CHANGING AND UNCHANGING OF THE DOMINATION NUMBER OF A GRAPH: PATH ADDITION NUMBERS

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Abstract

Given a graph $G = (V, E)$ and two its distinct vertices $u$ and $v$, the $(u, v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k-2}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_{k-1}$. We prove that $\gamma(G) - 1 \leq \gamma(G_{u,v,k})$ for all $k \geq 1$, and $\gamma(G_{u,v,k}) > \gamma(G)$ when $k \geq 5$. We also provide necessary and sufficient conditions for the equality $\gamma(G_{u,v,k}) = \gamma(G)$ to be valid for each pair $u, v \in V(G)$. In addition, we establish sharp upper and lower bounds for the minimum, respectively maximum, $k$ in a graph $G$ over all pairs of vertices $u$ and $v$ in $G$ such that the $(u, v)$-$P_k$-addition graph of $G$ has a larger domination number than $G$, which we consider separately for adjacent and non-adjacent pairs of vertices.

Keywords: domination number, path addition.

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1. Introduction

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [8]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The complement $\overline{G}$ of $G$ is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. We write $K_n$ for the complete graph of order $n$, $K_{m,n}$ for the complete bipartite graph with partite sets of order $m$ and $n$, and $P_n$ for the path on $n$ vertices. Let $C_m$ denote the cycle of length $m$. For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the
degree of $x$ is $\deg(x, G) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_G[A] = N_G(A) \cup A$. A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. Let $G$ be a graph and $uv$ be an edge of $G$. By subdividing the edge $uv$ we mean forming a graph $H$ from $G$ by adding a new vertex $w$ and replacing the edge $uv$ by $uw$ and $wv$. Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) \setminus \{uv\}) \cup \{uw, wv\}$. For a graph $G$, let $x \in S \subseteq V(G)$. A vertex $y \in V(G)$ is a $S$-private neighbor of $x$ if $N_G[y] \cap S = \{x\}$. The set of all $S$-private neighbors of $x$ is denoted by $pn_G[x,S]$.

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a comprehensive introduction to the theory we refer the reader to Haynes et al. [8]. A dominating set for a graph $G$ is a subset $D \subseteq V(G)$ of vertices such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. The concept of $\gamma$-bad/good vertices in graphs was introduced by Fricke et al. in [5]. A vertex $v$ of a graph $G$ is called

(i) [5] $\gamma$-good, if $v$ belongs to some $\gamma$-set of $G$, and
(ii) [5] $\gamma$-bad, if $v$ belongs to no $\gamma$-set of $G$.

A graph $G$ is said to be $\gamma$-excellent whenever all its vertices are $\gamma$-good [5]. Brigham et al. [3] defined a vertex $v$ of a graph $G$ to be $\gamma$-critical if $\gamma(G - v) < \gamma(G)$, and $G$ to be vertex domination-critical (from now on called vc-graph) if each vertex of $G$ is $\gamma$-critical. For a graph $G$ we define $V^-(G) = \{x \in V(G) \mid \gamma(G - x) < \gamma(G)\}$.

It is often of interest to known how the value of a graph parameter $\mu$ is affected when a change is made in a graph, for instance vertex or edge removal, edge addition, edge subdivision and edge contraction. In this connection, here we consider this question in the case $\mu = \gamma$ when a path is added to a graph.

Path-addition is an operation that takes a graph and adds an internally vertex-disjoint path between two vertices together with a set of supplementary edges. This operation can be considered as a natural generalization of the edge addition. Formally, let $u$ and $v$ be distinct vertices of a graph $G$. The $(u,v)$-$P_k$-addition graph of $G$ is the graph $G_{u,v,k-2}$ obtained from disjoint union of $G$ and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices $u$ and $x_0$, and identifying the vertices $v$ and $x_{k-1}$. When $k \geq 3$ we call $x_1, x_2, \ldots, x_{k-2}$ path-addition vertices. By $p_{\gamma}(u,v)$ we denote the minimum number $k$ such that $\gamma(G) < \gamma(G_{u,v,k})$. For every graph $G$ with at least 2 vertices we define

$\triangleright$ the $e$-path addition (or $\bar{e}$-path addition) number with respect to domination, de-
noted $epa_\gamma(G)$ ($\overline{epa}_\gamma(G)$, respectively), to be

- $epa_\gamma(G) = \min \{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}$,
- $\overline{epa}_\gamma(G) = \min \{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}$, and

the upper e-path addition (upper $\overline{e}$-path addition) number with respect to domination, denoted $Epa_\gamma(G)$ ($\overline{Epa}_\gamma(G)$, respectively), to be

- $Epa_\gamma(G) = \max \{pa_\gamma(u, v) \mid u, v \in V(G), uv \in E(G)\}$,
- $\overline{Epa}_\gamma(G) = \max \{pa_\gamma(u, v) \mid u, v \in V(G), uv \notin E(G)\}$.

If $G$ is complete, then we write $\overline{Epa}_\gamma(G) = \overline{epa}_\gamma(G) = \infty$, and if $G$ is edgeless then $epa_\gamma(G) = Epa_\gamma(G) = \infty$. In what follows the subscript $\gamma$ will be omitted from the notation.

The remainder of this paper is organized as follows. In Section 2, we prove that $1 \leq epa(G) \leq 3$ and $2 \leq Epa(G) \leq 5$, and we present necessary and sufficient conditions for $pa(u, v) = i, i = 1, 2, 3$, where $uv \in E(G)$. In Section 3, we show that $1 \leq \overline{epa}(G) \leq \overline{Epa}(G) \leq 5$, and we give necessary and sufficient conditions for $\overline{epa}(G) = \overline{Epa}(G) = j, 1 \leq j \leq 5$. We conclude in Section 4 with open problems.

We end this section with some known results which will be useful in proving our main results.

**Lemma 1** [2]. If $G$ is a graph and $H$ is any graph obtained from $G$ by subdividing some edges of $G$, then $\gamma(H) \geq \gamma(G)$.

**Lemma 2.** Let $G$ be a graph and $v \in V(G)$.

(i) [5] If $v$ is $\gamma$-bad, then $\gamma(G - v) = \gamma(G)$.

(ii) [3] $v$ is $\gamma$-critical if and only if $\gamma(G - v) = \gamma(G) - 1$.

(iii) [5] If $v$ is $\gamma$-critical, then all its neighbors are $\gamma$-bad vertices of $G - v$.

(iv) [11] If $e \in E(G)$, then $\gamma(G) - 1 \leq \gamma(G + e) \leq \gamma(G)$.

In most cases, Lemma 2 will be used in the sequel without specific reference.

2. **The Adjacent Case**

The aim of this section is to prove that $1 \leq pa(u, v) \leq 3$ and to find necessary and sufficient conditions for $pa(u, v) = i, i = 1, 2, 3$, where $uv \in E(G)$.

**Observation 3.** If $u$ and $v$ are adjacent vertices of a graph $G$, then $\gamma(G) = \gamma(G_{u,v,0}) \leq \gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 1$.

**Proof.** The equality $\gamma(G) = \gamma(G_{u,v,0})$ is obvious. For any $\gamma$-set $M$ of $G_{u,v,1}$ both $M_u = (M \setminus \{x_1\}) \cup \{u\}$ and $M_v = (M \setminus \{x_1\}) \cup \{v\}$ are dominating sets of $G$, and at
least one of them is a \( \gamma \)-set of \( G_{u,v,1} \). Hence \( \gamma(G) \leq \min\{|M_u|, |M_v|\} = \gamma(G_{u,v,1}) \).

The rest follows by Lemma 1.

**Theorem 4.** Let \( u \) and \( v \) be adjacent vertices of a graph \( G \). Then \( \gamma(G) \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \) and the following is true.

(i) \( \gamma(G) = \gamma(G_{u,v,1}) \) if and only if at least one of \( u \) and \( v \) is a \( \gamma \)-good vertex of \( G \).

(ii) \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \) if and only if both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \).

**Proof.** The left side inequality follows by Observation 3. If \( D \) is a \( \gamma \)-set of \( G \), then \( D \cup \{x_1\} \) is a dominating set of \( G_{u,v,1} \), which implies \( \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \).

If at least one of \( u \) and \( v \) belongs to some \( \gamma \)-set \( D_1 \) of \( G \), then \( D_1 \) is a dominating set of \( G_{u,v,1} \). This clearly implies \( \gamma(G) = \gamma(G_{u,v,1}) \).

Let now both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), and suppose that \( \gamma(G_{u,v,1}) = \gamma(G) \). In this case for any \( \gamma \)-set \( M \) of \( G_{u,v,1} \) fulfills \( u, v \not\in M \) and \( x_1 \in M \).

But then \( (M \setminus \{x_1\}) \cup \{u\} \) is a \( \gamma \)-set for both \( G \) and \( G_{u,v,1} \), a contradiction.

**Corollary 5.** Let \( G \) be a graph with edges. Then \( \text{Epa}(G) \geq 2 \) and \( \text{epa}(G) = 1 \) if and only if the set of all \( \gamma \)-bad vertices of \( G \) is neither empty nor independent.

**Theorem 6.** Let \( u \) and \( v \) be adjacent vertices of a graph \( G \). Then \( \gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \). Moreover,

(A) \( \gamma(G_{u,v,2}) = \gamma(G) + 1 \) if and only if at least one of the following holds:

(i) both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \),

(ii) at least one of \( u \) and \( v \) is \( \gamma \)-good, \( u, v \not\in V^-(G) \) and each \( \gamma \)-set of \( G \) contains at most one of \( u \) and \( v \).

(B) \( \gamma(G_{u,v,2}) = \gamma(G) \) if and only if at least one of the following is true:

(iii) there exists a \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \),

(iv) at least one of \( u \) and \( v \) is in \( V^-(G) \).

**Proof.** The left side inequality follows by Observation 3. If \( D \) is an arbitrary \( \gamma \)-set of \( G \), then \( D \cup \{x_1\} \) is a dominating set of \( G_{u,v,2} \). Hence \( \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \).

(A) \( \Rightarrow \) Assume that the equality \( \gamma(G_{u,v,2}) = \gamma(G) + 1 \) holds. By Theorem 4 we know that \( \gamma(G_{u,v,1}) \in \{\gamma(G), \gamma(G) + 1\} \). If \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \), then again by Theorem 4, both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \). So let \( \gamma(G) = \gamma(G_{u,v,1}) \).

Then at least one of \( u \) and \( v \) is a \( \gamma \)-good vertex of \( G \) (Theorem 4). Clearly there is no \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \). If \( u \in V^-(G) \) and \( U \) is a \( \gamma \)-set of \( G - u \), then \( U \cup \{x_1\} \) is a dominating set of \( G_{u,v,2} \) and \( |U \cup \{x_1\}| = \gamma(G) \), a contradiction. Thus \( u, v \not\in V^-(G) \).

(A) \( \Leftarrow \) If both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), then \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \) (Theorem 4). But we know that \( \gamma(G_{u,v,1}) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \); hence
\(\gamma(G_{u,v,2}) = \gamma(G) + 1\). Finally let (ii) hold and \(M\) be a \(\gamma\)-set of \(G_{u,v,2}\). If \(x_1, x_2 \notin M\), then \(u, v \in M\) which leads to \(\gamma(G_{u,v,2}) > \gamma(G)\). If \(x_1, x_2 \in M\), then \((M \setminus \{x_1, x_2\}) \cup \{u, v\}\) is a dominating set of \(G\) of cardinality more than \(\gamma(G)\). Now let without loss of generality \(x_1 \in M\) and \(x_2 \notin M\). If \(M \setminus \{x_1\}\) is a dominating set of \(G\), then \(\gamma(G) + 1 \leq |M| = \gamma(G_{u,v,2}) \leq \gamma(G) + 1\). So, let \(M \setminus \{x_1\}\) be no dominating set of \(G\). Hence \(M \setminus \{x_1\}\) is a dominating set of \(G - u\).

Since \(u \notin V^-(G)\), \(\gamma(G) \leq \gamma(G - u) \leq |M \setminus \{x_1\}| < \gamma(G_{u,v,2})\).

\((\exists) \Rightarrow \) Let \(\gamma(G_{u,v,2}) = \gamma(G)\). Suppose that neither (iii) nor (iv) is valid. Hence \(u, v \notin V^-(G)\) and no \(\gamma\)-set of \(G\) contains both \(u\) and \(v\). But then at least one of (i) and (ii) holds, and from (A) we conclude that \(\gamma(G_{u,v,2}) = \gamma(G) + 1\), a contradiction.

\((\exists) \Leftarrow \) Let at least one of (iii) and (iv) be hold. Then neither (i) nor (ii) is fulfilled. Now by (A) we have \(\gamma(G_{u,v,2}) \neq \gamma(G) + 1\). Since \(\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1\), we obtain \(\gamma(G) = \gamma(G_{u,v,2})\).

The independent domination number of a graph \(G\), denoted by \(i(G)\), is the minimum size of an independent dominating set of \(G\). It is obvious that \(i(G) \geq \gamma(G)\). In a graph \(G\), \(i(G)\) is strongly equal to \(\gamma(G)\), written \(i(G) \equiv \gamma(G)\), if each \(\gamma\)-set of \(G\) is independent. It remains an open problem to characterize the graphs \(G\) with \(i(G) \equiv \gamma(G)\) [7].

**Corollary 7.** Let \(G\) be a graph with edges. Then (a) \(e\text{pa}(G) \geq 2\) if and only if the set of all \(\gamma\)-bad vertices is either empty or independent, and (b) \(E\text{pa}(G) = 2\) if and only if \(i(G) \equiv \gamma(G)\).

**Proof.** (a) Immediately by Corollary 5.

(b) \(\Rightarrow\) Let \(E\text{pa}(G) = 2\). If \(D\) is a \(\gamma\)-set of \(G\) and \(u, v \in D\) are adjacent, then \(D\) is a dominating set of \(G_{u,v,2}\), a contradiction.

(b) \(\Leftarrow\) Let all \(\gamma\)-sets of \(G\) be independent. Suppose \(u \in V^-(G)\) and \(D\) is a \(\gamma\)-set of \(G - u\). Then \(D_1 = D \cup \{v\}\) is a \(\gamma\)-set of \(G\), where \(v\) is any neighbor of \(u\). But \(D_1\) is not independent. Hence \(V^-(G)\) is empty. Thus, for any 2 adjacent vertices \(u\) and \(v\) of \(G\) is fulfilled either (A)(i) or (A)(ii) of Theorem 6. Therefore \(E\text{pa}(G) \leq 2\). The result now follows by Corollary 5.

Denote by \(\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}\) the additive group of order \(n\). Let \(S\) be a subset of \(\mathbb{Z}_n\) such that \(0 \notin S\) and \(x \in S\) implies \(-x \in S\). The circulant graph with distance set \(S\) is the graph \(C(n; S)\) with vertex set \(\mathbb{Z}_n\) and vertex \(x\) adjacent to vertex \(y\) if and only if \(x - y \in S\).

Let \(n \geq 3\) and \(k \in \mathbb{Z}_n \setminus \{0\}\). The generalized Petersen graph \(P(n, k)\) is the graph on the vertex-set \(\{x_i, y_i \mid i \in \mathbb{Z}_n\}\) with adjacencies \(x_i x_{i+1}, x_i y_i,\) and \(y_i y_{i+k}\) for all \(i\).

**Example 8.** A special case of graphs \(G\) with \(E\text{pa}(T) = 2\) are graphs for which each \(\gamma\)-set is efficient dominating (an efficient dominating set in a graph \(G\) is a
set $S$ such that $\{N[s] \mid s \in S\}$ is a partition of $V(G)$. We list several examples of such graphs [10].

(a) A crown graph $H_{n,n}$, $n \geq 3$, which is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

(b) Circulant graphs $G = C(n=(2k+1)t;\{1,\ldots,k\}\cup\{n-1,\ldots,n-k\})$, where $k,t \geq 1$.

(c) Circulant graphs $G = C(n;\{\pm 1,\pm s\})$, where $2 \leq s \leq n-2$, $s \neq n/2$, $5 \mid n$ and $s \equiv \pm 2 \pmod{5}$.

(d) The generalized Petersen graph $P(n,k)$, where $n \equiv 0 \pmod{4}$ and $k$ is odd.

**Theorem 9.** If $u$ and $v$ are adjacent vertices of a graph $G$, then $\gamma(G_{u,v,3}) = \gamma(G) + 1$.

**Proof.** If $D$ is a $\gamma$-set of $G$, then $D \cup \{x_2\}$ is a dominating set of $G$. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$.

Let $M$ be a $\gamma$-set of $G_{u,v,3}$. Then at least one of $x_1, x_2$ and $x_3$ is in $M$. If $x_2 \in M$, then clearly $\gamma(G_{u,v,3}) = \gamma(G) + 1$. If $x_2 \notin M$ and $x_1, x_3 \in M$, then $(M\{x_1,x_3\}) \cup \{u\}$ is a dominating set of $G$. If $x_2, x_3 \notin M$ and $x_1 \in M$, then $v \in M$ and $M\{x_1\}$ is a dominating set of $G$. All this leads to $\gamma(G_{u,v,3}) = \gamma(G) + 1$.

**Corollary 10.** Let $G$ be a graph with edges. Then $epa(G) \leq Epa(G) \leq 3$. Moreover, $Epa(G) = 3$ if and only if $G$ has a $\gamma$-set that is not independent, and $epa(G) = 3$ if and only if for each pair of adjacent vertices $u$ and $v$ at least one of the following is valid.

(i) There exists a $\gamma$-set of $G$ which contains both $u$ and $v$.

(ii) At least one of $u$ and $v$ is in $V^-(G)$.

**Proof.** By Corollary 5 and Theorem 9 we have $1 \leq epa(G) \leq Epa(G) \leq 3$ and $2 \leq Epa(G)$. Since $Epa(G) = 2$ if and only if $i(G) = \gamma(G)$ (by Corollary 7), $Epa(G) = 3$ if and only if $G$ has a $\gamma$-set that is not independent.

Clearly $epa(G) = 3$ if and only if $\gamma(G_{u,v,2}) = \gamma(G)$ for each pair of adjacent vertices $u$ and $v$ of $G$. Then because of Theorem 6(3), we have that $epa(G) = 3$ if and only if for each pair of adjacent vertices $u$ and $v$ of $G$ at least one of (i) and (ii) holds.

**Corollary 11.** Let $G$ be a graph with edges. If $V^-(G)$ has a subset which is a vertex cover of $G$, then $epa(G) = 3$. In particular, if $G$ is a vc-graph then $epa(G) = 3$.

We define the following classes of graphs $G$ with $\Delta(G) \geq 1$.

- $\mathcal{A} = \{G \mid epa(G) = 3\}$,
• \( A_1 = \{ G \mid V^-(G) \) is a vertex cover of \( G \}, \\
• \( A_2 = \{ G \mid \) each two adjacent vertices belongs to some \( \gamma \)-set of \( G \}, \\
• \( A_3 = \{ G \mid G \) is a vc-graph \}.

Clearly, \( A_3 \subseteq A_1 \) and by Corolaries 10 and 11, \( A_1 \cup A_2 \subseteq A \). These relationships are illustrated in the Venn diagram in Figure 1(left). To continue we relabel this diagram in six regions \( R_0 - R_5 \) as shown in Figure 1(right). In what follows in this section we show that none of \( R_0 - R_5 \) is empty. The corona of a graph \( H \) is the graph \( G = H \circ K_1 \) obtained from \( H \) by adding a degree-one neighbor to every vertex of \( H \). If \( F \) and \( H \) are disjoint graphs, \( v_F \in V(F) \) and \( v_H \in V(H) \), then the coalescence \( (F \cdot H)(v_F,v_H : v) \) of \( F \) and \( H \) via \( v_F \) and \( v_H \), is the graph obtained from the union of \( F \) and \( H \) by identifying \( v_F \) and \( v_H \) in a vertex labeled \( v \).

Figure 1. Left: Classes of graphs with \( epa = 3 \). Right: Regions of Venn diagram.

**Remark 12.** It is easy to see that all the following hold.

(i) If \( H \) is a connected graph of order \( n \geq 2 \), then \( G = H \circ K_1 \in R_0 \).

(ii) Let \( G_k^1 \) be a graph obtained from the cycle \( C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0, k \geq 2 \), by adding a vertex \( y \) and edges \( yx_0, yx_2 \). Then \( \gamma(G_k^1) = k + 1 \), \( G_k^1 \) is \( \gamma \)-excellent, \( V^-(G_k^1) = \{x_0, x_2\} \cup \bigcup_{r=1}^{k-1} \{x_{3r+1}, x_{3r+2}\} \) is a vertex cover of \( G \), and there is no \( \gamma \)-set of \( G_k^1 \) that contains both \( x_{3r+1} \) and \( x_{3r+2} \). Thus \( G_k^1 \) is in \( R_1 \).

(iii) The graph \( H_{10} \) depicted in Figure 2 is in \( A_3 \) and \( \gamma(H_{10}) = 3 \). It is obvious that no \( \gamma \)-set of \( H_{10} \) contains both \( u \) and \( v \). Hence \( H_{10} \in R_2 \). Consider now the graph \( G_k^2 = (C_{3k+1} \cdot H_{10})(x_0, w : z) \), where \( C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0, k \geq 2 \), is a cycle on \( 3k+1 \) vertices and \( w \) is any of the two common neighbors of \( u \) and \( v \) in \( H_{10} \). Since both \( C_{3k+1} \) and \( H_{10} \) are vc-graphs, by [4] we have that \( G_k^2 \) is vc-graph and \( \gamma(G_k^2) = \gamma(C_{3k+1}) + \gamma(H_{10}) - 1 \). Let \( D \) be an arbitrary \( \gamma \)-set of \( G_k^2 \), \( D_1 = D \cap V(H_{10}) \) and \( D_2 = D \cap V(C_{3k+1}) \). Then exactly one of the following holds.

(a) \( z \in D \), \( D_1 \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \) is a \( \gamma \)-set of \( C_{3k+1} \).

(b) \( z \notin D \), \( D_1 \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \cup \{x_0\} \) is a \( \gamma \)-set of \( C_{3k+1} \).
(c) \( z \not\in D, D_1 \cup \{w\} \) is a \( \gamma \)-set of \( H_{10} \) and \( D_2 \) is a \( \gamma \)-set of \( C_{3k+1} \).

Since no \( \gamma \)-set of \( H_{10} \) contains both \( u \) and \( v \), by (a), (b) and (c) we conclude that at most one of \( u \) and \( v \) is in \( D \). Thus \( G_2^k \in R_2 \).

(iv) \( C_{3k+1} \in R_3 \) for all \( k \geq 1 \).
(v) \( K_{2,n} \in R_4 \) for all \( n \geq 3 \).
(vi) \( K_{n,n} \in R_5 \) for all \( n \geq 3 \).

Thus all regions \( R_0, R_1, R_2, R_3, R_4, R_5 \) are nonempty.

![Figure 2. Graph \( H_{10} \) is in \( R_2 \).](image)

3. The Nonadjacent Case

In this section we show that \( 1 \leq \tau pa(G) \leq Epa(G) \leq 5 \) and we obtain necessary and sufficient conditions for \( \tau pa(G) = Epa(G) = j, 1 \leq j \leq 5 \).

We begin with an easy observation which is an immediate consequence by Lemma 2(iv) and Lemma 1.

**Observation 13.** Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). Then \( \gamma(G) - 1 \leq \gamma(G_{u,v,0}) \leq \gamma(G) \) and \( \gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1}) \) for \( k \geq 0 \).

**Theorem 14.** Let \( u \) and \( v \) be nonadjacent vertices of a graph \( G \). Then \( \gamma(G)-1 \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1 \). Moreover,

(i) \( \gamma(G) - 1 = \gamma(G_{u,v,1}) \) if and only if \( \gamma(G - \{u, v\}) = \gamma(G) - 2 \).

(ii) \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \) if and only if both \( u \) and \( v \) are \( \gamma \)-bad vertices of \( G \), \( u \not\in V^- (G - v) \) and \( v \not\in V^- (G - u) \). If \( \gamma(G_{u,v,1}) = \gamma(G) + 1 \), then \( x_1 \in V^- (G_{u,v,1}) \).

**Proof.** The left side inequality follows by Observation 13.

(i) \( \Rightarrow \) Assume the equality \( \gamma(G) - 1 = \gamma(G_{u,v,1}) \) holds and let \( M \) be any \( \gamma \)-set of \( G_{u,v,1} \). Then at least one and not more than two of \( x_1, u \) and \( v \) must be in \( M \). Hence \( M_1 = (M \setminus \{x_1\}) \cup \{u, v\} \) is a dominating set of \( G \) and \( \gamma(G) \leq |M_1| \leq |M| + 1 = \gamma(G_{u,v,1}) + 1 = \gamma(G) \). This immediately implies that \( M_1 \) is a \( \gamma \)-set of \( G \). Hence \( x_1 \in M \) and \( pm[x_1, M] = \{x_1, u, v\} \). Since \( M_1 \setminus \{u, v\} \) is a dominating set of \( G - \{u, v\} \), we have \( \gamma(G) - 2 \leq \gamma(G - \{u, v\}) \leq |M_1 \setminus \{u, v\}| = \gamma(G) - 2 \).
(i) Suppose now $\gamma(G - \{u, v\}) = \gamma(G) - 2$. Then for any $\gamma$-set $U$ of $G-\{u, v\}$, the set $U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$. This leads to $\gamma(G_{u,v,1}) \leq |U \cup \{x_1\}| = \gamma(G) - 1 \leq \gamma(G_{u,v,1})$.

Now we will prove the right side inequality. Let $D$ be any $\gamma$-set of $G$. If at least one of $u$ and $v$ is in $D$, then $D$ is a dominating set $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G)$. So, let neither $u$ nor $v$ belong to some $\gamma$-set of $G$. Then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G) + 1$.

(ii) Assume that $\gamma(G_{u,v,1}) = \gamma(G) + 1$. Then $u$ and $v$ are $\gamma$-bad vertices of $G$ and for any $\gamma$-set $D$ of $G$, $D \cup \{x_1\}$ is a $\gamma$-set of $G_{u,v,1}$. Hence $x_1 \in V^-(G_{u,v,1})$. Suppose $u \in V^-(G - v)$ and let $U$ be a $\gamma$-set of $G - \{u, v\}$. Then $U_1 = U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(U_1) = \gamma(G_{u,v,1}) \leq |U_1| = 1 + \gamma((G - v) - u) = \gamma(G - v) = \gamma(G)$, a contradiction. Thus $u \not\in V^-(G - v)$ and by symmetry, $v \not\in V^-(G - u)$.

(ii) Let both $u$ and $v$ be $\gamma$-bad vertices of $G$, $u \not\in V^-(G - v)$ and $v \not\in V^-(G - u)$. Hence $\gamma(G - \{u, v\}) \geq \gamma(G)$. Consider any $\gamma$-set $M$ of $G_{u,v,1}$. If one of $u$ and $v$ belongs to $M$, then $\gamma(G) + 1 = \gamma(G_{u,v,1})$. So, let $x_1$ is in each $\gamma$-set of $G_{u,v,1}$. But then $\gamma_n(x_1, M) = \{x_1, u, v\}$. Hence $\gamma(G_{u,v,1}) - 1 = \gamma(G - \{u, v\}) \geq \gamma(G) \geq \gamma(G_{u,v,1}) - 1$.

**Corollary 15.** Let $G$ be a noncomplete graph. Then $1 \leq \gamma_{pa}(G) \leq \overline{\gamma_{pa}}(G)$ and the following assertions hold.

(i) $\gamma_{pa}(G) = 1$ if and only if there are nonadjacent $\gamma$-bad vertices $u$ and $v$ of $G$ such that $u \not\in V^-(G - v)$ and $v \not\in V^-(G - u)$.

(ii) $\overline{\gamma_{pa}}(G) = 1$ if and only if $\gamma(G) = 1$.

**Proof.** Observation 13 implies $1 \leq \gamma_{pa}(G)$.

(i) Immediately by Theorem 14.

(ii) If $\gamma(G) = 1$, then clearly $\overline{\gamma_{pa}}(G) = 1$. If $\gamma(G) \geq 2$, then $G$ has 2 nonadjacent vertices at least one of which is $\gamma$-good. By Theorem 14, $\overline{\gamma_{pa}}(G) \geq 2$.

**Theorem 16.** Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Moreover,

(C) $\gamma(G_{u,v,2}) = \gamma(G)$ if and only if one of the following holds.

(i) There is a $\gamma$-set of $G$ which contains both $u$ and $v$.

(ii) At least one of $u$ and $v$ is in $V^-(G)$.

(D) $\gamma(G_{u,v,2}) = \gamma(G) + 1$ if and only if $u, v \not\in V^-(G)$ and any $\gamma$-set of $G$ contains at most one of $u$ and $v$.

**Proof.** For any $\gamma$-set $D$ of $G$, $D \cup \{x_2\}$ is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Suppose $\gamma(G_{u,v,2}) \leq \gamma(G) - 1$ and let $M$ be a $\gamma$-set of $G_{u,v,2}$. Then at least one of $x_1$ and $x_2$ is in $M$. If $x_1, x_2 \in M$, then $M_1 = (M \setminus \{x_1, x_2\}) \cup$
\{u, v\} is a dominating set of \(G\) and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. So let without loss of generality, \(x_1 \in M\) and \(x_2 \notin M\). If \(u \in M\) or \(v \in M\), then again \(M_1\) is a dominating set of \(G\) and \(|M_1| \leq \gamma(G_{u,v,2})\), a contradiction. Thus \(x_1 \in M\) and \(u, v \notin M\). But then \((M \setminus \{x_1\}) \cup \{u\}\) is a dominating set of \(G\), contradicting \(\gamma(G_{u,v,2}) < \gamma(G)\). Thus \(\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1\).

\((\mathbb{C}) \Rightarrow \) Let \(\gamma(G_{u,v,2}) = \gamma(G)\). Assume that neither (i) nor (ii) hold. Let \(M\) be a \(\gamma\)-set of \(G_{u,v,2}\). If \(x_1, x_2 \in M\), then \(M_1 = (M \setminus \{x_1, x_2\}) \cup \{u, v\}\) is a dominating set of \(G\) of cardinality not more than \(\gamma(G)\) and \(u, v \in M_1\), a contradiction. Let without loss of generality \(x_1 \in M\) and \(x_2 \notin M\). Since \(M \setminus \{x_1\}\) is no dominating set of \(G\), \(u \in \text{pn}[x_1, M]\). But then \(M_3 = (M \setminus \{x_1\}) \cup \{u\}\) is a \(\gamma\)-set of \(G\) and \(u \in V^-(G)\), a contradiction. Thus at least one of (i) and (ii) is valid.

\((\mathbb{C}) \Leftarrow \) If both \(u\) and \(v\) belong to some \(\gamma\)-set \(D\) of \(G\), then \(D\) is a dominating set of \(G_{u,n,2}\). Hence \(\gamma(G_{u,v,2}) = \gamma(G)\). Finally let \(u \in V^-(G)\) and \(D\) a \(\gamma\)-set of \(G - u\). Then \(D \cup \{x_1\}\) is a dominating set of \(G_{u,v,2}\) of cardinality \(\gamma(G)\). Thus \(\gamma(G_{u,v,2}) = \gamma(G)\).

\((\mathbb{D})\) Immediately by (\(\mathbb{C}\)) and \(\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1\).

\textbf{Corollary 17}. Let \(G\) be a noncomplete graph. Then the following assertions hold.

(i) \(\varepsilon_{\text{pa}}(G) \leq 2\) if and only if there are nonadjacent vertices \(u, v \in V(G) \setminus V^-(G)\) such that any \(\gamma\)-set of \(G\) contains at most one of them.

(ii) \(\overline{\text{Epa}}(G) = 2\) if and only if \(\gamma(G) \geq 2\) and each \(\gamma\)-set of \(G\) is a clique.

\textbf{Proof}. (i) Immediately by Theorem 16.

(ii) \(\Rightarrow \) Let \(\overline{\text{Epa}}(G) = 2\). By Corollary 15, \(\gamma(G) \geq 2\). Suppose \(G\) has a \(\gamma\)-set, say \(D\), which is not a clique. Then there are nonadjacent \(u, v \in D\). By Theorem 16(\(\mathbb{C}\)), \(\gamma(G_{u,v,2}) = \gamma(G)\), which contradict \(\overline{\text{Epa}}(G) = 2\). Thus, each \(\gamma\)-set of \(G\) is a clique.

(ii) \(\Leftarrow \) Let \(\gamma(G) \geq 2\) and let each \(\gamma\)-set of \(G\) be a clique. If \(G\) has a vertex \(z \in V^-(G)\) and \(M_z\) is a \(\gamma\)-set of \(G - z\), then \(M = M_z \cup \{z\}\) is a \(\gamma\)-set of \(G\) and \(z\) is an isolated vertex of the graph induced by \(M\), a contradiction. Thus \(V^-(G)\) is empty. Now by Theorem 16(\(\mathbb{D}\)), \(\overline{\text{Epa}}(G) = 2\).

\textbf{Example 18}. The join of two graphs \(G_1\) and \(G_2\) with disjoint vertex sets is the graph, denoted by \(G_1 + G_2\), with the vertex set \(V(G_1) \cup V(G_2)\) and edge set \(E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}\). Let \(\gamma(G_i) \geq 3\), \(i = 1, 2\). Then \(\gamma(G_1 + G_2) = 2\) and each \(\gamma\)-set of \(G_1 + G_2\) contains exactly one vertex of \(G_i\), \(i = 1, 2\). Hence \(\overline{\text{Epa}}(G_1 + G_2) = 2\). In particular, \(\overline{\text{Epa}}(K_{m,n}) = 2\) when \(m, n \geq 3\).

\textbf{Theorem 19}. Let \(u\) and \(v\) be nonadjacent vertices of a graph \(G\). Then \(\gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G) + 1\). Moreover, \(\gamma(G_{u,v,3}) = \gamma(G)\) if and only if at least one of the following holds.
(i) \( u \in V^-(G) \) and \( v \) is a \( \gamma \)-good vertex of \( G - u \),
(ii) \( v \in V^-(G) \) and \( u \) is a \( \gamma \)-good vertex of \( G - v \).

**Proof.** If \( D \) is a dominating set of \( G \), then \( D \cup \{x_2\} \) is a dominating set of \( G_{u,v,3} \). Hence \( \gamma(G_{u,v,2}) \leq \gamma(G) + 1 \). We already know that \( \gamma(G) \leq \gamma(G_{u,v,2}) \) and \( \gamma(G_{u,v,2}) \leq \gamma(G_{u,v,3}) \). Thus \( \gamma(G) \leq \gamma(G_{u,v,3}) \).

\( \Rightarrow \) Let \( \gamma(G_{u,v,3}) = \gamma(G) \) and let \( M \) be a \( \gamma \)-set of \( G_{u,v,3} \) such that \( Q = M \cap \{x_1,x_2,x_3\} \) has minimum cardinality. Clearly \(|Q| = 1\). If \( \{x_2\} = Q \), then \( M\{x_2\} \) is a dominating set of \( G \), contradicting \( \gamma(G_{u,v,3}) = \gamma(G) \). Let without loss of generality \( \{x_1\} = Q \). This implies \( v \in M \), \( x_3 \in pn[v,M] \) and \( pn[x_1,M] = \{u,x_1,x_2\} \). Then \( M_2 = (M\{x_1\}) \cup \{u\} \) is a \( \gamma \)-set of \( G \), \( pn[u,M_2] = \{u\} \) and \( v \in M_2 \); hence (i) holds.

\( \Leftarrow \) Let without loss of generality (i) is true. Then there is a \( \gamma \)-set \( D \) of \( G \) such that \( u,v \in D \) and \( D \setminus \{u\} \) is a \( \gamma \)-set of \( G - u \). But then \( (D \setminus \{u\}) \cup \{x_1\} \) is a dominating set of \( G_{u,v,3} \), which implies \( \gamma(G) \geq \gamma(G_{u,v,3}) \).

**Corollary 20.** Let \( G \) be a noncomplete graph. Then the following holds.

(E) \( \overline{\text{pa}}(G) \leq 3 \) if and only if there is a pair of nonadjacent vertices \( u \) and \( v \) such that neither (i) nor (ii) is valid, where

(i) \( u \in V^-(G) \) and \( v \) is a \( \gamma \)-good vertex of \( G - u \),
(ii) \( v \in V^-(G) \) and \( u \) is a \( \gamma \)-good vertex of \( G - v \).

(F) \( \overline{\text{pa}}(G) = \overline{\text{Epa}}(G) = 3 \) if and only if all vertices of \( G \) are \( \gamma \)-good, \( V^-(G) \) is empty and for every 2 nonadjacent vertices \( u \) and \( v \) of \( G \) there is a \( \gamma \)-set of \( G \) which contains them both.

**Proof.** (F) \( \Rightarrow \) Let \( \overline{\text{pa}}(G) = \overline{\text{Epa}}(G) = 3 \). If \( u \in V^-(G) \) and \( D \) is a \( \gamma \)-set of \( G - u \), then for \( u \) and each \( v \in D \) is fulfilled (i) of Theorem 19. But then \( \overline{\text{Epa}}(G) \neq 3 \), a contradiction. So, \( V^-(G) \) is empty. Suppose that \( G \) has \( \gamma \)-bad vertices. Then there is a \( \gamma \)-bad vertex which is nonadjacent to some other vertex of \( G \). But Theorem 16(D) implies \( \overline{\text{pa}}(G) < 3 \), a contradiction. Thus all vertices of \( G \) are \( \gamma \)-good. Now let \( u,v \in V(G) \) be nonadjacent. If there is no \( \gamma \)-set of \( G \) which contains both \( u \) and \( v \), then by Theorem 16(D) we have \( \gamma(G_{u,v,2}) = \gamma(G) + 1 \), a contradiction.

(F) \( \Leftarrow \) Let \( V^-(G) \) be empty and for each pair \( u,v \) of nonadjacent vertices of \( G \) there is a \( \gamma \)-set \( D_{uv} \) of \( G \) with \( u,v \in D_{uv} \). By Theorem 19, \( \gamma(G_{u,v,3}) = \gamma(G) + 1 \), and by Theorem 16, \( \gamma(G_{u,v,2}) = \gamma(G) \). Hence \( pa(u,v) = 3 \).

**Example 21.** Denote by \( \mathcal{U} \) the class of all graphs \( G \) with \( \overline{\text{pa}}(G) = \overline{\text{Epa}}(G) = 3 \). Then all the following holds. (a) Circulant graphs \( C(2k + 1; \{\pm 1, \pm 2, \ldots, \pm (k - 1)\}) \in \mathcal{U} \) for all \( k \geq 1 \). (b) Let \( G \) be a nonconnected graph. Then \( G \in \mathcal{U} \) if and only if \( G \) has no isolated vertices and each its component is either in \( \mathcal{U} \) or is complete.
Theorem 22. Let $u$ and $v$ be nonadjacent vertices of a graph $G$. Then $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. Moreover, the following assertions are valid.

1. $\gamma(G_{u,v,4}) = \gamma(G) + 2$ if and only if $\gamma(G_{u,v,1}) = \gamma(G) + 1$.
2. If $\gamma(G_{u,v,1}) = \gamma(G)$ and $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{2, 3\}$, then $\gamma(G_{u,v,4}) = \gamma(G) + 1$.
3. Let $\gamma(G_{u,v,3}) = \gamma(G)$. Then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$ and the equality holds if and only if $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$.
4. $\gamma(G_{u,v,4}) = \gamma(G)$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Proof. Since $\gamma(G) \leq \gamma(G_{u,v,3})$ (by Theorem 19) and $\gamma(G_{u,v,3}) \leq \gamma(G_{u,v,4})$ (by Observation 13), we have $\gamma(G) \leq \gamma(G_{u,v,4})$. Let $S$ be a $\gamma$-set of $G$. Then $S \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$, which leads to $\gamma(G_{u,v,4}) \leq \gamma(G) + 2$.

Claim 1. If $\gamma(G_{u,v,1}) \leq \gamma(G)$, then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Proof. Assume that $v$ is a $\gamma$-bad vertex of $G$, $u \in V^-(G - v)$ and $R$ a $\gamma$-set of $G - \{u, v\}$. Then $|R| = \gamma((G - v) - u) = \gamma(G - v) - 1 = \gamma(G) - 1$ and $R \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) \leq |R| + 2 = \gamma(G) + 1$.

Assume now that $D$ is a $\gamma$-set of $G$ with $u \in D$. Then $D \cup \{x_3\}$ is a dominating set of $G_{u,v,4}$. Hence again $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. Now by Theorem 14 we immediately obtain the required.

(G) Let $\gamma(G_{u,v,4}) = \gamma(G) + 2$. By Claim 1, $\gamma(G_{u,v,1}) > \gamma(G)$ and by Theorem 14, $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

Let now $\gamma(G_{u,v,1}) = \gamma(G) + 1$. By Theorem 14, $u$ and $v$ are $\gamma$-bad vertices of $G$, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$. Let $M$ be a $\gamma$-set of $G_{u,v,4}$ such that $R = M \cap \{x_1, x_2, x_3, x_4\}$ has minimum cardinality. Clearly $|R| \in \{1, 2\}$. Assume first $|R| = 1$ and without loss of generality $\{x_2\} = M$. Then $M \setminus \{x_2\}$ is a dominating set of $G$ with $v \in M \setminus \{x_2\}$. Since $v$ is a $\gamma$-bad vertex of $G$, $|M \setminus \{x_2\}| > \gamma(G)$ and then $\gamma(G_{u,v,4}) = |M| > \gamma(G) + 1$. Let now $|R| = 2$ and without loss of generality $x_1, x_4 \in M$. Since $|M \cap \{x_1, x_2, x_3, x_4\}|$ is minimum, $u, v \notin M$ and $M \setminus \{x_1, x_4\}$ is a dominating set of $G - \{u, v\}$. But then $\gamma(G_{u,v,4}) = 2 + |M \setminus \{x_1, x_4\}| \geq 2 + \gamma((G - u) - v) \geq 2 + \gamma(G - u) = 2 + \gamma(G)$.

(H) Let $\gamma(G_{u,v,1}) = \gamma(G)$. By Claim 1, $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. If $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{1, 2\}$, then since $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,i})$, we obtain $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(i) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Hence at least one of (i) and (ii) of Theorem 19 holds, and by (H), $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Assume that the equality holds. If $\gamma(G - \{u, v\}) = \gamma(G) - 2$, then for any $\gamma$-set $U$ of $G - \{u, v\}$, $U \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) = \gamma(G)$, a contradiction.
Let now $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and without loss of generality condition (i) of Theorem 19 be satisfied. Suppose $\gamma(G_{u,v,4}) = \gamma(G)$. Hence for each $\gamma$-set $M$ of $G_{u,v,4}$ are fulfilled: $x_1, x_4 \in M$, $x_2, x_3, u, v \notin M$, $pn[x_1, M] = \{x_1, x_2, u\}$ and $pn[x_4, M] = \{x_3, x_4, v\}$. But then $\gamma(G - \{u, v\}) = \gamma(G) - 2$, a contradiction. Thus $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

(\[) If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,3}) = \gamma(G)$ and by (\[G), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Now let $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Thus $\gamma(G_{u,v,4}) = \gamma(G)$.

**Theorem 23.** Let $u$ and $v$ be nonadjacent vertices of a graph $G$. If $\gamma(G_{u,v,k}) = \gamma(G)$, then $k \leq 4$. If $k \geq 5$, then $\gamma(G_{u,v,k}) > \gamma(G)$. If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,5}) = \gamma(G) + 1$.

**Proof.** By Theorem 22, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. If $\gamma(G_{u,v,4}) > \gamma(G)$, then $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$ because of Observation 13. So, let $\gamma(G_{u,v,4}) = \gamma(G)$. By Theorem 22(\[H), $\gamma(G - \{u, v\}) = \gamma(G) - 2$. But then for each $\gamma$-set $D$ of $G - \{u, v\}$, the set $D \cup \{x_1, x_3, x_5\}$ is a dominating set of $G_{u,v,5}$. Hence $\gamma(G_{u,v,5}) \leq \gamma(G) + 1$. Let now $M$ be a $\gamma$-set of $G_{u,v,5}$. Then at least one of $x_2, x_3, x_4$ is in $M$ and hence $\gamma(G_{u,v,5}) = |M| \geq \gamma(G) + 1$. Thus $\gamma(G_{u,v,5}) = \gamma(G) + 1$.

Now using again Observation 13 we conclude that $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$.

**Corollary 24.** Let $G$ be a noncomplete graph. Then $\overline{\text{epa}}(G) \leq \overline{\text{epa}}(G) \leq 5$. Moreover, the following holds.

(i) $\overline{\text{epa}}(G) = 5$ if and only if there are nonadjacent vertices $u$ and $v$ of $G$ with $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(ii) $\overline{\text{epa}}(G) = 5$ if and only if $G$ is edgeless.

(iii) $\text{epa}(G) = \overline{\text{epa}}(G) = 4$ if and only if for each pair $u, v$ of nonadjacent vertices of $G$, $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and at least one of the following holds:

(a) $u \in V^-(G)$ and $v$ is a $\gamma$-good vertex of $G - u$,

(b) $v \in V^-(G)$ and $u$ is a $\gamma$-good vertex of $G - v$.

**Proof.** By Theorem 23, $\overline{\text{epa}}(G) \leq 5$.

(i) $\Rightarrow$ Let $\overline{\text{epa}}(G) = 5$. Then there is a pair $u, v$ of nonadjacent vertices of $G$ such that $\gamma(G_{u,v,4}) = \gamma(G)$. Now by Theorem 22(\[H), $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

(i) $\Leftarrow$ Let $\gamma(G - \{u, v\}) = \gamma(G) - 2$ and $D$ be a $\gamma$-set of $G - \{u, v\}$, where $u$ and $v$ are nonadjacent vertices of $G$. Hence $D_1 = D \cup \{x_1, x_3\}$ is a dominating set of $G_{u,v,4}$ and $|D_1| = \gamma(G)$. This implies $\gamma(G_{u,v,4}) = \gamma(G)$. The result now follows by Theorem 23.

(ii) If $G$ has no edges, then the result is obvious. So let $G$ have edges and $\overline{\text{epa}}(G) = 5$. Then for any 2 nonadjacent vertices $u$ and $v$ of $G$ is satisfied
\[ \gamma(G - \{u, v\}) = \gamma(G) - 2 \text{ (by (i))}. \] Hence we can choose \(u\) and \(v\) so that they have a neighbor in common, say \(w\). But then \(w\) is a \(\gamma\)-bad vertex of \(G - u\) which implies \(v \notin V^{-}(G - u)\). This leads to \(\gamma(G - \{u, v\}) \geq \gamma(G) - 1\), a contradiction.

(iii) \(\Rightarrow\) Let \(\tau pa(G) = Epa(G) = 4\). Then for each two nonadjacent \(u, v \in V(G)\) we have \(\gamma(G) = \gamma(G_{u,v,3}) < \gamma(G_{u,v,4})\). Now by Theorem 22(\(G\)), \(\gamma(G - \{u, v\}) \geq \gamma(G) - 1\) and by Theorem 19, at least one of (a) and (b) is valid.

(iii) \(\Leftrightarrow\) Consider any two nonadjacent vertices \(u, v\) of \(G\). Then \(\gamma(G - \{u, v\}) \geq \gamma(G) - 1\) and at least one of (a) and (b) is valid. Theorem 19 now implies \(\gamma(G) = \gamma(G_{u,v,3})\), and by Theorem 22, \(pa(u, v) = 4\).

Example 25. Let \(G_n\) be the Cartesian product of two copies of \(K_n, n \geq 2\). We consider \(G_n\) as an \(n \times n\) array of vertices \(\{x_{i,j} | 1 \leq i \leq j \leq n\}\), where the closed neighborhood of \(x_{i,j}\) is the union of the sets \(\{x_{1,j}, x_{2,j}, \ldots, x_{n,j}\}\) and \(\{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}\). Note that \(V(G_n) = V^{-}(G_n)\) and \(\gamma(G_n) = n\) [6]. It is easy to see that the following sets are \(\gamma\)-sets of \(G_n - x_{1,1}: D_i = \{x_{2,i}, x_{3,i}, \ldots, x_{n,i}\}\), \(i = 2, 3, \ldots, n\), where \(x_{k,j} = x_{k,j-n+1}\) for \(j > n\) and \(2 \leq k \leq n\). Since \(D = \bigcup_{i=2}^{n} D_i = V(G_n) \setminus N[x_{1,1}]\), all \(\gamma\)-bad vertices of \(G_n - x_{1,1}\) are the neighbors of \(x_{1,1}\) in \(G_n\). Since each vertex of \(D\) is adjacent to some neighbor of \(x_{1,1}\), \(V^{-}(G_n - x_{1,1})\) is empty. Now by Theorem 19 we have \(pa(x_{1,1}, y) \geq 4\), and by Theorem 22(\(G\)), \(pa(x_{1,1}, y) < 5\). Thus \(pa(x_{1,1}, y) = 4\). By reason of symmetry, we obtain \(\tau pa(G_n) = Epa(G_n) = 4\).

4. Observations and Open Problems

A constructive characterization of the trees \(T\) with \(i(T) \equiv \gamma(T)\), and therefore a constructive characterization of the trees \(T\) with \(Epa(T) = 2\) (by Corollary 7), was provided in [9].

Problem 26. Characterize all unicyclic graphs \(G\) with \(Epa(G) = 2\).

Problem 27. Find results on \(\gamma\)-excellent graphs \(G\) with \(Epa(G) = 2\).

Problem 28. Characterize all graphs \(G\) with \(\tau pa(G) = Epa(G) = 4\).

Corollary 29. Let \(G\) be a connected noncomplete graph with edges. Then

(i) \(2 \leq EPA(G) + \text{Epa}(G) \leq 8\),

(ii) \(2 \leq EPA(G) + \tau pa(G) \leq 7\),
(iii) $3 \leq E_{pa}(G) + \overline{E}_{pa}(G) \leq 8$,
(iv) $3 \leq E_{pa}(G) + \tau_{pa}(G) \leq 7$.

**Proof.** (i)–(iv) The left-side inequalities immediately follow by Corollary 5 and Corollary 15. The right-side inequalities hold because of Corollary 10 and Corollary 24.

Note that all bounds stated in Corollary 29 are attainable. We leave finding examples demonstrating this to the reader.

**Problem 30.** Characterize all graphs $G$ that attain the bounds in Corollary 29.

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**References**


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