u,v,0}) \leq \gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 1$.

**Proof.** The equality $\gamma(G) = \gamma(G_{u,v,0})$ is obvious. For any $\gamma$-set $M$ of $G_{u,v,1}$ both $M_u = (M \setminus \{x_1\}) \cup \{u\}$ and $M_v = (M \setminus \{x_1\}) \cup \{v\}$ are dominating sets of $G$, and at
least one of them is a $\gamma$-set of $G_{u,v,1}$. Hence $\gamma(G) \leq \min\{|M_u|, |M_v|\} = \gamma(G_{u,v,1})$. The rest follows by Lemma 1.

**Theorem 4.** Let $u$ and $v$ be adjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1$ and the following is true.

(i) $\gamma(G) = \gamma(G_{u,v,1})$ if and only if at least one of $u$ and $v$ is a $\gamma$-good vertex of $G$.

(ii) $\gamma(G_{u,v,1}) = \gamma(G) + 1$ if and only if both $u$ and $v$ are $\gamma$-bad vertices of $G$.

**Proof.** The left side inequality follows by Observation 3. If $D$ is a $\gamma$-set of $G$, then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$, which implies $\gamma(G_{u,v,1}) \leq \gamma(G) + 1$.

If at least one of $u$ and $v$ belongs to some $\gamma$-set $D_1$ of $G$, then $D_1$ is a dominating set of $G_{u,v,1}$. This clearly implies $\gamma(G) = \gamma(G_{u,v,1})$.

Let now both $u$ and $v$ are $\gamma$-bad vertices of $G$, and suppose that $\gamma(G_{u,v,1}) = \gamma(G)$. In this case for any $\gamma$-set $M$ of $G_{u,v,1}$ is fulfilled $u, v \notin M$ and $x_1 \in M$. But then $(M \setminus \{x_1\}) \cup \{u\}$ is a $\gamma$-set for both $G$ and $G_{u,v,1}$, a contradiction.

**Corollary 5.** Let $G$ be a graph with edges. Then $\text{Epa}(G) \geq 2$ and $\text{epa}(G) = 1$ if and only if the set of all $\gamma$-bad vertices of $G$ is neither empty nor independent.

**Theorem 6.** Let $u$ and $v$ be adjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Moreover,

(A) $\gamma(G_{u,v,2}) = \gamma(G) + 1$ if and only if at least one of the following holds:

(i) both $u$ and $v$ are $\gamma$-bad vertices of $G$,

(ii) at least one of $u$ and $v$ is $\gamma$-good, $u, v \notin V^-(G)$ and each $\gamma$-set of $G$ contains at most one of $u$ and $v$.

(B) $\gamma(G_{u,v,2}) = \gamma(G)$ if and only if at least one of the following is true:

(iii) there exists a $\gamma$-set of $G$ which contains both $u$ and $v$,

(iv) at least one of $u$ and $v$ is in $V^-(G)$.

**Proof.** The left side inequality follows by Observation 3. If $D$ is an arbitrary $\gamma$-set of $G$, then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) \leq \gamma(G) + 1$.

$(A) \Rightarrow$ Assume that the equality $\gamma(G_{u,v,2}) = \gamma(G) + 1$ holds. By Theorem 4 we know that $\gamma(G_{u,v,1}) \in \{\gamma(G), \gamma(G) + 1\}$. If $\gamma(G_{u,v,1}) = \gamma(G) + 1$, then again by Theorem 4, both $u$ and $v$ are $\gamma$-bad vertices of $G$. So let $\gamma(G) = \gamma(G_{u,v,1})$. Then at least one of $u$ and $v$ is a $\gamma$-good vertex of $G$ (Theorem 4). Clearly there is no $\gamma$-set of $G$ which contains both $u$ and $v$. If $u \in V^-(G)$ and $U$ is a $\gamma$-set of $G - u$, then $U \cup \{x_1\}$ is a dominating set of $G_{u,v,2}$ and $|U \cup \{x_1\}| = \gamma(G)$, a contradiction. Thus $u, v \notin V^-(G)$.

$(A) \Leftarrow$ If both $u$ and $v$ are $\gamma$-bad vertices of $G$, then $\gamma(G_{u,v,1}) = \gamma(G) + 1$ (Theorem 4). But we know that $\gamma(G_{u,v,1}) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$; hence
\( \gamma(G_{u,v}, 2) = \gamma(G) + 1 \). Finally let (ii) hold and \( M \) be a \( \gamma \)-set of \( G_{u,v}, 2 \). If \( x_1, x_2 \notin M \), then \( u, v \in M \) which leads to \( \gamma(G_{u,v}, 2) > \gamma(G) \). If \( x_1, x_2 \in M \), then \((M \setminus \{x_1, x_2\}) \cup \{u, v\}\) is a dominating set of \( G \) of cardinality more than \( \gamma(G) \). Now let without loss of generality \( x_1 \in M \) and \( x_2 \notin M \). If \( M \setminus \{x_1\} \) is a dominating set of \( G \), then \( \gamma(G) + 1 \leq |M| = \gamma(G_{u,v}, 2) \leq \gamma(G) + 1 \). So, let \( M \setminus \{x_1\} \) be no dominating set of \( G \). Hence \( M \setminus \{x_1\} \) is a dominating set of \( G - u \).

Since \( u \notin V^{-}(G) \), \( \gamma(G) \leq \gamma(G - u) \leq |M \setminus \{x_1\}| < \gamma(G_{u,v}, 2) \).

\((\exists) \Rightarrow \) Let \( \gamma(G_{u,v}, 2) = \gamma(G) \). Suppose that neither (iii) nor (iv) is valid. Hence \( u, v \notin V^{-}(G) \) and no \( \gamma \)-set of \( G \) contains both \( u \) and \( v \). But then at least one of (i) and (ii) holds, and from (A) we conclude that \( \gamma(G_{u,v}, 2) = \gamma(G) + 1 \), a contradiction.

\((\exists) \Leftarrow \) Let at least one of (iii) and (iv) be hold. Then neither (i) nor (ii) is fulfilled. Now by (A) we have \( \gamma(G_{u,v}, 2) \neq \gamma(G) + 1 \). Since \( \gamma(G) \leq \gamma(G_{u,v}, 2) \leq \gamma(G) + 1 \), we obtain \( \gamma(G) = \gamma(G_{u,v}, 2) \).

The independent domination number of a graph \( G \), denoted by \( i(G) \), is the minimum size of an independent dominating set of \( G \). It is obvious that \( i(G) \geq \gamma(G) \). In a graph \( G \), \( i(G) \) is strongly equal to \( \gamma(G) \), written \( i(G) \equiv \gamma(G) \), if each \( \gamma \)-set of \( G \) is independent. It remains an open problem to characterize the graphs \( G \) with \( i(G) \equiv \gamma(G) \) [7].

**Corollary 7.** Let \( G \) be a graph with edges. Then (a) \( epa(G) \geq 2 \) if and only if the set of all \( \gamma \)-bad vertices is either empty or independent, and (b) \( Epa(G) = 2 \) if and only if \( i(G) \equiv \gamma(G) \).

**Proof.** (a) Immediately by Corollary 5.

(b) \( \Rightarrow \) Let \( Epa(G) = 2 \). If \( D \) is a \( \gamma \)-set of \( G \) and \( u, v \in D \) are adjacent, then \( D \) is a dominating set of \( G_{u,v}, 2 \), a contradiction.

(b) \( \Leftarrow \) Let all \( \gamma \)-sets of \( G \) be independent. Suppose \( u \in V^{-}(G) \) and \( D \) is a \( \gamma \)-set of \( G - u \). Then \( D_1 = D \cup \{v\} \) is a \( \gamma \)-set of \( G \), where \( v \) is any neighbor of \( u \). But \( D_1 \) is not independent. Hence \( V^{-}(G) \) is empty. Thus, for any 2 adjacent vertices \( u \) and \( v \) of \( G \) is fulfilled either (A)(i) or (A)(ii) of Theorem 6. Therefore \( Epa(G) \leq 2 \). The result now follows by Corollary 5.

Denote by \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) the additive group of order \( n \). Let \( S \) be a subset of \( \mathbb{Z}_n \) such that 0 \( \notin \mathbb{Z} \) and \( x \in S \) implies \( -x \in S \). The circulant graph with distance set \( S \) is the graph \( C(n; S) \) with vertex set \( \mathbb{Z}_n \) and vertex \( x \) adjacent to vertex \( y \) if and only if \( x - y \in S \).

Let \( n \geq 3 \) and \( k \in \mathbb{Z}_n \setminus \{0\} \). The generalized Petersen graph \( P(n, k) \) is the graph on the vertex-set \( \{x_i, y_i \mid i \in \mathbb{Z}_n\} \) with adjacencies \( x_i x_{i+1}, x_i y_i, \) and \( y_i y_{i+k} \) for all \( i \).

**Example 8.** A special case of graphs \( G \) with \( Epa(T) = 2 \) are graphs for which each \( \gamma \)-set is efficient dominating (an efficient dominating set in a graph \( G \) is a
set $S$ such that $\{N[s] | s \in S\}$ is a partition of $V(G)$. We list several examples of such graphs [10].

(a) A crown graph $H_{n,n}$, $n \geq 3$, which is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.
(b) Circulant graphs $G = C(n = (2k+1)t; \{1, \ldots, k\} \cup \{n-1, \ldots, n-k\})$, where $k, t \geq 1$.
(c) Circulant graphs $G = C(n; \{\pm 1, \pm s\})$, where $2 \leq s \leq n-2$, $s \neq n/2$, $5 | n$ and $s \equiv \pm 2 \pmod{5}$.
(d) The generalized Petersen graph $P(n, k)$, where $n \equiv 0 \pmod{4}$ and $k$ is odd.

**Theorem 9.** If $u$ and $v$ are adjacent vertices of a graph $G$, then $\gamma(G_{u,v,3}) = \gamma(G) + 1$.

**Proof.** If $D$ is a $\gamma$-set of $G$, then $D \cup \{x_2\}$ is a dominating set of $G$. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$.

Let $M$ be a $\gamma$-set of $G_{u,v,3}$. Then at least one of $x_1, x_2$ and $x_3$ is in $M$. If $x_2 \in M$, then clearly $\gamma(G_{u,v,3}) = \gamma(G) + 1$. If $x_2 \notin M$ and $x_1, x_3 \in M$, then $(M \setminus \{x_1, x_3\}) \cup \{u\}$ is a dominating set of $G$. If $x_2, x_3 \notin M$ and $x_1 \in M$, then $v \in M$ and $M \setminus \{x_1\}$ is a dominating set of $G$. All this leads to $\gamma(G_{u,v,3}) = \gamma(G) + 1$.

**Corollary 10.** Let $G$ be a graph with edges. Then $\text{epa}(G) \leq \text{Epa}(G) \leq 3$. Moreover, $\text{Epa}(G) = 3$ if and only if $G$ has a $\gamma$-set that is not independent, and $\text{epa}(G) = 3$ if and only if for each pair of adjacent vertices $u$ and $v$ at least one of the following is valid.

(i) There exists a $\gamma$-set of $G$ which contains both $u$ and $v$.
(ii) At least one of $u$ and $v$ is in $V^-(G)$.

**Proof.** By Corollary 5 and Theorem 9 we have $1 \leq \text{epa}(G) \leq \text{Epa}(G) \leq 3$ and $2 \leq \text{Epa}(G)$. Since $\text{Epa}(G) = 2$ if and only if $i(G) \equiv \gamma(G)$ (by Corollary 7), $\text{Epa}(G) = 3$ if and only if $G$ has a $\gamma$-set that is not independent.

Clearly $\text{epa}(G) = 3$ if and only if $\gamma(G_{u,v,2}) = \gamma(G)$ for each pair of adjacent vertices $u$ and $v$ of $G$. Then because of Theorem 6(B), we have that $\text{epa}(G) = 3$ if and only if for each pair of adjacent vertices $u$ and $v$ of $G$ at least one of (i) and (ii) holds.

**Corollary 11.** Let $G$ be a graph with edges. If $V^-(G)$ has a subset which is a vertex cover of $G$, then $\text{epa}(G) = 3$. In particular, if $G$ is a $	ext{vc}$-graph then $\text{epa}(G) = 3$.

We define the following classes of graphs $G$ with $\Delta(G) \geq 1$.

- $A = \{G | \text{epa}(G) = 3\}$,
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- \( \mathcal{A}_1 = \{ G \mid V^-(G) \text{ is a vertex cover of } G \} \),
- \( \mathcal{A}_2 = \{ G \mid \text{each two adjacent vertices belongs to some } \gamma\text{-set of } G \} \),
- \( \mathcal{A}_3 = \{ G \mid G \text{ is a vc-graph} \} \).

Clearly, \( \mathcal{A}_3 \subseteq \mathcal{A}_1 \) and by Corolaries 10 and 11, \( \mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A} \). These relationships are illustrated in the Venn diagram in Figure 1(left). To continue we relabel this diagram in six regions \( \mathbf{R}_0 - \mathbf{R}_5 \) as shown in Figure 1(right). In what follows in this section we show that none of \( \mathbf{R}_0 - \mathbf{R}_5 \) is empty. The corona of a graph \( H \) is the graph \( G = H \circ K_1 \) obtained from \( H \) by adding a degree-one neighbor to every vertex of \( H \). If \( F \) and \( H \) are disjoint graphs, \( v_F \in V(F) \) and \( v_H \in V(H) \), then the coalescence \( (F \cdot H)(v_F,v_H : v) \) of \( F \) and \( H \) via \( v_F \) and \( v_H \), is the graph obtained from the union of \( F \) and \( H \) by identifying \( v_F \) and \( v_H \) in a vertex labeled \( v \).

\[ \text{Figure 1. Left: Classes of graphs with } epa = 3. \text{ Right: Regions of Venn diagram.} \]

Remark 12. It is easy to see that all the following hold.

(i) If \( H \) is a connected graph of order \( n \geq 2 \), then \( G = H \circ K_1 \in \mathbf{R}_0 \).

(ii) Let \( G^1_k \) be a graph obtained from the cycle \( C_{3k+1} : x_0,x_1,x_2,\ldots,x_{3k},x_0 \), \( k \geq 2 \), by adding a vertex \( y \) and edges \( yx_0,yx_2 \). Then \( \gamma(G^1_k) = k + 1 \), \( G^1_k \) is \( \gamma \)-excellent, \( V^-(G^1_k) = \{x_0,x_2\} \cup \bigcup_{r=1}^{k-1}\{x_{3r+1},x_{3r+2}\} \) is a vertex cover of \( G \), and there is no \( \gamma \)-set of \( G^1_k \) that contains both \( x_{3r+1} \) and \( x_{3r+2} \). Thus \( G^1_k \) is in \( \mathbf{R}_1 \).

(iii) The graph \( H_{10} \) depicted in Figure 2 is in \( \mathcal{A}_3 \) and \( \gamma(H_{10}) = 3 \) [1]. It is obvious that no \( \gamma \)-set of \( H_{10} \) contains both \( u \) and \( v \). Hence \( H_{10} \in \mathbf{R}_2 \). Consider now the graph \( G^2_k = (C_{3k+1} \cdot H_{10})(x_0,w : z) \), where \( C_{3k+1} : x_0,x_1,x_2,\ldots,x_{3k},x_0 \), \( k \geq 2 \), is a cycle on \( 3k+1 \) vertices and \( w \) is any of the two common neighbors of \( u \) and \( v \) in \( H_{10} \). Since both \( C_{3k+1} \) and \( H_{10} \) are vc-graphs, by [4] we have that \( G^2_k \) is vc-graph and \( \gamma(G^2_k) = \gamma(C_{3k+1}) + \gamma(H_{10}) - 1 \). Let \( D \) be an arbitrary \( \gamma \)-set of \( G^2_k \), \( D_1 = D \cap V(H_{10}) \) and \( D_2 = D \cap V(C_{3k+1}) \). Then exactly one of the following holds.

\( a) \ z \in D, \ D_1 \text{ is a } \gamma \text{-set of } H_{10} \text{ and } D_2 \text{ is a } \gamma \text{-set of } C_{3k+1} \).

\( b) \ z \notin D, \ D_1 \text{ is a } \gamma \text{-set of } H_{10} \text{ and } D_2 \cup \{x_0\} \text{ is a } \gamma \text{-set of } C_{3k+1} \).
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(c) \( z \notin D \), \( D_1 \cup \{w\} \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \) is a \( \gamma \)-set of \( C_{3k+1} \).

Since no \( \gamma \)-set of \( H_{10} \) contains both \( u \) and \( v \), by (a), (b) and (c) we conclude that at most one of \( u \) and \( v \) is in \( D \). Thus \( G_2^2 \in R_2 \).

(iv) \( C_{3k+1} \in R_3 \) for all \( k \geq 1 \).

(v) \( K_{2,n} \in R_4 \) for all \( n \geq 3 \).

(vi) \( K_{n,n} \in R_5 \) for all \( n \geq 3 \).
Thus all regions \( R_0, R_1, R_2, R_3, R_4, R_5 \) are nonempty.

Figure 2. Graph \( H_{10} \) is in \( R_2 \).

3. The Nonadjacent Case

In this section we show that \( 1 \leq \tau_{pa}(G) \leq \tau_{ba}(G) \leq 5 \) and we obtain necessary and sufficient conditions for \( \tau_{pa}(G) = \tau_{ba}(G) = j, 1 \leq j \leq 5 \).

We begin with an easy observation which is an immediate consequence by Lemma 2(iv) and Lemma 1.

Observation 13. Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). Then \( \gamma(G) - 1 \leq \gamma(G, u, v, 0) \leq \gamma(G) \) and \( \gamma(G, u, v, k) \leq \gamma(G, u, v, k+1) \) for \( k \geq 0 \).

Theorem 14. Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). Then \( \gamma(G) - 1 \leq \gamma(G, u, v, 1) \leq \gamma(G) + 1 \). Moreover,

(i) \( \gamma(G) - 1 = \gamma(G, u, v, 1) \) if and only if \( \gamma(G - \{u, v\}) = \gamma(G) - 2 \).

(ii) \( \gamma(G, u, v, 1) = \gamma(G) + 1 \) if and only if both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), \( u \notin V^-(G - v) \) and \( v \notin V^-(G - u) \). If \( \gamma(G, u, v, 1) = \gamma(G) + 1 \), then \( x_1 \in V^-(G, u, v, 1) \).

Proof. The left side inequality follows by Observation 13.

(i) \( \Rightarrow \) Assume the equality \( \gamma(G) - 1 = \gamma(G, u, v, 1) \) holds and let \( M \) be any \( \gamma \)-set of \( G_{u,v,1} \). Then at least one and not more than two of \( x_1, u \) and \( v \) must be in \( M \). Hence \( M_1 = (M \setminus \{x_1\}) \cup \{u, v\} \) is a dominating set of \( G \) and \( \gamma(G) \leq |M_1| \leq |M| + 1 = \gamma(G_{u,v,1}) + 1 = \gamma(G) \). This immediately implies that \( M_1 \) is a \( \gamma \)-set of \( G \). Hence \( x_1 \in M \) and \( pm[x_1, M] = \{x_1, u, v\} \). Since \( M_1 \setminus \{u, v\} \) is a dominating set of \( G - \{u, v\} \), we have \( \gamma(G) - 2 \leq \gamma(G - \{u, v\}) \leq |M_1 \setminus \{u, v\}| = \gamma(G) - 2 \).
(i) Suppose now $\gamma(G - \{u, v\}) = \gamma(G) - 2$. Then for any $\gamma$-set $U$ of $G - \{u, v\}$, the set $U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$. This leads to $\gamma(G_{u,v,1}) \leq |U \cup \{x_1\}| = \gamma(G) - 1 \leq \gamma(G_{u,v,1})$.

Now we will prove the right side inequality. Let $D$ be any $\gamma$-set of $G$. If at least one of $u$ and $v$ is in $D$, then $D$ is a dominating set $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G)$. So, let neither $u$ nor $v$ belong to some $\gamma$-set of $G$. Then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G) + 1$.

(ii) Assume that $\gamma(G_{u,v,1}) = \gamma(G) + 1$. Then $u$ and $v$ are $\gamma$-bad vertices of $G$ and for any $\gamma$-set $D$ of $G$, $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$. Hence $x_1 \in V^-(G_{u,v,1})$. Suppose $u \in V^-(G - v)$ and let $U$ be a $\gamma$-set of $G - \{u, v\}$. Then $U_1 = U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G) + 1 = \gamma(G_{u,v,1}) \leq |U_1| = 1 + \gamma((G - v) - u) = \gamma(G - v) = \gamma(G)$, a contradiction. Thus $u \notin V^-(G - v)$ and by symmetry, $v \notin V^-(G - u)$.

(ii) Let both $u$ and $v$ be $\gamma$-bad vertices of $G$, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$. Hence $\gamma(G - \{u, v\}) \geq \gamma(G)$. Consider any $\gamma$-set $M$ of $G_{u,v,1}$. If one of $u$ and $v$ belongs to $M$, then $\gamma(G) + 1 = \gamma(G_{u,v,1})$. So, let $x_1$ is in each $\gamma$-set of $G_{u,v,1}$. But then $\gamma(G_{u,v,1}) - 1 = \gamma(G - \{u, v\}) \geq \gamma(G) \geq \gamma(G_{u,v,1}) - 1$.

**Corollary 15.** Let $G$ be a noncomplete graph. Then $1 \leq \overline{\gamma}pa(G) \leq \overline{E}pa(G)$ and the following assertions hold.

(i) $\overline{\gamma}pa(G) = 1$ if and only if there are nonadjacent $\gamma$-bad vertices $u$ and $v$ of $G$ such that $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$.

(ii) $\overline{E}pa(G) = 1$ if and only if $\gamma(G) = 1$.

**Proof.** Observation 13 implies $1 \leq \overline{\gamma}pa(G)$.

(i) Immediately by Theorem 14.

(ii) If $\gamma(G) = 1$, then clearly $\overline{E}pa(G) = 1$. If $\gamma(G) \geq 2$, then $G$ has 2 nonadjacent vertices at least one of which is $\gamma$-good. By Theorem 14, $\overline{E}pa(G) \geq 2$.

**Theorem 16.** Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Moreover,

(C) $\gamma(G_{u,v,2}) = \gamma(G)$ if and only if one of the following holds.

(i) There is a $\gamma$-set of $G$ which contains both $u$ and $v$.

(ii) At least one of $u$ and $v$ is in $V^-(G)$.

(D) $\gamma(G_{u,v,2}) = \gamma(G) + 1$ if and only if $u, v \notin V^-(G)$ and any $\gamma$-set of $G$ contains at most one of $u$ and $v$.

**Proof.** For any $\gamma$-set $D$ of $G$, $D \cup \{x_2\}$ is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Suppose $\gamma(G_{u,v,2}) \leq \gamma(G) - 1$ and let $M$ be a $\gamma$-set of $G_{u,v,2}$. Then at least one of $x_1$ and $x_2$ is in $M$. If $x_1, x_2 \in M$, then $M_1 = (M \setminus \{x_1, x_2\}) \cup$
\{u, v\} is a dominating set of G and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. So let without loss of generality, \(x_1 \in M\) and \(x_2 \notin M\). If \(u \in M\) or \(v \in M\), then again \(M_1\) is a dominating set of G and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. Thus \(x_1 \in M\) and \(u, v \notin M\). But then \((M \setminus \{x_1\}) \cup \{u\}\) is a dominating set of G, contradicting \(\gamma(G_{u,v,2}) < \gamma(G)\). Thus \(\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1\).

(\(C\)) \Rightarrow Let \(\gamma(G_{u,v,2}) = \gamma(G)\). Assume that neither (i) nor (ii) hold. Let M be a \(\gamma\)-set of \(G_{u,v,2}\). If \(x_1, x_2 \in M\), then \(M_1 = (M \setminus \{x_1, x_2\}) \cup \{u, v\}\) is a dominating set of G of cardinality not more than \(\gamma(G)\) and \(u, v \in M_1\), a contradiction. Let without loss of generality \(x_1 \in M\) and \(x_2 \notin M\). Since \(M \setminus \{x_1\}\) is no dominating set of G, \(u \in pn[x_1, M]\). But then \(M_3 = (M \setminus \{x_1\}) \cup \{u\}\) is a \(\gamma\)-set of G and \(u \in V^-(G)\), a contradiction. Thus at least one of (i) and (ii) is valid.

(\(C\)) \Leftarrow If both \(u\) and \(v\) belong to some \(\gamma\)-set \(D\) of G, then \(D\) is a dominating set of \(G_{u,v,2}\). Hence \(\gamma(G_{u,v,2}) = \gamma(G)\). Finally let \(u \in V^-(G)\) and \(D\) a \(\gamma\)-set of \(G - u\). Then \(D \cup \{x_1\}\) is a dominating set of \(G_{u,v,2}\) of cardinality \(\gamma(G)\). Thus \(\gamma(G_{u,v,2}) = \gamma(G)\).

(\(D\)) Immediately by (\(C\)) and \(\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1\).

**Corollary 17.** Let G be a noncomplete graph. Then the following assertions hold.

(i) \(\overline{p}(G) \leq 2\) if and only if there are nonadjacent vertices \(u, v \in V(G)\setminus V^-(G)\) such that any \(\gamma\)-set of G contains at most one of them.

(ii) \(\overline{E}(G) = 2\) if and only if \(\gamma(G) \geq 2\) and each \(\gamma\)-set of G is a clique.

**Proof.** (i) Immediately by Theorem 16.

(ii) \(\Rightarrow\) Let \(\overline{E}(G) = 2\). By Corollary 15, \(\gamma(G) \geq 2\). Suppose G has a \(\gamma\)-set, say D, which is not a clique. Then there are nonadjacent \(u, v \in D\). By Theorem 16(\(C\)), \(\gamma(G_{u,v,2}) = \gamma(G)\), which contradict \(\overline{E}(G) = 2\). Thus, each \(\gamma\)-set of G is a clique.

(ii) \(\Leftarrow\) Let \(\gamma(G) \geq 2\) and let each \(\gamma\)-set of G be a clique. If G has a vertex \(z \in V^-(G)\) and \(M_z\) is a \(\gamma\)-set of \(G - z\), then \(M = M_z \cup \{z\}\) is a \(\gamma\)-set of G and \(z\) is an isolated vertex of the graph induced by \(M\), a contradiction. Thus \(V^-(G)\) is empty. Now by Theorem 16(\(D\)), \(\overline{E}(G) = 2\).

**Example 18.** The join of two graphs \(G_1\) and \(G_2\) with disjoint vertex sets is the graph, denoted by \(G_1 + G_2\), with the vertex set \(V(G_1) \cup V(G_2)\) and edge set \(E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}\). Let \(\gamma(G_i) \geq 3\), \(i = 1, 2\). Then \(\gamma(G_1 + G_2) = 2\) and each \(\gamma\)-set of \(G_1 + G_2\) contains exactly one vertex of \(G_i\), \(i = 1, 2\). Hence \(\overline{E}(G_1 + G_2) = 2\). In particular, \(\overline{E}(K_{m,n}) = 2\) when \(m, n \geq 3\).

**Theorem 19.** Let \(u\) and \(v\) be nonadjacent vertices of a graph G. Then \(\gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G) + 1\). Moreover, \(\gamma(G_{u,v,3}) = \gamma(G)\) if and only if at least one of the following holds.
(i) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,
(ii) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

**Proof.** If $D$ is a dominating set of $G$, then $D \cup \{x_2\}$ is a dominating set of $G_{u,v,3}$. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$. We already know that $\gamma(G) \leq \gamma(G_{u,v,2})$ and $\gamma(G_{u,v,2}) \leq \gamma(G_{u,v,3})$. But then $\gamma(G) \leq \gamma(G_{u,v,3})$.

$\Rightarrow$ Let $\gamma(G_{u,v,3}) = \gamma(G)$ and let $M$ be a $\gamma$-set of $G_{u,v,3}$ such that $Q = M \cap \{x_1, x_2, x_3\}$ has minimum cardinality. Clearly $|Q| = 1$. If $\{x_2\} = Q$, then $M \setminus \{x_2\}$ is a dominating set of $G$, contradicting $\gamma(G_{u,v,3}) = \gamma(G)$. Let without loss of generality $\{x_1\} = Q$. This implies $v \in M$, $x_3 \in pn[v, M]$ and $pn[x_1, M] = \{u, x_1, x_2\}$. Then $M_2 = (M \setminus \{x_1\}) \cup \{u\}$ is a $\gamma$-set of $G$, $pn[u, M_2] = \{u\}$ and $v \in M_2$; hence (i) holds.

$\Leftarrow$ Let without loss of generality (i) is true. Then there is a $\gamma$-set $D$ of $G$ such that $u, v \in D$ and $D \setminus \{u\}$ is a $\gamma$-set of $G - u$. But then $(D \setminus \{u\}) \cup \{x_1\}$ is a dominating set of $G_{u,v,3}$, which implies $\gamma(G) \geq \gamma(G_{u,v,3})$.

**Corollary 20.** Let $G$ be a noncomplete graph. Then the following holds.

(E) $\overline{\text{epa}}(G) \leq 3$ if and only if there is a pair of nonadjacent vertices $u$ and $v$ such that neither (i) nor (ii) is valid, where

(i) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,
(ii) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

(F) $\overline{\text{epa}}(G) = \text{Epa}(G) = 3$ if and only if all vertices of $G$ are $\gamma$-good, $V^-(G)$ is empty and for every 2 nonadjacent vertices $u$ and $v$ of $G$ there is a $\gamma$-set of $G$ which contains them both.

**Proof.** (F)$\Rightarrow$ Let $\overline{\text{epa}}(G) = \text{Epa}(G) = 3$. If $u \in V^-(G)$ and $D$ is a $\gamma$-set of $G - u$, then for $u$ and each $v \in D$ is fulfilled (i) of Theorem 19. But then $\text{Epa}(G) \neq 3$, a contradiction. So, $V^-(G)$ is empty. Suppose that $G$ has $\gamma$-bad vertices. Then there is a $\gamma$-bad vertex which is nonadjacent to some other vertex of $G$. But Theorem 16(D) implies $\overline{\text{epa}}(G) < 3$, a contradiction. Thus all vertices of $G$ are $\gamma$-good. Now let $u, v \in V(G)$ be nonadjacent. If there is no $\gamma$-set of $G$ which contains both $u$ and $v$, then by Theorem 16(D) we have $\gamma(G_{u,v,2}) = \gamma(G) + 1$, a contradiction.

(F)$\Leftarrow$ Let $V^-(G)$ be empty and for each pair $u, v$ of nonadjacent vertices of $G$ there is a $\gamma$-set $D_{uv}$ of $G$ with $u, v \in D_{uv}$. By Theorem 19, $\gamma(G_{u,v,3}) = \gamma(G) + 1$, and by Theorem 16, $\gamma(G_{u,v,2}) = \gamma(G)$. Hence $\text{pa}(u, v) = 3$.

**Example 21.** Denote by $\mathcal{U}$ the class of all graphs $G$ with $\overline{\text{epa}}(G) = \text{Epa}(G) = 3$. Then all the following holds. (a) Circulant graphs $C(2k + 1; \{\pm 1, \pm 2, \ldots, \pm (k - 1)\}) \in \mathcal{U}$ for all $k \geq 1$. (b) Let $G$ be a nonconnected graph. Then $G \in \mathcal{U}$ if and only if $G$ has no isolated vertices and each its component is either in $\mathcal{U}$ or is complete.
Theorem 22. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u, v, 4}) \leq \gamma(G) + 2$. Moreover, the following assertions are valid.

(1) $\gamma(G_{u,v,4}) = \gamma(G) + 2$ if and only if $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

(2) If $\gamma(G_{u,v,1}) = \gamma(G)$ and $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{2, 3\}$, then $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(3) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$ and the equality holds if and only if $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$.

(4) $\gamma(G_{u,v,4}) = \gamma(G)$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Proof. Since $\gamma(G) \leq \gamma(G_{u,v,3})$ (by Theorem 19) and $\gamma(G_{u,v,3}) \leq \gamma(G_{u,v,4})$ (by Observation 13), we have $\gamma(G) \leq \gamma(G_{u,v,4})$. Let $S$ be a $\gamma$-set of $G$. Then $S \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$, which leads to $\gamma(G_{u,v,4}) \leq \gamma(G) + 2$.

Claim 1. If $\gamma(G_{u,v,1}) \leq \gamma(G)$, then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Proof. Assume that $v$ is a $\gamma$-bad vertex of $G$, $u \in V^-(G - v)$ and $R$ a $\gamma$-set of $G - \{u, v\}$. Then $|R| = \gamma(G - v) - u = \gamma(G) - 1 = \gamma(G) - 1$ and $R \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) \leq |R| + 2 = \gamma(G) + 1$.

Assume now that $D$ is a $\gamma$-set of $G$ with $u \in D$. Then $D \cup \{x_3\}$ is a dominating set of $G_{u,v,4}$. Hence again $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. Now by Theorem 14 we immediately obtain the required.

(1) Let $\gamma(G_{u,v,4}) = \gamma(G) + 2$. By Claim 1, $\gamma(G_{u,v,1}) > \gamma(G)$ and by Theorem 14, $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

Let now $\gamma(G_{u,v,1}) = \gamma(G) + 1$. By Theorem 14, $u$ and $v$ are $\gamma$-bad vertices of $G$, $u \not\in V^-(G - v)$ and $v \not\in V^-(G - u)$. Let $M$ be a $\gamma$-set of $G_{u,v,4}$ such that $R = M \cap \{x_1, x_2, x_3, x_4\}$ has minimum cardinality. Clearly $|R| \in \{1, 2\}$. Assume first $|R| = 2$ and without loss of generality $\{x_2\} = M$. Then $M \setminus \{x_2\}$ is a dominating set of $G$ with $v \in M \setminus \{x_2\}$. Since $v$ is a $\gamma$-bad vertex of $G$, $|M \setminus \{x_2\}| > \gamma(G)$ and then $\gamma(G_{u,v,4}) = |M| > \gamma(G) + 1$. Let now $|R| = 2$ and without loss of generality $x_1, x_4 \in M$. Since $|M \cap \{x_1, x_2, x_3, x_4\}|$ is minimum, $u, v \not\in M$ and $M \setminus \{x_1, x_4\}$ is a dominating set of $G - \{u, v\}$. But then $\gamma(G_{u,v,4}) = 2 + |M \setminus \{x_1, x_4\}| \geq 2 + \gamma(G - u) \geq 2 + \gamma(G - u) = 2 + \gamma(G)$.

(2) Let $\gamma(G_{u,v,1}) = \gamma(G)$. By Claim 1, $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. If $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{1, 2\}$, then since $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,i})$, we obtain $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(3) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Hence at least one of (i) and (ii) of Theorem 19 holds, and by (3), $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Assume that the equality holds. If $\gamma(G - \{u, v\}) = \gamma(G) - 2$, then for any $\gamma$-set $U$ of $G - \{u, v\}$, $U \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) = \gamma(G)$, a contradiction.
Let now $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and without loss of generality condition (i) of Theorem 19 be satisfied. Suppose $\gamma(G_{u,v,4}) = \gamma(G)$. Hence for each $\gamma$-set $M$ of $G_{u,v,4}$ are fulfilled: $x_1, x_4 \in M$, $x_2, x_3, u, v \notin M$, $pn[x_1, M] = \{x_1, x_2, u\}$ and $pn[x_4, M] = \{x_3, x_4, v\}$. But then $\gamma(G - \{u, v\}) = \gamma(G) - 2$, a contradiction. Thus $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

\[ (\text{iii}) \] If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,3}) = \gamma(G)$ and by (ii), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Now let $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Thus $\gamma(G_{u,v,4}) = \gamma(G)$. \hfill \blacksquare

**Theorem 23.** Let $u$ and $v$ be nonadjacent vertices of a graph $G$. If $\gamma(G_{u,v,k}) = \gamma(G)$, then $k \leq 4$. If $k \geq 5$, then $\gamma(G_{u,v,k}) > \gamma(G)$. If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,5}) = \gamma(G) + 1$.

**Proof.** By Theorem 22, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. If $\gamma(G_{u,v,4}) > \gamma(G)$, then $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$ because of Observation 13. So, let $\gamma(G_{u,v,4}) = \gamma(G)$. By Theorem 22(\textit{i}), $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_3, x_5\}$ is a dominating set of $G_{u,v,5}$. Hence $\gamma(G_{u,v,5}) \leq \gamma(G) + 1$. Let now $M$ be a $\gamma$-set of $G_{u,v,5}$. Then at least one of $x_2, x_3, x_4$ is in $M$ and hence $\gamma(G_{u,v,5}) = |M| \geq \gamma(G) + 1$. Thus $\gamma(G_{u,v,5}) = \gamma(G) + 1$. Now using again Observation 13 we conclude that $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$. \hfill \blacksquare

**Corollary 24.** Let $G$ be a noncomplete graph. Then $\overline{epa}(G) \leq \overline{Epa}(G) \leq 5$. Moreover, the following holds.

(i) $\overline{Epa}(G) = 5$ if and only if there are nonadjacent vertices $u$ and $v$ of $G$ with $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(ii) $\overline{epa}(G) = 5$ if and only if $G$ is edgeless.

(iii) $\overline{epa}(G) = \overline{Epa}(G) = 4$ if and only if for each pair $u, v$ of nonadjacent vertices of $G$, $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and at least one of the following holds:

(a) $u \in V^{-}(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,

(b) $v \in V^{-}(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

**Proof.** By Theorem 23, $\overline{Epa}(G) \leq 5$.

(i) $\Rightarrow$ Let $\overline{Epa}(G) = 5$. Then there is a pair $u, v$ of nonadjacent vertices of $G$ such that $\gamma(G_{u,v,4}) = \gamma(G)$. Now by Theorem 22(\textit{iii}), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(ii) $\Leftarrow$ Let $\gamma(G - \{u, v\}) = \gamma(G) - 2$ and $D$ be a $\gamma$-set of $G - \{u, v\}$, where $u$ and $v$ are nonadjacent vertices of $G$. Hence $D_1 = D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$ and $|D_1| = \gamma(G)$. This implies $\gamma(G_{u,v,4}) = \gamma(G)$. The result now follows by Theorem 23.

(ii) If $G$ has no edges, then the result is obvious. So let $G$ have edges and $\overline{epa}(G) = 5$. Then for any 2 nonadjacent vertices $u$ and $v$ of $G$ is satisfied
γ(G − {u, v}) = γ(G) − 2 (by (i)). Hence we can choose u and v so that they have a neighbor in common, say w. But then w is a γ-bad vertex of G − u which implies v /∈ V^−(G − u). This leads to γ(G − {u, v}) ≥ γ(G) − 1, a contradiction.

(iii) ⇒ Let τ_p(G) = E_p(G) = 4. Then for each two nonadjacent u, v ∈ V(G) we have γ(G) = γ(G_u,v,3) < γ(G_u,v,4). Now by Theorem 22(G), γ(G − {u, v}) ≥ γ(G) − 1 and by Theorem 19, at least one of (a) and (b) is valid.

(iii) ⇐ Consider any two nonadjacent vertices u, v of G. Then γ(G − {u, v}) ≥ γ(G) − 1 and at least one of (a) and (b) is valid. Theorem 19 now implies γ(G) = γ(G_u,v,3), and by Theorem 22, pa(u, v) = 4.

Example 25. Let G_n be the Cartesian product of two copies of K_n, n ≥ 2. We consider G_n as an n × n array of vertices \{x_{i,j} | 1 ≤ i ≤ j ≤ n\}, where the closed neighborhood of x_{i,j} is the union of the sets \{x_{1,j}, x_{2,j}, \ldots , x_{n,j}\} and \{x_{i,1}, x_{i,2}, \ldots , x_{i,n}\}. Note that V(G_n) = V^−(G_n) and γ(G_n) = n [6]. It is easy to see that the following sets are γ-sets of G_n − x_{1,1}: D_i = \{x_{2,i}, x_{3,i+1}, \ldots , x_{n,n+i−2}\}, i = 2, 3, \ldots , n, where x_{k,j} = x_{k,j−n+1} for j > n and 2 ≤ k ≤ n. Since D = \bigcup_{i=2}^{n} D_i = V(G_n)\setminus N[x_{1,1}], all γ-bad vertices of G_n − x_{1,1} are the neighbors of x_{1,1} in G_n. Since each vertex of D is adjacent to some neighbor of x_{1,1}, V^−(G_n − x_{1,1}) is empty. Now by Theorem 19 we have pa(x_{1,1}, y) ≥ 4, and by Theorem 22(II), pa(x_{1,1}, y) < 5. Thus pa(x_{1,1}, y) = 4. By reason of symmetry, we obtain τ_p(G_n) = E_p(G_n) = 4.

4. Observations and Open Problems

A constructive characterization of the trees T with i(T) ≡ γ(T), and therefore a constructive characterization of the trees T with E_p(T) = 2 (by Corollary 7), was provided in [9].

Problem 26. Characterize all unicyclic graphs G with E_p(G) = 2.

Problem 27. Find results on γ-excellent graphs G with E_p(G) = 2.

Problem 28. Characterize all graphs G with τ_p(G) = E_p(G) = 4.

Corollary 29. Let G be a connected noncomplete graph with edges. Then

(i) 2 ≤ e_p(G) + E_p(G) ≤ 8,
(ii) 2 ≤ e_p(G) + τ_p(G) ≤ 7,
(iii) \(3 \leq E_{pa}(G) + \overline{E_{pa}(G)} \leq 8\),
(iv) \(3 \leq E_{pa}(G) + \overline{\tau_{pa}(G)} \leq 7\).

**Proof.** (i)–(iv) The left-side inequalities immediately follow by Corollary 5 and Corollary 15. The right-side inequalities hold because of Corollary 10 and Corollary 24.

Note that all bounds stated in Corollary 29 are attainable. We leave finding examples demonstrating this to the reader.

**Problem 30.** Characterize all graphs \(G\) that attain the bounds in Corollary 29.

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**References**


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