Abstract

Given a graph $G = (V, E)$ and two its distinct vertices $u$ and $v$, the $(u, v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_{k-1}$. We prove that $\gamma(G) - 1 \leq \gamma(G_{u,v,k})$ for all $k \geq 1$, and $\gamma(G_{u,v,k}) > \gamma(G)$ when $k \geq 5$. We also provide necessary and sufficient conditions for the equality $\gamma(G_{u,v,k}) = \gamma(G)$ to be valid for each pair $u, v \in V(G)$. In addition, we establish sharp upper and lower bounds for the minimum, respectively maximum, $k$ in a graph $G$ over all pairs of vertices $u$ and $v$ in $G$ such that the $(u, v)$-$P_k$-addition graph of $G$ has a larger domination number than $G$, which we consider separately for adjacent and non-adjacent pairs of vertices.

Keywords: domination number, path addition.

2010 Mathematics Subject Classification: 05C69.
degree of $x$ is $\deg(x, G) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_G[A] = N_G(A) \cup A$. A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. Let $G$ be a graph and $uv$ be an edge of $G$. By subdividing the edge $uv$ we mean forming a graph $H$ from $G$ by adding a new vertex $w$ and replacing the edge $uv$ by $uw$ and $wv$. Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) \setminus \{uv\}) \cup \{uw, wv\}$. For a graph $G$, let $x \in S \subseteq V(G)$. A vertex $y \in V(G)$ is a $S$-private neighbor of $x$ if $N_G[y] \cap S = \{x\}$. The set of all $S$-private neighbors of $x$ is denoted by $pn_G[x, S]$.

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a comprehensive introduction to the theory of domination in graphs we refer the reader to Haynes et al. [8]. A dominating set for a graph $G$ is a subset $D \subseteq V(G)$ of vertices such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. The concept of $\gamma$-bad/good vertices in graphs was introduced by Fricke et al. in [5]. A vertex $v$ of a graph $G$ is called

(i) [5] $\gamma$-good, if $v$ belongs to some $\gamma$-set of $G$, and
(ii) [5] $\gamma$-bad, if $v$ belongs to no $\gamma$-set of $G$.

A graph $G$ is said to be $\gamma$-excellent whenever all its vertices are $\gamma$-good [5]. Brigham et al. [3] defined a vertex $v$ of a graph $G$ to be $\gamma$-critical if $\gamma(G - v) < \gamma(G)$, and $G$ to be vertex domination-critical (from now on called vc-graph) if each vertex of $G$ is $\gamma$-critical. For a graph $G$ we define $V^−(G) = \{x \in V(G) \mid \gamma(G - x) < \gamma(G)\}$.

It is often of interest to known how the value of a graph parameter $\mu$ is affected when a change is made in a graph, for instance vertex or edge removal, edge addition, edge subdivision and edge contraction. In this connection, here we consider this question in the case $\mu = \gamma$ when a path is added to a graph.

Path-addition is an operation that takes a graph and adds an internally vertex-disjoint path between two vertices together with a set of supplementary edges. This operation can be considered as a natural generalization of the edge addition. Formally, let $u$ and $v$ be distinct vertices of a graph $G$. The $(u, v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k-2}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_{k-1}$. When $k \geq 3$ we call $x_1, x_2, \ldots, x_{k-2}$ path-addition vertices. By $pa_\mu(u, v)$ we denote the minimum number $k$ such that $\gamma(G) < \gamma(G_{u,v,k})$. For every graph $G$ with at least 2 vertices we define

$\triangleright$ the $e$-path addition (or $\tau$-path addition) number with respect to domination, de-
Path Addition Numbers

noted \(epa_\gamma(G)\) (\(\overline{epa}_\gamma(G)\)), respectively, to be

- \(epa_\gamma(G) = \min\{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}\),
- \(\overline{epa}_\gamma(G) = \min\{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}\), and

the upper \(e\)-path addition (upper \(\overline{e}\)-path addition) number with respect to domination, denoted \(Epa_\gamma(G)\) (\(\overline{Epa}_\gamma(G)\)), respectively, to be

- \(Epa_\gamma(G) = \max\{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}\),
- \(\overline{Epa}_\gamma(G) = \max\{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}\).

If \(G\) is complete, then we write \(\overline{Epa}_\gamma(G) = \overline{epa}_\gamma(G) = \infty\), and if \(G\) is edgeless then \(epa_\gamma(G) = Epa_\gamma(G) = \infty\). In what follows the subscript \(\gamma\) will be omitted from the notation.

The remainder of this paper is organized as follows. In Section 2, we prove that \(1 \leq epa(G) \leq 3\) and \(2 \leq Epa(G) \leq 3\), and we present necessary and sufficient conditions for \(pa(u, v) = i, i = 1, 2, 3\), where \(uv \in E(G)\). In Section 3, we show that \(1 \leq \overline{epa}(G) \leq \overline{Epa}(G) \leq 5\), and we give necessary and sufficient conditions for \(\overline{epa}(G) = \overline{Epa}(G) = j, 1 \leq j \leq 5\). We conclude in Section 4 with open problems.

We end this section with some known results which will be useful in proving our main results.

**Lemma 1** [2]. If \(G\) is a graph and \(H\) is any graph obtained from \(G\) by subdividing some edges of \(G\), then \(\gamma(H) \geq \gamma(G)\).

**Lemma 2.** Let \(G\) be a graph and \(v \in V(G)\).

(i) [5] If \(v\) is \(\gamma\)-bad, then \(\gamma(G - v) = \gamma(G)\).

(ii) [3] \(v\) is \(\gamma\)-critical if and only if \(\gamma(G - v) = \gamma(G) - 1\).

(iii) [5] If \(v\) is \(\gamma\)-critical, then all its neighbors are \(\gamma\)-bad vertices of \(G - v\).

(iv) [11] If \(e \in E(G)\), then \(\gamma(G) - 1 \leq \gamma(G + e) \leq \gamma(G)\).

In most cases, Lemma 2 will be used in the sequel without specific reference.

2. The Adjacent Case

The aim of this section is to prove that \(1 \leq pa(u, v) \leq 3\) and to find necessary and sufficient conditions for \(pa(u, v) = i, i = 1, 2, 3\), where \(uv \in E(G)\).

**Observation 3.** If \(u\) and \(v\) are adjacent vertices of a graph \(G\), then \(\gamma(G) = \gamma(G_{u,v,0}) \leq \gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})\) for \(k \geq 1\).

**Proof.** The equality \(\gamma(G) = \gamma(G_{u,v,0})\) is obvious. For any \(\gamma\)-set \(M\) of \(G_{u,v,1}\) both \(M_u = (M \setminus \{x_1\}) \cup \{u\}\) and \(M_v = (M \setminus \{x_1\}) \cup \{v\}\) are dominating sets of \(G\), and at
least one of them is a \( \gamma \)-set of \( G_{u,v,1} \). Hence \( \gamma(G) \leq \min\{|M_u|, |M_v|\} = \gamma(G_{u,v,1}) \).

The rest follows by Lemma 1.

**Theorem 4.** Let \( u \) and \( v \) be adjacent vertices of a graph \( G \). Then \( \gamma(G) \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \) and the following is true.

(i) \( \gamma(G) = \gamma(G_{u,v,1}) \) if and only if at least one of \( u \) and \( v \) is a \( \gamma \)-good vertex of \( G \).

(ii) \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \) if and only if both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \).

**Proof.** The left side inequality follows by Observation 3. If \( D \) is a \( \gamma \)-set of \( G \), then \( D \cup \{x_1\} \) is a dominating set of \( G_{u,v,1} \), which implies \( \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \).

If at least one of \( u \) and \( v \) belongs to some \( \gamma \)-set \( D_1 \) of \( G \), then \( D_1 \) is a dominating set of \( G_{u,v,1} \). This clearly implies \( \gamma(G) = \gamma(G_{u,v,1}) \).

Let now both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), and suppose that \( \gamma(G_{u,v,1}) = \gamma(G) \). In this case for any \( \gamma \)-set \( M \) of \( G_{u,v,1} \) is fulfilled \( u, v \not\in M \) and \( x_1 \in M \). But then \( (M \setminus \{x_1\}) \cup \{u\} \) is a \( \gamma \)-set for both \( G \) and \( G_{u,v,1} \), a contradiction.

**Corollary 5.** Let \( G \) be a graph with edges. Then \( \text{Epa}(G) \geq 2 \) and \( \text{epa}(G) = 1 \) if and only if the set of all \( \gamma \)-bad vertices of \( G \) is neither empty nor independent.

**Theorem 6.** Let \( u \) and \( v \) be adjacent vertices of a graph \( G \). Then \( \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \). Moreover,

(A) \( \gamma(G_{u,v,2}) = \gamma(G) + 1 \) if and only if at least one of the following holds:

(i) both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \),

(ii) at least one of \( u \) and \( v \) is \( \gamma \)-good, \( u, v \not\in V^-(G) \) and each \( \gamma \)-set of \( G \) contains at most one of \( u \) and \( v \).

(B) \( \gamma(G_{u,v,2}) = \gamma(G) \) if and only if at least one of the following is true:

(iii) there exists a \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \),

(iv) at least one of \( u \) and \( v \) is in \( V^-(G) \).

**Proof.** The left side inequality follows by Observation 3. If \( D \) is an arbitrary \( \gamma \)-set of \( G \), then \( D \cup \{x_1\} \) is a dominating set of \( G_{u,v,2} \). Hence \( \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \).

(A) \( \Rightarrow \) Assume that the equality \( \gamma(G_{u,v,2}) = \gamma(G) + 1 \) holds. By Theorem 4 we know that \( \gamma(G_{u,v,1}) \in \{\gamma(G), \gamma(G) + 1\} \). If \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \), then again by Theorem 4, both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \). So let \( \gamma(G) = \gamma(G_{u,v,1}) \).

Then at least one of \( u \) and \( v \) is a \( \gamma \)-good vertex of \( G \) (Theorem 4). Clearly there is no \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \). If \( u \in V^-(G) \) and \( U \) is a \( \gamma \)-set of \( G - u \), then \( U \cup \{x_1\} \) is a dominating set of \( G_{u,v,2} \) and \( |U \cup \{x_1\}| = \gamma(G) \), a contradiction. Thus \( u, v \not\in V^-(G) \).

(A) \( \Leftarrow \) If both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), then \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \) (Theorem 4). But we know that \( \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \); hence
\[ \gamma(G_{u,v,2}) = \gamma(G) + 1. \] Finally let (ii) hold and \( M \) be a \( \gamma \)-set of \( G_{u,v,2} \). If \( x_1, x_2 \not\in M \), then \( u, v \in M \) which leads to \( \gamma(G_{u,v,2}) > \gamma(G) \). If \( x_1, x_2 \in M \), then \( (M \setminus \{x_1, x_2\}) \cup \{u, v\} \) is a dominating set of \( G \) of cardinality more than \( \gamma(G) \). Now let without loss of generality \( x_1 \in M \) and \( x_2 \not\in M \). If \( M \setminus \{x_1\} \) is a dominating set of \( G \), then \( \gamma(G) + 1 \leq |M| = \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \). So, let \( M \setminus \{x_1\} \) be no dominating set of \( G \). Hence \( M \setminus \{x_1\} \) is a dominating set of \( G - u \). Since \( u \not\in V(G) \), \( \gamma(G) \leq \gamma(G - u) \leq |M \setminus \{x_1\}| < \gamma(G_{u,v,2}) \).

\[ (\exists) \Rightarrow \text{Let } \gamma(G_{u,v,2}) = \gamma(G). \text{ Suppose that neither (iii) nor (iv) is valid. Hence } u, v \not\in V(G) \text{ and no } \gamma \text{-set of } G \text{ contains both } u \text{ and } v. \text{ But then at least one of (i) and (ii) holds, and from (A) we conclude that } \gamma(G_{u,v,2}) = \gamma(G) + 1, \text{ a contradiction.} \]

\[ (\exists) \Leftarrow \text{Let at least one of (iii) and (iv) be hold. Then neither (i) nor (ii) is fulfilled. Now by (A) we have } \gamma(G_{u,v,2}) \neq \gamma(G) + 1. \text{ Since } \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1, \text{ we obtain } \gamma(G) = \gamma(G_{u,v,2}). \]

The independent domination number of a graph \( G \), denoted by \( i(G) \), is the minimum size of an independent dominating set of \( G \). It is obvious that \( i(G) \geq \gamma(G) \). In a graph \( G, i(G) \) is strongly equal to \( \gamma(G) \), written \( i(G) \equiv \gamma(G) \), if each \( \gamma \)-set of \( G \) is independent. It remains an open problem to characterize the graphs \( G \) with \( i(G) \equiv \gamma(G) \) [7].

**Corollary 7.** Let \( G \) be a graph with edges. Then (a) \( EPA(G) \geq 2 \) if and only if the set of all \( \gamma \)-bad vertices is either empty or independent, and (b) \( EPA(G) = 2 \) if and only if \( i(G) \equiv \gamma(G) \).

**Proof.** (a) Immediately by Corollary 5.

(b) \( \Rightarrow \) Let \( EPA(G) = 2 \). If \( D \) is a \( \gamma \)-set of \( G \) and \( u, v \in D \) are adjacent, then \( D \) is a dominating set of \( G_{u,v,2} \), a contradiction.

(b) \( \Leftarrow \) Let all \( \gamma \)-sets of \( G \) be independent. Suppose \( u \in V(G) \) and \( D \) is a \( \gamma \)-set of \( G - u \). Then \( D = D \cup \{v\} \) is a \( \gamma \)-set of \( G \), where \( v \) is any neighbor of \( u \). But \( D \) is not independent. Hence \( V(G) \) is empty. Thus, for any 2 adjacent vertices \( u \) and \( v \) of \( G \) is fulfilled either (A)(i) or (A)(ii) of Theorem 6. Therefore \( EPA(G) \leq 2 \). The result now follows by Corollary 5.

Denote by \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) the additive group of order \( n \). Let \( S \) be a subset of \( \mathbb{Z}_n \) such that \( 0 \not\in S \) and \( x \in S \) implies \(-x \not\in S \). The circulant graph with distance set \( S \) is the graph \( C(n; S) \) with vertex set \( \mathbb{Z}_n \) and vertex \( x \) adjacent to vertex \( y \) if and only if \( x - y \in S \).

Let \( n \geq 3 \) and \( k \in \mathbb{Z}_n \setminus \{0\} \). The generalized Petersen graph \( P(n,k) \) is the graph on the vertex-set \( \{x, y\} \) with adjaecencies \( x_i x_{i+1}, x_i y_i, y_i y_{i+k} \) for all \( i \).

**Example 8.** A special case of graphs \( G \) with \( EPA(T) = 2 \) are graphs for which each \( \gamma \)-set is efficient dominating (an efficient dominating set in a graph \( G \) is a
set \( S \) such that \( \{ N[s] \mid s \in S \} \) is a partition of \( V(G) \). We list several examples of such graphs [10].

(a) A crown graph \( H_{n,n}, n \geq 3 \), which is obtained from the complete bipartite graph \( K_{n,n} \) by removing a perfect matching.

(b) Circulant graphs \( G = C(n = (2k+1)t; \{1, \ldots, k\} \cup \{n-1, \ldots, n-k\}) \), where \( k, t \geq 1 \).

(c) Circulant graphs \( G = C(n; \{\pm 1, \pm s\}) \), where \( 2 \leq s \leq n-2 \), \( s \neq n/2 \), \( 5 \mid n \) and \( s \equiv \pm 2 \pmod{5} \).

(d) The generalized Petersen graph \( P(n, k) \), where \( n \equiv 0 \pmod{4} \) and \( k \) is odd.

**Theorem 9.** If \( u \) and \( v \) are adjacent vertices of a graph \( G \), then \( \gamma(G_{u,v,3}) = \gamma(G) + 1 \).

**Proof.** If \( D \) is a \( \gamma \)-set of \( G \), then \( D \cup \{x_2\} \) is a dominating set of \( G \). Hence \( \gamma(G_{u,v,3}) \leq \gamma(G) + 1 \).

Let \( M \) be a \( \gamma \)-set of \( G_{u,v,3} \). Then at least one of \( x_1, x_2 \) and \( x_3 \) is in \( M \). If \( x_2 \in M \), then clearly \( \gamma(G_{u,v,3}) = \gamma(G) + 1 \). If \( x_2 \notin M \) and \( x_1, x_3 \in M \), then \( (M \setminus \{x_1, x_3\}) \cup \{u\} \) is a dominating set of \( G \). If \( x_2, x_3 \notin M \) and \( x_1 \in M \), then \( v \in M \) and \( M \setminus \{x_1\} \) is a dominating set of \( G \). All this leads to \( \gamma(G_{u,v,3}) = \gamma(G) + 1 \).

**Corollary 10.** Let \( G \) be a graph with edges. Then \( \text{epa}(G) \leq \text{Epa}(G) \leq 3 \). Moreover, \( \text{Epa}(G) = 3 \) if and only if \( G \) has a \( \gamma \)-set that is not independent, and \( \text{epa}(G) = 3 \) if and only if for each pair of adjacent vertices \( u \) and \( v \) at least one of the following is valid.

(i) There exists a \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \).

(ii) At least one of \( u \) and \( v \) is in \( V^-(G) \).

**Proof.** By Corollary 5 and Theorem 9 we have \( 1 \leq \text{epa}(G) \leq \text{Epa}(G) \leq 3 \) and \( 2 \leq \text{Epa}(G) \). Since \( \text{Epa}(G) = 2 \) if and only if \( i(G) \equiv \gamma(G) \) (by Corollary 7), \( \text{Epa}(G) = 3 \) if and only if \( G \) has a \( \gamma \)-set that is not independent.

Clearly \( \text{epa}(G) = 3 \) if and only if \( \gamma(G_{u,v,2}) = \gamma(G) \) for each pair of adjacent vertices \( u \) and \( v \) of \( G \). Then because of Theorem 6(ii), we have that \( \text{epa}(G) = 3 \) if and only if for each pair of adjacent vertices \( u \) and \( v \) of \( G \) at least one of (i) and (ii) holds.

**Corollary 11.** Let \( G \) be a graph with edges. If \( V^-(G) \) has a subset which is a vertex cover of \( G \), then \( \text{epa}(G) = 3 \). In particular, if \( G \) is a vc-graph then \( \text{epa}(G) = 3 \).

We define the following classes of graphs \( G \) with \( \Delta(G) \geq 1 \).

- \( \mathcal{A} = \{G \mid \text{epa}(G) = 3\} \),
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\[ A_1 = \{ G \mid V^-(G) \text{ is a vertex cover of } G \}, \]
\[ A_2 = \{ G \mid \text{each two adjacent vertices belongs to some } \gamma\text{-set of } G \}, \]
\[ A_3 = \{ G \mid G \text{ is a vc-graph} \}. \]

Clearly, \( A_3 \subseteq A_1 \) and by Corolaries 10 and 11, \( A_1 \cup A_2 \subseteq A \). These relationships are illustrated in the Venn diagram in Figure 1(left). To continue we relabel this diagram in six regions \( R_0 - R_5 \) as shown in Figure 1(right). In what follows in this section we show that none of \( R_0 - R_5 \) is empty. The corona of a graph \( H \) is the graph \( G = H \circ K_1 \) obtained from \( H \) by adding a degree-one neighbor to every vertex of \( H \). If \( F \) and \( H \) are disjoint graphs, \( v_F \in V(F) \) and \( v_H \in V(H) \), then the coalescence \( (F \cdot H)(v_F, v_H : v) \) of \( F \) and \( H \) via \( v_F \) and \( v_H \), is the graph obtained from the union of \( F \) and \( H \) by identifying \( v_F \) and \( v_H \) in a vertex labeled \( v \).

![Venn Diagram](image)

**Figure 1.** Left: Classes of graphs with \( epa = 3 \). Right: Regions of Venn diagram.

**Remark 12.** It is easy to see that all the following hold.

(i) If \( H \) is a connected graph of order \( n \geq 2 \), then \( G = H \circ K_1 \in R_0 \).

(ii) Let \( G_k^1 \) be a graph obtained from the cycle \( C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0 \) \( k \geq 2 \), by adding a vertex \( y \) and edges \( yx_0, yx_2 \). Then \( \gamma(G_k^1) = k + 1 \), \( G_k^1 \) is \( \gamma \)-excellent, \( V^-(G_k^1) = \{ x_0, x_2 \} \cup \bigcup_{r=1}^{k-1} \{ x_{3r+1}, x_{3r+2} \} \) is a vertex cover of \( G \), and there is no \( \gamma \)-set of \( G_k^1 \) that contains both \( x_{3r+1} \) and \( x_{3r+2} \). Thus \( G_k^1 \) is in \( R_1 \).

(iii) The graph \( H_{10} \) depicted in Figure 2 is in \( A_4 \) and \( \gamma(H_{10}) = 3 \) [1]. It is obvious that no \( \gamma \)-set of \( H_{10} \) contains both \( u \) and \( v \). Hence \( H_{10} \in R_2 \). Consider now the graph \( G_k^2 = (C_{3k+1} \cdot H_{10})(x_0, w : z) \), where \( C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0 \), \( k \geq 2 \), is a cycle on \( 3k+1 \) vertices and \( w \) is any of the two common neighbors of \( u \) and \( v \) in \( H_{10} \). Since both \( C_{3k+1} \) and \( H_{10} \) are vc-graphs, by [4] we have that \( G_k^2 \) is vc-graph and \( \gamma(G_k^2) = \gamma(C_{3k+1}) + \gamma(H_{10}) - 1 \). Let \( D \) be an arbitrary \( \gamma \)-set of \( G_k^2 \). \( D_1 = D \cap V(H_{10}) \) and \( D_2 = D \cap V(C_{3k+1}) \). Then exactly one of the following holds.

(a) \( z \in D \), \( D_1 \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \) is a \( \gamma \)-set of \( C_{3k+1} \).

(b) \( z \notin D \), \( D_1 \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \cup \{x_0\} \) is a \( \gamma \)-set of \( C_{3k+1} \).
(c) $z \notin D$, $D_1 \cup \{w\}$ is a $\gamma$-set of $H_{10}$ and $D_2$ is a $\gamma$-set of $C_{3k+1}$.

Since no $\gamma$-set of $H_{10}$ contains both $u$ and $v$, by (a), (b) and (c) we conclude that at most one of $u$ and $v$ is in $D$. Thus $G_2^0 \in \mathcal{R}_2$.

(iv) $C_{3k+1} \in \mathcal{R}_3$ for all $k \geq 1$.
(v) $K_{2,n} \in \mathcal{R}_4$ for all $n \geq 3$.
(vi) $K_{n,n} \in \mathcal{R}_5$ for all $n \geq 3$.

Thus all regions $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ are nonempty.

Figure 2. Graph $H_{10}$ is in $\mathcal{R}_2$.

3. The Nonadjacent Case

In this section we show that $1 \leq \tau(G) \leq \overline{Lpa}(G) \leq 5$ and we obtain necessary and sufficient conditions for $\tau(G) = \overline{Lpa}(G) = j$, $1 \leq j \leq 5$.

We begin with an easy observation which is an immediate consequence by Lemma 2(iv) and Lemma 1.

Observation 13. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) - 1 \leq \gamma(G_{u,v,0}) \leq \gamma(G)$ and $\gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 0$.

Theorem 14. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) - 1 \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1$. Moreover,

(i) $\gamma(G) - 1 = \gamma(G_{u,v,1})$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(ii) $\gamma(G_{u,v,1}) = \gamma(G) + 1$ if and only if both $u$ and $v$ are $\gamma$-bad vertices of $G$, $u \notin V^{-}(G - v)$ and $v \notin V^{-}(G - u)$. If $\gamma(G_{u,v,1}) = \gamma(G) + 1$, then $x_1 \in V^{-}(G_{u,v,1})$.

Proof. The left side inequality follows by Observation 13.

(i) Assume the equality $\gamma(G) - 1 = \gamma(G_{u,v,1})$ holds and let $M$ be any $\gamma$-set of $G_{u,v,1}$. Then at least one and not more than two of $x_1, u$ and $v$ must be in $M$. Hence $M_1 = (M \setminus \{x_1\}) \cup \{u, v\}$ is a dominating set of $G$ and $\gamma(G) \leq |M_1| \leq |M| + 1 = \gamma(G_{u,v,1}) + 1 = \gamma(G)$. This immediately implies that $M_1$ is a $\gamma$-set of $G$. Hence $x_1 \in M$ and $pn[x_1, M] = \{x_1, u, v\}$. Since $M_1 \setminus \{u, v\}$ is a dominating set of $G - \{u, v\}$, we have $\gamma(G) - 2 \leq \gamma(G - \{u, v\}) \leq |M_1 \setminus \{u, v\}| = \gamma(G) - 2$. 


(i) Suppose now \( \gamma(G - \{u, v\}) = \gamma(G) - 2 \). Then for any \( \gamma \)-set \( U \) of \( G - \{u, v\} \), the set \( U \cup \{x_1\} \) is a dominating set of \( G_{u,v,1} \). This leads to \( \gamma(G_{u,v,1}) \leq |U \cup \{x_1\}| = \gamma(G) - 1 \leq \gamma(G_{u,v,1}) \).

Now we will prove the right side inequality. Let \( D \) be any \( \gamma \)-set of \( G \). If at least one of \( u \) and \( v \) is in \( D \), then \( D \) is a dominating set of \( G_{u,v,1} \) and \( \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \).

(ii) Assume that \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \). Then \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \) and for any \( \gamma \)-set \( D \) of \( G_{u,v,1} \). Hence \( x_1 \in V^{-}(G_{u,v,1}) \). Suppose \( x_1 \in V^{-}(G - v) \) and let \( U \) be a \( \gamma \)-set of \( G - \{u, v\} \). Then \( U_1 = U \cup \{x_1\} \) is a dominating set of \( G_{u,v,1} \) and \( \gamma(G_{u,v,1}) = 1 + \gamma((G - v) - u) = \gamma(G - v) = \gamma(G) \), a contradiction. Thus \( u \notin V^{-}(G - v) \) and by symmetry, \( v \notin V^{-}(G - u) \).

(ii) \( \Rightarrow \) Let both \( u \) and \( v \) be \( \gamma \)-bad vertices of \( G \), \( u \notin V^{-}(G - v) \) and \( v \notin V^{-}(G - u) \). Hence \( \gamma(G - \{u, v\}) \geq \gamma(G) \). Consider any \( \gamma \)-set \( M \) of \( G_{u,v,1} \). If one of \( u \) and \( v \) belongs to \( M \), then \( \gamma(G) + 1 = \gamma(G_{u,v,1}) \). So, let \( x_1 \) is in each \( \gamma \)-set of \( G_{u,v,1} \). But then \( \gamma(G_{u,v,1}) - 1 = \gamma(G - \{u, v\}) \geq \gamma(G) \geq \gamma(G_{u,v,1}) - 1 \).

\[ \text{Corollary 15.} \text{ Let } G \text{ be a noncomplete graph. Then } 1 \leq \overline{\gamma}(G) \leq E_{\gamma}(G) \text{ and the following assertions hold.} \]

(i) \( \overline{\gamma}(G) = 1 \) if and only if there are nonadjacent \( \gamma \)-bad vertices \( u \) and \( v \) of \( G \) such that \( u \notin V^{-}(G - v) \) and \( v \notin V^{-}(G - u) \).

(ii) \( E_{\gamma}(G) = 1 \) if and only if \( \gamma(G) = 1 \).

**Proof.** Observation 13 implies \( 1 \leq \overline{\gamma}(G) \).

(i) Immediately by Theorem 14.

(ii) If \( \gamma(G) = 1 \), then clearly \( E_{\gamma}(G) = 1 \). If \( \gamma(G) \geq 2 \), then \( G \) has 2 nonadjacent vertices at least one of which is \( \gamma \)-good. By Theorem 14, \( E_{\gamma}(G) \geq 2 \).

\[ \text{Theorem 16. Let } u \text{ and } v \text{ be nonadjacent vertices of a graph } G. \text{ Then } \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1. \text{ Moreover,} \]

\[ (C) \ \gamma(G_{u,v,2}) = \gamma(G) \text{ if and only if one of the following holds.} \]

(i) There is a \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \).

(ii) At least one of \( u \) and \( v \) is in \( V^{-}(G) \).

\[ (D) \ \gamma(G_{u,v,2}) = \gamma(G) + 1 \text{ if and only if } u, v \notin V^{-}(G) \text{ and any } \gamma \text{-set of } G \text{ contains at most one of } u \text{ and } v. \]

**Proof.** For any \( \gamma \)-set \( D \) of \( G \), \( D \cup \{x_2\} \) is a dominating set of \( G_{u,v,2} \). Hence \( \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \). Suppose \( \gamma(G_{u,v,2}) \leq \gamma(G) - 1 \) and let \( M \) be a \( \gamma \)-set of \( G_{u,v,2} \). Then at least one of \( x_1 \) and \( x_2 \) is in \( M \). If \( x_1, x_2 \in M \), then \( M_1 = (M \setminus \{x_1, x_2\}) \cup \)
\{u, v\} is a dominating set of \( G \) and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. So let without loss of generality, \( x_1 \in M \) and \( x_2 \notin M \). If \( u \in M \) or \( v \in M \), then again \( M_1 \) is a dominating set of \( G \) and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. Thus \( x_1 \in M \) and \( u, v \notin M \). But then \( (M \setminus \{x_1\}) \cup \{u\} \) is a dominating set of \( G \), contradicting \( \gamma(G_{u,v,2}) < \gamma(G) \). Thus \( \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \).

\((C) \Rightarrow \) Let \( \gamma(G_{u,v,2}) = \gamma(G) \). Assume that neither (i) nor (ii) hold. Let \( M \) be a \( \gamma \)-set of \( G_{u,v,2} \). If \( x_1, x_2 \in M \), then \( M_1 = (M \setminus \{x_1, x_2\}) \cup \{u, v\} \) is a dominating set of \( G \) of cardinality not more than \( \gamma(G) \) and \( u, v \in M_1 \), a contradiction. Let without loss of generality \( x_1 \in M \) and \( x_2 \notin M \). Since \( M \setminus \{x_1\} \) is no dominating set of \( G \), \( u \in pn[x_1, M] \). But then \( M_3 = (M \setminus \{x_1\}) \cup \{u\} \) is a \( \gamma \)-set of \( G \) and \( u \in V^-(G) \), a contradiction. Thus at least one of (i) and (ii) is valid.

\((C) \Leftarrow \) If both \( u \) and \( v \) belong to some \( \gamma \)-set \( D \) of \( G \), then \( D \) is a dominating set of \( G_{u,v,2} \). Hence \( \gamma(G_{u,v,2}) = \gamma(G) \). Finally let \( u \in V^-(G) \) and \( D \) a \( \gamma \)-set of \( G - u \). Then \( D \cup \{x_1\} \) is a dominating set of \( G_{u,v,2} \) of cardinality \( \gamma(G) \). Thus \( \gamma(G_{u,v,2}) = \gamma(G) \).

\((D) \) Immediately by \((C) \) and \( \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \).

\begin{corollary}
Let \( G \) be a noncomplete graph. Then the following assertions hold.

\( (i) \) \( \varepsilon_{pa}(G) \leq 2 \) if and only if there are nonadjacent vertices \( u, v \in V(G) \setminus V^-(G) \) such that any \( \gamma \)-set of \( G \) contains at most one of them.

\( (ii) \) \( \overline{E}_{pa}(G) = 2 \) if and only if \( \gamma(G) \geq 2 \) and each \( \gamma \)-set of \( G \) is a clique.

\begin{proof}
\( (i) \) Immediately by Theorem 16.

\( (ii) \Rightarrow \) Let \( \overline{E}_{pa}(G) = 2 \). By Corollary 15, \( \gamma(G) \geq 2 \). Suppose \( G \) has a \( \gamma \)-set, say \( D \), which is not a clique. Then there are nonadjacent \( u, v \in D \). By Theorem 16\((C) \), \( \gamma(G_{u,v,2}) = \gamma(G) \), which contradict \( \overline{E}_{pa}(G) = 2 \). Thus, each \( \gamma \)-set of \( G \) is a clique.

\( (ii) \Leftarrow \) Let \( \gamma(G) \geq 2 \) and let each \( \gamma \)-set of \( G \) be a clique. If \( G \) has a vertex \( z \in V^-(G) \) and \( M_z \) is a \( \gamma \)-set of \( G - z \), then \( M = M_z \cup \{z\} \) is a \( \gamma \)-set of \( G \) and \( z \) is an isolated vertex of the graph induced by \( M \), a contradiction. Thus \( V^-(G) \) is empty. Now by Theorem 16\((D) \), \( \overline{E}_{pa}(G) = 2 \).
\end{proof}

\begin{example}
The join of two graphs \( G_1 \) and \( G_2 \) with disjoint vertex sets is the graph, denoted by \( G_1 \cup G_2 \), with the vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\} \). Let \( \gamma(G_i) \geq 3 \), \( i = 1, 2 \). Then \( \gamma(G_1 + G_2) = 2 \) and each \( \gamma \)-set of \( G_1 + G_2 \) contains exactly one vertex of \( G_i \), \( i = 1, 2 \). Hence \( \overline{E}_{pa}(G_1 + G_2) = 2 \). In particular, \( \overline{E}_{pa}(K_{m,n}) = 2 \) when \( m, n \geq 3 \).
\end{example}

\begin{theorem}
Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). Then \( \gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G) + 1 \). Moreover, \( \gamma(G_{u,v,3}) = \gamma(G) \) if and only if at least one of the following holds.
\end{theorem}
(i) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,
(ii) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

**Proof.** If $D$ is a dominating set of $G$, then $D \cup \{x_2\}$ is a dominating set of $G_{u,v,3}$. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$. We already know that $\gamma(G) \leq \gamma(G_{u,v,2})$ and $\gamma(G_{u,v,2}) \leq \gamma(G_{u,v,3})$. But then $\gamma(G) \leq \gamma(G_{u,v,3})$.

$\Rightarrow$ Let $\gamma(G_{u,v,3}) = \gamma(G)$ and let $M$ be a $\gamma$-set of $G_{u,v,3}$ such that $Q = M \cap \{x_1, x_2, x_3\}$ has minimum cardinality. Clearly $|Q| = 1$. If $\{x_2\} = Q$, then $M \setminus \{x_2\}$ is a dominating set of $G$, contradicting $\gamma(G_{u,v,3}) = \gamma(G)$. Let without loss of generality $\{x_1\} = Q$. This implies $v \in M$, $x_3 \in pn[v, M]$ and $pn[x_1, M] = \{u, x_1, x_2\}$. Then $M_2 = (M \setminus \{x_1\}) \cup \{u\}$ is a $\gamma$-set of $G$, $pn[u, M_2] = \{u\}$ and $v \in M_2$; hence (i) holds.

$\Leftarrow$ Let without loss of generality (i) is true. Then there is a $\gamma$-set $D$ of $G$ such that $u, v \in D$ and $D \setminus \{u\}$ is a $\gamma$-set of $G - u$. But then $(D \setminus \{u\}) \cup \{x_1\}$ is a dominating set of $G_{u,v,3}$, which implies $\gamma(G) \geq \gamma(G_{u,v,3})$.

**Corollary 20.** Let $G$ be a noncomplete graph. Then the following holds.

**(E)** $\text{epa}(G) \leq 3$ if and only if there is a pair of nonadjacent vertices $u$ and $v$ such that neither (i) nor (ii) is valid, where

(i) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,
(ii) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

**(F)** $\text{epa}(G) = \overline{\text{epa}}(G) = 3$ if and only if all vertices of $G$ are $\gamma$-good, $V^-(G)$ is empty and for every 2 nonadjacent vertices $u$ and $v$ of $G$ there is a $\gamma$-set of $G$ which contains them both.

**Proof.** (E) $\Rightarrow$ Let $\text{epa}(G) = \overline{\text{epa}}(G) = 3$. If $u \in V^-(G)$ and $D$ is a $\gamma$-set of $G - u$, then for $u$ and each $v \in D$ is fulfilled (i) of Theorem 19. But then $\text{epa}(G) \neq 3$, a contradiction. So, $V^-(G)$ is empty. Suppose that $G$ has $\gamma$-bad vertices. Then there is a $\gamma$-bad vertex which is nonadjacent to some other vertex of $G$. But Theorem 16(D) implies $\text{epa}(G) < 3$, a contradiction. Thus all vertices of $G$ are $\gamma$-good. Now let $u, v \in G$ be nonadjacent. If there is no $\gamma$-set of $G$ which contains both $u$ and $v$, then by Theorem 16(D) we have $\gamma(G_{u,v,2}) = \gamma(G) + 1$, a contradiction.

**(F) $\Leftarrow$ Let $V^-(G)$ be empty and for each pair $u, v$ of nonadjacent vertices of $G$ there is a $\gamma$-set $D_{uv}$ of $G$ with $u, v \in D_{uv}$. By Theorem 19, $\gamma(G_{u,v,3}) = \gamma(G) + 1$, and by Theorem 16, $\gamma(G_{u,v,2}) = \gamma(G)$. Hence $\text{pa}(u, v) = 3$.

**Example 21.** Denote by $U$ the class of all graphs $G$ with $\text{epa}(G) = \overline{\text{epa}}(G) = 3$. Then all the following holds. (a) Circulant graphs $C(2k + 1; \{\pm 1, \pm 2, \ldots, \pm (k - 1)\}) \in U$ for all $k \geq 1$. (b) Let $G$ be a nonconnected graph. Then $G \in U$ if and only if $G$ has no isolated vertices and each its component is either in $U$ or is complete.
Theorem 22. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. Moreover, the following assertions are valid.

(G) $\gamma(G_{u,v,4}) = \gamma(G) + 2$ if and only if $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

(ii) If $\gamma(G_{u,v,1}) = \gamma(G)$ and $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{2,3\}$, then $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(i) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$ and the equality holds if and only if $\gamma(G - \{u,v\}) \geq \gamma(G) - 1$.

(iii) $\gamma(G_{u,v,4}) = \gamma(G)$ if and only if $\gamma(G - \{u,v\}) = \gamma(G) - 2$.

Proof. Since $\gamma(G) \leq \gamma(G_{u,v,3})$ (by Theorem 19) and $\gamma(G_{u,v,3}) \leq \gamma(G_{u,v,4})$ (by Observation 13), we have $\gamma(G) \leq \gamma(G_{u,v,4})$. Let $S$ be a $\gamma$-set of $G$. Then $S \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$, which leads to $\gamma(G_{u,v,4}) \leq \gamma(G) + 2$.

Claim 1. If $\gamma(G_{u,v,1}) \leq \gamma(G)$, then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Proof. Assume that $v$ is a $\gamma$-bad vertex of $G$, $u \in V^-(G - v)$ and $R$ a $\gamma$-set of $G - \{u,v\}$. Then $|R| = \gamma((G - v) - u) = \gamma(G - v) - 1 = \gamma(G) - 1$ and $R \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) \leq |R| + 2 = \gamma(G) + 1$.

Assume now that $D$ is a $\gamma$-set of $G$ with $u \in D$. Then $D \cup \{x_3\}$ is a dominating set of $G_{u,v,4}$. Hence again $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. Now by Theorem 14 we immediately obtain the required.

(G) Let $\gamma(G_{u,v,4}) = \gamma(G) + 2$. By Claim 1, $\gamma(G_{u,v,1}) > \gamma(G)$ and by Theorem 14, $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

Let now $\gamma(G_{u,v,1}) = \gamma(G) + 1$. By Theorem 14, $u$ and $v$ are $\gamma$-bad vertices of $G$, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$. Let $M$ be a $\gamma$-set of $G_{u,v,4}$ such that $R = M \cap \{x_1, x_2, x_3, x_4\}$ has minimum cardinality. Clearly $|R| \in \{1,2\}$. Assume first $|R| = 1$ and without loss of generality $\{x_2\} = M$. Then $M \setminus \{x_2\}$ is a dominating set of $G$ with $v \in M \setminus \{x_2\}$. Since $v$ is a $\gamma$-bad vertex of $G$, $|M \setminus \{x_2\}| > \gamma(G)$ and then $\gamma(G_{u,v,4}) = |M| > \gamma(G) + 1$. Let now $|R| = 2$ and without loss of generality $x_1, x_4 \in M$. Since $|M \cap \{x_1, x_2, x_3, x_4\}|$ is minimum, $u, v \notin M$ and $M \setminus \{x_1, x_4\}$ is a dominating set of $G - \{u,v\}$. But then $\gamma(G_{u,v,4}) = 2 + |M \setminus \{x_1, x_4\}| \geq 2 + \gamma(G - u) - v \geq 2 + \gamma(G - u) = 2 + \gamma(G)$. 

(ii) Let $\gamma(G_{u,v,1}) = \gamma(G)$. By Claim 1, $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. If $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{1,2\}$, then since $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,i})$, we obtain $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(i) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Hence at least one of (i) and (ii) of Theorem 19 holds, and by (ii), $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Assume that the equality holds. If $\gamma(G - \{u,v\}) = \gamma(G) - 2$, then for any $\gamma$-set $U$ of $G - \{u,v\}$, $U \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) = \gamma(G)$, a contradiction.
Let now $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and without loss of generality condition (i) of Theorem 19 be satisfied. Suppose $\gamma(G_{u,v,4}) = \gamma(G)$. Hence for each $\gamma$-set $M$ of $G_{u,v,4}$ are fulfilled: $x_1, x_4 \in M$, $x_2, x_3, u, v \notin M$, $pn[x_1, M] = \{x_1, x_2, u\}$ and $pn[x_4, M] = \{x_3, x_4, v\}$. But then $\gamma(G - \{u, v\}) = \gamma(G) - 2$, a contradiction. Thus $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(iii) If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,3}) = \gamma(G)$ and by (G), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Now let $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Thus $\gamma(G_{u,v,4}) = \gamma(G)$.

Theorem 23. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. If $\gamma(G_{u,v,k}) = \gamma(G)$, then $k \leq 4$. If $k \geq 5$, then $\gamma(G_{u,v,k}) > \gamma(G)$. If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,5}) = \gamma(G) + 1$.

Proof. By Theorem 22, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. If $\gamma(G_{u,v,4}) > \gamma(G)$, then $\gamma(G_{u,v,5}) > \gamma(G)$ for all $k \geq 5$ because of Observation 13. So, let $\gamma(G_{u,v,4}) = \gamma(G)$. By Theorem 22(I), $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_3, x_5\}$ is a dominating set of $G_{u,v,5}$. Hence $\gamma(G_{u,v,5}) \leq \gamma(G) + 1$. Let now $M$ be a $\gamma$-set of $G_{u,v,5}$. Then at least one of $x_2, x_3, x_4$ is in $M$ and hence $\gamma(G_{u,v,5}) = |M| \geq \gamma(G) + 1$. Thus $\gamma(G_{u,v,5}) = \gamma(G) + 1$. Now using again Observation 13 we conclude that $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$.

Corollary 24. Let $G$ be a noncomplete graph. Then $\overline{epa}(G) \leq \overline{Epa}(G) \leq 5$. Moreover, the following holds.

(i) $\overline{Epa}(G) = 5$ if and only if there are nonadjacent vertices $u$ and $v$ of $G$ with $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(ii) $\overline{pa}(G) = 5$ if and only if $G$ is edgeless.

(iii) $\overline{pa}(G) = \overline{Epa}(G) = 4$ if and only if for each pair $u, v$ of nonadjacent vertices of $G$, $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and at least one of the following holds:

(a) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G$.

(b) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G$.

Proof. By Theorem 23, $\overline{Epa}(G) \leq 5$.

(i) $\Rightarrow$ Let $\overline{Epa}(G) = 5$. Then there is a pair $u, v$ of nonadjacent vertices of $G$ such that $\gamma(G_{u,v,4}) = \gamma(G)$. Now by Theorem 22(I), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(i) $\Leftarrow$ Let $\gamma(G - \{u, v\}) = \gamma(G) - 2$ and $D$ be a $\gamma$-set of $G - \{u, v\}$, where $u$ and $v$ are nonadjacent vertices of $G$. Hence $D_1 = D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$ and $|D_1| = \gamma(G)$. This implies $\gamma(G_{u,v,4}) = \gamma(G)$. The result now follows by Theorem 23.

(ii) If $G$ has no edges, then the result is obvious. So let $G$ have edges and $\overline{pa}(G) = 5$. Then for any 2 nonadjacent vertices $u$ and $v$ of $G$ is satisfied
\( \gamma(G - \{u, v\}) = \gamma(G) - 2 \) (by (i)). Hence we can choose \( u \) and \( v \) so that they have a neighbor in common, say \( w \). But then \( w \) is a \( \gamma \)-bad vertex of \( G - u \) which implies \( v \not\in V^-(G - u) \). This leads to \( \gamma(G - \{u, v\}) \geq \gamma(G) - 1 \), a contradiction.

(iii) \( \Rightarrow \) Let \( \epsilon_{pa}(G) = \overline{Epa}(G) = 4 \). Then for each two nonadjacent \( u, v \in V(G) \) we have \( \gamma(G) = \gamma(G_{u,v,3}) < \gamma(G_{u,v,4}) \). Now by Theorem 22(\( G \)), \( \gamma(G - \{u, v\}) \geq \gamma(G) - 1 \) and by Theorem 19, at least one of (a) and (b) is valid.

(iii) \( \Leftarrow \) Consider any two nonadjacent vertices \( u, v \) of \( G \). Then \( \gamma(G - \{u, v\}) \geq \gamma(G) - 1 \) and at least one of (a) and (b) is valid. Theorem 19 now implies \( \gamma(G) = \gamma(G_{u,v,3}) \), and by Theorem 22, \( pa(u, v) = 4 \).

\[ \text{Example 25.} \] Let \( G_n \) be the Cartesian product of two copies of \( K_n, n \geq 2 \). We consider \( G_n \) as an \( n \times n \) array of vertices \( \{x_{i,j} \mid 1 \leq i \leq j \leq n\} \), where the closed neighborhood of \( x_{i,j} \) is the union of the sets \( \{x_{1,j}, x_{2,j}, \ldots, x_{n,j}\} \) and \( \{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\} \). Note that \( V(G_n) = V^-(G_n) \) and \( \gamma(G_n) = n \) [6]. It is easy to see that the following sets are \( \gamma \)-sets of \( G_n - x_{1,1} \): \( D_i = \{x_{2,i}, x_{3,i+1}, \ldots, x_{n,n+i-2}\} \), \( i = 2, 3, \ldots, n \), where \( x_{k,j} = x_{k,j-n+1} \) for \( j > n \) and \( 2 \leq k \leq n \). Since \( D = \bigcup_{i=2}^{n} D_i = V(G_n) \setminus N[x_{1,1}] \), all \( \gamma \)-bad vertices of \( G_n - x_{1,1} \) are the neighbors of \( x_{1,1} \) in \( G_n \). Since each vertex of \( D \) is adjacent to some neighbor of \( x_{1,1} \), \( V^-(G_n - x_{1,1}) \) is empty. Now by Theorem 19 we have \( pa(x_{1,1}, y) \geq 4 \), and by Theorem 22(\( \overline{H} \)), \( pa(x_{1,1}, y) < 5 \). Thus \( pa(x_{1,1}, y) = 4 \). By reason of symmetry, we obtain \( \overline{\epsilon}_{pa}(G_n) = \overline{Epa}(G_n) = 4 \).

4. Observations and Open Problems

A constructive characterization of the trees \( T \) with \( i(T) \equiv \gamma(T) \), and therefore a constructive characterization of the trees \( T \) with \( Epa(T) = 2 \) (by Corollary 7), was provided in [9].

\[ \text{Problem 26.} \] Characterize all unicyclic graphs \( G \) with \( Epa(G) = 2 \).

\[ \text{Problem 27.} \] Find results on \( \gamma \)-excellent graphs \( G \) with \( \overline{Epa}(G) = 2 \).

\[ \text{Problem 28.} \] Characterize all graphs \( G \) with \( \overline{\epsilon}_{pa}(G) = \overline{Epa}(G) = 4 \).

\[ \text{Corollary 29.} \] Let \( G \) be a connected noncomplete graph with edges. Then

(i) \( 2 \leq \epsilon_{pa}(G) + \overline{Epa}(G) \leq 8 \),

(ii) \( 2 \leq \epsilon_{pa}(G) + \overline{\epsilon}_{pa}(G) \leq 7 \),
(iii) $3 \leq Epa(G) + \bar{Epa}(G) \leq 8$,  
(iv) $3 \leq Epa(G) + \tau pa(G) \leq 7$.

**Proof.** (i)–(iv) The left-side inequalities immediately follow by Corollary 5 and Corollary 15. The right-side inequalities hold because of Corollary 10 and Corollary 24.

Note that all bounds stated in Corollary 29 are attainable. We leave finding examples demonstrating this to the reader.

**Problem 30.** Characterize all graphs $G$ that attain the bounds in Corollary 29.

**Acknowledgments**

The author wishes to express his sincere thanks to the reviewers for their very helpful comments and remarks.

**References**


Received 18 June 2018
Revised 2 November 2018
Accepted 3 November 2018