LIST EDGE COLORING OF PLANAR GRAPHS WITHOUT 6-CYCLES WITH TWO CHORDS

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Abstract

A graph $G$ is edge-$L$-colorable if for a given edge assignment $L = \{L(e) : e \in E(G)\}$, there exists a proper edge-coloring $\varphi$ of $G$ such that $\varphi(e) \in L(e)$ for all $e \in E(G)$. If $G$ is edge-$L$-colorable for every edge assignment $L$ such that $|L(e)| \geq k$ for all $e \in E(G)$, then $G$ is said to be edge-$k$-choosable. In this paper, we prove that if $G$ is a planar graph without 6-cycles with two chords, then $G$ is edge-$k$-choosable, where $k = \max\{7, \Delta(G) + 1\}$, and is edge-$t$-choosable, where $t = \max\{9, \Delta(G)\}$.

Keywords: planar graph, edge choosable, list edge chromatic number, chord.

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1. Introduction

Graphs considered in this paper are finite, simple and undirected. The terminologies and notations used but undefined in this paper can be found in [2]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $\Delta$ and $\delta$) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively. A cycle $C$ of length $k$ is called a $k$-cycle in the graph $G$. If $xy \in E(G) \setminus E(C)$ and $x, y \in V(C)$, $xy$ is called to be a chord of $C$ in the graph $G$.

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An edge coloring of a graph $G$ is a mapping $\varphi$ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ for some positive integer $k$. An edge coloring is called proper if every two adjacent edges receive different colors. The edge chromatic number $\chi'(G)$ is the smallest integer $k$ such that $G$ has a proper edge-coloring into the set $\{1, 2, \ldots, k\}$.

We say that $L$ is an edge assignment for the graph $G$ if it assigns a list $L(e)$ of possible colors to each edge $e$ of $G$. If $G$ has a proper edge-coloring $\varphi$ such that $\varphi(e) \in L(e)$ for each edge $e$ of $G$, then we say that $G$ is edge-$L$-colorable or $\varphi$ is an edge-$L$-coloring of $G$. The graph $G$ is edge-$k$-choosable if it is edge-$L$-colorable for every edge assignment $L$ satisfying $|L(e)| \geq k$ for all $e \in E(G)$. The list edge chromatic number $\chi'_{\text{list}}(G)$ of $G$ is the smallest $k$ such that $G$ is edge-$k$-choosable.

On the list edge coloring of a graph, there is a celebrated conjecture known as the list edge coloring conjecture, which was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris (see [8, 13]).

**Conjecture 1** [9]. If $G$ is a multigraph, then $\chi'_{\text{list}}(G) = \chi'(G)$.

The conjecture has been proved for a few classes of graphs, such as graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [4], outerplanar graphs [19], bipartite multigraphs [4, 7], complete graphs of odd order [9]. Vizing [15] proposed a weaker conjecture than Conjecture 1.

**Conjecture 2** [9]. Every graph $G$ is edge-$(\Delta(G) + 1)$-choosable.

Harris [10] showed that $\chi'_{\text{list}}(G) \leq 2\Delta(G) - 2$ if $G$ is a graph with $\Delta(G) \geq 3$. This implies Conjecture 2 for the case $\Delta(G) = 3$. Juvan et al. [14] settled the case for $\Delta(G) = 4$ in 1999. And there are some other special cases of Conjecture 2 which have been confirmed, such as complete graphs [8], graphs with girth at least $8\Delta(G)(\ln \Delta(G)(G) + 1.1)$ [15], planar graphs with $\Delta(G) \geq 8$ [1], and planar graphs with $\Delta(G) \neq 5$ and without intersecting 3-cycles [20]. Suppose that $G$ is a planar graph without $k$-cycles for some fixed integer $3 \leq k \leq 6$. Then it was proved that Conjecture 2 holds if $G$ satisfies one of the four following conditions:

(i) either $k = 3$ or $k = 4$ and $\Delta(G) \neq 5$ [22],

(ii) $k = 4$ [17],

(iii) $k = 5$ [20],

(iv) $k = 6$ and $\Delta(G) \neq 5$ [18].

Other related known results on this topic can be found in [5, 11, 12, 16].

Cai [6] proved that if $G$ is a planar graph without chordal 6-cycles, then $G$ is edge-$k$-choosable, where $k = \max\{8, \Delta(G) + 1\}$. In this paper, we will strengthen this result and obtain that if $G$ is a planar graph and each 6-cycle of $G$ contains at most one chord, then $\chi'_{\text{list}}(G) \leq \max\{7, \Delta(G) + 1\}$ and $\chi'_{\text{list}}(G) \leq \max\{9, \Delta(G)\}$.
2. Main Results and Their Proofs

In the section, we always assume that all graphs are planar graphs that have been embedded in the plane and $G$ is a planar graph without 6-cycles with two chords. We use $d_G(x)$, or simply $d(x)$, to denote the degree of a vertex $x$ in $G$. For $f \in F(G)$, if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk, then we write $f = u_1u_2 \cdots u_nu_1$. The degree of a $f$, denoted by $d(f)$, is the number of edges incident with $f$, where each cut-edge is counted twice. We denote by $d(f)$ the minimum degree of vertices incident with the face $f$. A vertex (face) $x$ is called to be a $k$-vertex ($k$-face), $k^+$-vertex ($k^+$-face) and $k^-$-vertex ($k^-$-face), if $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, respectively. $f_i(v)$ is the number of $i$-faces incident with $v$ for each $v \in V(G)$.

First, we give some properties on $G$.

Lemma 3. If $v$ is a $5^+$-vertex of $G$, then $f_5(v) \leq \lceil \frac{3}{2} d(v) \rceil$.

Proof. Since $G$ contains no 6-cycles with two chords, $v$ is not incident with four consecutive 3-faces. So $f_3(v) \leq \lceil \frac{3}{2} d(v) \rceil$. \hfill \blacksquare

Lemma 4. Let $u$ be a 4-vertex of $G$.

1. If $f_3(u) = 3$, then $f_4(u) = 0$, that is, $u$ is incident with a $5^+$-face.
2. If $f_3(u) = 2$, then $f_4(u) \leq 1$.

Proof. Let neighbors of $u$ be $u_1, u_2, u_3, u_4$ and faces incident with $u$ be $f_1, f_2, f_3, f_4$ in the clockwise order, where $f_1$ is incident with $u_1, u_2$.

1. Without loss of generality, we assume that $f_1, f_2, f_3$ are 3-faces. If $f_4$ is a 4-face $uu_1vu_4$, then the 6-cycle $uu_2vu_3vu_1u$ contains two chords $uu_3$ and $uu_4$, a contradiction. So $d(f_1) \geq 5$, that is, $f_5(u) = 1$.

2. Suppose that two 3-faces incident with $u$ are not adjacent, without loss of generality, we assume that $f_1, f_3$ are 3-faces. If $f_2$ is a 4-face $uu_2vu_3u$, then the 6-cycle $uu_1vu_2vu_3u$ contains two chords $uu_2$ and $uu_3$, a contradiction. So $d(f_2) \geq 5$. By the same argument, we have $d(f_4) \geq 5$.

Suppose that two 3-faces incident with $u$ are adjacent, without loss of generality, we assume that $f_1, f_2$ are 3-faces. If $f_3$ is a 4-face $uu_3vu_4u$, then we must have $v = u_1$. Since $d(u) = 4$, $d(u_4) \geq 5$. Thus if $f_4$ is a 4-face $uu_1vu_4u$, then we also have $w = u_3$, it is impossible. So $d(f_4) \geq 5$. By the same argument, if $d(f_4) = 4$, then $d(f_5) \geq 5$. Hence $f_4(u) \leq 1$. \hfill \blacksquare

Lemma 5. $G$ satisfies at least one of the following conditions.

1. $G$ has an edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$.
2. $G$ has an even cycle $C = v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$. 


(3) $G$ has a 6-vertex $u$ with five neighbors $v, w, x, y, z$ such that $d(v) = d(y) = 3$ and $vw, xy, yz \in E(G)$ (see Figure 1).

![Figure 1. The subgraph for Lemma 5(3).](image)

**Proof.** Let $G$ be a minimal counterexample to the lemma. It is easy to check that $G$ is connected. By the choice of $G$, we have the following observations.

(P1) For any edge $uv$, $d(u) + d(v) \geq \max\{9, \Delta(G) + 3\}$ by (1). Then $\delta(G) \geq 3$ and all neighbors of a $i$-vertex must be $(9 - i)^+$-vertices, where $i = 3, 4$ or 5.

(P2) Let $G_3$ be the subgraph induced by the edges incident with 3-vertices of $G$. Then $G_3$ is a forest.

By (P1), every two 3-vertices are not adjacent, and it follows that $G_3$ is a bipartite subgraph. By (2), $G_3$ contains no even cycles. So $G_3$ is a forest and (P2) holds. Let $V_1$ be the set of 3-vertices of $G$. Thus for any component of $G_3$, we select a vertex $u \notin V_1$ as a root of the tree. Then every 3-vertex has exactly two children. If $uv \in E(G_3)$, $u \in V_1$ and $v$ is a child of $u$, then $v$ is called a 3-master of $u$. Note that each 3-vertex has exactly two 3-masters and each vertex of degree at least 6 can be the 3-master of at most one 3-vertex.

According to the Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ of a planar graph $G$, we have

$$
\sum_{v \in V(G)} (3d(v)-10) + \sum_{f \in F(G)} (2d(f)-10) = -10(|V(G)|-|E(G)|+|F(G)|) = -20 < 0.
$$

Now we define the initial weight function on $V(G) \cup F(G)$ by letting $w(x) = 3d(x) - 10$ for any $x \in V(G)$ and $w(x) = 2d(x) - 10$ for any $x \in F(G)$. Thus the total sum of weights is the negative number $-20$. We use the following rules to redistribute the initial charge that leads to a new charge $w'(x)$.

**R1.** Every 3-vertex $v$ receives $\frac{1}{2}$ from each of its 3-masters.

**R2.** Let $f = uu'vv'$ be a 4-face in $G$ with $d(u) \leq \min\{d(u'), d(v), d(v')\}$. If $d(u) \geq 4$, then $f$ receives $\frac{1}{2}$ from each of its incident vertices. Otherwise, $f$ receives nothing from $u$, receives $\frac{1}{2}$ from $v$, $\frac{3}{4}$ from $u'$ and $\frac{3}{4}$ from $v'$. 
R3. Let $f$ be a 3-face incident with a $4^+$-vertex $v$. Then $f$ receives $a$ from $v$.

R3.1. If $d(v) = 4$, then

$$
a = \begin{cases} 
\frac{1}{2} & \text{if } f_4^-(v) = 4 \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located in the middle of three consecutive } 3\text{-faces incident with } v, \\
\frac{3}{4} & \text{if } f_3(v) = 2 \text{ and } f_4(v) = 1, \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located in one side of three consecutive } 3\text{-faces incident with } v, \\
1 & \text{otherwise.}
\end{cases}
$$

R3.2. If $d(v) = 5$, then

$$
a = \begin{cases} 
\frac{3}{2} & \text{if } f_3(v) = 3 \text{ and one of the following conditions holds:} \\
& \text{(i) } f_4(v) = 1, \\
& \text{(ii) } f_4(v) = 0 \text{ and } f \text{ is located in the middle of three consecutive } 3\text{-faces incident with } v, \\
& \text{(iii) } \text{two faces adjacent to } f \text{ at } v \text{ are } 5^+\text{-faces,} \\
\frac{7}{4} & \text{otherwise.}
\end{cases}
$$

R3.3. If $d(v) \geq 6$, then

$$
a = \begin{cases} 
\frac{3}{2} & \text{if } f \text{ is adjacent to two non-adjacent } (3, 6, 6^+)\text{-faces at } v \\
& \text{and } d(v) = 6, \\
\frac{7}{4} & \text{if } f \text{ is incident with a } 3\text{-vertex,} \\
\end{cases}
$$

In the following, we will check that $w'(x) \geq 0$ for all elements $x \in V(G) \cup F(G)$ to obtain the following obvious contradiction.

$$0 \leq \sum_{v \in V \cup F} w'(x) = \sum_{v \in V \cup F} w(x) = -20.$$

First, we consider the final charge of any face $f$. If $d(f) \geq 5$, then it retains its initial charge and it follows that $w'(f) = w(f) = 2d(f) - 10 \geq 0$. Suppose that $d(f) = 4$. Then $w(f) = 8 - 10 = -2$. If $\delta(f) = 3$, then $w'(f) = w(f) + \frac{3}{2} + \frac{1}{2} + \frac{3}{4} = 0$ by R2. Otherwise $w'(f) = w(f) + 4 \times \frac{1}{2} = 0$. So $w'(f) \geq 0$ if $d(f) = 4$.

Suppose that $d(f) = 3$. Then $w(f) = 6 - 10 = -4$. If $\delta(f) = 3$, then $f$ is incident with two $6^+$-vertices by (P1) and it follows that $w'(f) = w(f) + 2 + 2 = 0$ by R3.3. If $\delta(f) \geq 5$, then $f$ receives at least $\frac{3}{2}$ from each of its incident vertices by R3.2 and R3.3, so $w'(f) \geq w(f) + 3 \times \frac{3}{2} > 0$. In the following, we assume that $\delta(f) = 4$. Let $f$ be a 3-face $uvw$ such that $d(u) = 4$. Then $d(v) \geq 5$ and $d(w) \geq 5$ by (P1). According to R3.1, we consider the following three cases.

Case 1. $f$ receives $\frac{1}{2}$ from $u$, that is, $f_4^-(u) = 4$ or $f_3(u) = 3$ and $f$ is located in the middle of three consecutive 3-faces incident with $u$. 

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It suffices to check that \( f \) receives at least \( \frac{7}{4} \) from each of \( v \) and \( w \). Thus \( w'(f) \geq w(f) + \frac{1}{2} + \frac{3}{2} + \frac{7}{4} = 0 \), a contradiction.

**Subcase 1.1.** \( f_4(u) = 4 \), that is, \( u \) is incident with four faces of degree at most 4. Then \( f_3(u) = 4 \) or \( f_3(u) = 1 \) by Lemma 4. If \( f_3(u) = 1 \), then all faces adjacent to \( f \) are \( 4^+ \)-faces, and it follows from R3.2 and R3.3 that \( f \) receives at least \( \frac{7}{4} \) from \( v, w \) respectively. If \( f_3(u) = 4 \), then any 3-face incident with \( u \) must be adjacent to a \( 5^+ \)-face and it follows from R3.2 and R3.3 that \( f \) receives at least \( \frac{7}{4} \) from \( v, w \) respectively.

**Subcase 1.2.** \( f_3(u) = 3 \) and \( f \) is located in the middle of three consecutive 3-faces incident with \( u \). If \( d(v) \geq 6 \), then two faces adjacent to \( f \) at \( v \) are not \( (3, 6, 6^+) \)-faces (since \( d(u) = 4 \) and \( uv \) is incident with two \( (4, 5^+, 6^+) \)-faces) and it follows from R3.3 that \( f \) receives at least \( \frac{7}{4} \) from \( v \). Suppose that \( d(v) = 5 \). Let five faces incident with \( v \) be \( f, f_1, \ldots, f_4 \) in clockwise order, where \( uv \) is incident with \( f \) and \( f_1 \) (see Figure 2). Then \( d(f_4) \geq 5 \) since \( G \) contains no 6-cycles with two chords. If \( f_3(v) = 3 \), then \( f_4(v) = 0 \), and \( f \) is not located in the middle of three consecutive 3-faces incident with \( v \) (since \( d(f_4) \geq 5 \)), and only one face adjacent to \( f \) at \( v \) is a \( 5^+ \)-face (since \( d(f_1) = 3 \)). So \( f \) receives at least \( \frac{7}{4} \) from \( v \) by R3.2. By symmetry, \( f \) receives at least \( \frac{7}{4} \) from \( w \).

![Figure 2. \( d(u) = 4 \), \( f_3(u) = 3 \) and \( f \) is located in the middle of three consecutive 3-faces incident with \( u \).](image)

**Case 2.** \( f \) receives \( \frac{3}{4} \) from \( u \). Then \( f_3(u) = 2 \) and \( f_4(u) = 1 \), or \( f_3(v) = 3 \) and \( f \) is located in the one side of these 3-faces by R3.1. Suppose that \( f_3(u) = 2 \) and \( f_4(u) = 1 \). Then the induced subgraph of \( u \) and its neighbors must be isomorphic to a configuration as Figure 3, where \( w = x \) or \( w = y \). If \( vx \) is incident with two 3-faces \( uwv \) and \( vxx'v \), then the 6-cycle \( xx'vyuzx \) contains two chords \( uv \) and \( uy \), a contradiction. If \( vx \) is incident with a 4-face \( vxx'x''v \), then the 6-cycle \( xx'x''vyuzx \) contains two chords \( uv \) and \( xv \), a contradiction, too. So \( vx \) is incident with a \( 5^+ \)-face. By the same argument, \( vy \) is incident with a \( 5^+ \)-face, too. By R3.2 and R3.3, \( f \) receives at least \( \frac{7}{4} \) from \( v \), at least \( \frac{3}{2} \) from \( w \). So \( w'(f) \geq w(f) + \frac{3}{4} + \frac{3}{2} + \frac{7}{4} = 0 \) by R3.
Suppose that $u$ is incident with three 3-faces and $f$ is located in the one side of these 3-faces. Then $u$ is incident with a $5^+$-face by Lemma 4. Without loss of generality, we assume that $uv$ is incident with two 3-faces. By the similar arguments with Subcase 1.2, $v$ sends at least $\frac{7}{4}$ to $f$. So $w'(f) \geq w(f) + \frac{3}{4} + \frac{3}{2} + \frac{7}{4} = 0$.

**Case 3.** $f$ receives 1 from $u$. Since $d(v) \geq 5$, $v$ sends at least $\frac{3}{2}$ to $f$ by R3.2 and R3.3. Similarly, $w$ sends at least $\frac{3}{2}$ to $f$. So $w'(f) \geq w(f) + 1 + \frac{3}{2} + \frac{3}{2} = 0$.

Till now, we have checked that $w'(f) \geq 0$ for any face $f \in F(G)$. Next, we begin to check the new charge of all vertices of $G$. Let $v$ be a vertex of $G$. If $d(v) = 3$, then $w'(v) \geq w(v) + 2 \times \frac{1}{2} = 0$ by R1 since $v$ has exactly two 3-masters. Suppose that $d(v) = 4$. If $f_4^-(v) \leq 2$, then $w'(v) = w(v) - 2 \times 1 = 0$ by R3.1. If $f_4^-(v) = 4$, then $w'(v) = w(v) - 4 \times \frac{1}{2} = 0$ by R3.1. If $f_4^-(v) = 3$, then $f_3^-(v) = 3$ and $f_4^+(v) = 0$, or $f_4^+(v) = 1$ and $f_3^+(v) = 2$ by Lemma 4. So $w'(v) \geq w(v) - \frac{1}{2} - 2 \times \frac{3}{4} = 0$.

Suppose that $d(v) = 5$. Then $w(v) = 15 - 10 = 5$ and $f_3^+(v) \leq 3$ by Lemma 3. If $f_3^+(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times \frac{7}{4} - 3 \times \frac{1}{2} = 0$ by R2 and R3.2. Suppose that $f_3^+(v) = 3$. If $f_4^+(v) = 1$, then $w'(v) \leq 3$ and it follows that $w'(v) \geq w(v) - 3 \times \frac{3}{2} - \frac{1}{2} = 0$ by R2 and R3.2. Otherwise $f_4^+(v) = 2$ and it follows that $w'(v) \geq w(v) - 2 \times \frac{3}{2} - \frac{1}{2} = 0$ by R3.2.

Suppose that $d(v) = 6$. Then $w(v) = 18 - 10 = 8$ and $f_3^+(v) \leq \left\lfloor \frac{3}{4} \times 6 \right\rfloor = 4$ by Lemma 3. It follows from (P2) that it may be the 3-master of some 3-vertex $u$, that is, $v$ needs to send at most $\frac{1}{2}$ to its neighbors by R1. If $f_3^+(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0$ by R1–R3. If $f_3^+(v) = 3$, then $f_5^+(v) \geq 1$ and it follows that $w'(v) \geq w(v) - 3 \times 2 - 2 \times \frac{3}{4} - \frac{1}{2} = 0$. Suppose that $f_3^+(v) = 4$. Then $f_4^+(v) = 0$. If $v$ is incident with at most two $(3, 6, 6^+)$-faces, then $w'(v) \geq w(v) - 2 \times 2 - 2 \times \frac{3}{2} - \frac{1}{2} = 0$. Otherwise, $v$ is incident with three $(3, 6, 6^+)$-faces by (P2) and (3) of the lemma, and $v$ is incident with three consecutive 3-faces in which the middle 3-face is incident with two non-adjacent $(3, 6, 6^+)$-faces. So $w'(v) \geq w(v) - 3 \times 2 - \frac{3}{2} - \frac{1}{2} = 0$ by R1 and R3.3.

Suppose that $d(v) = 7$. Then $f_3^+(v) \leq 5$ by Lemma 3. If $f_3^+(v) = 5$, then
\( f_4(v) = 0 \) and \( w'(v) \geq w(v) - 5 \times 2 - \frac{1}{2} > 0 \). If \( f_3(v) = 4 \), then \( f_4(v) \leq 1 \) and \( w'(v) \geq w(v) - 4 \times 2 - \frac{3}{4} - \frac{1}{2} > 0 \). If \( f_3(v) \leq 3 \), then \( w'(v) \geq w(v) - 3 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0 \).

If \( d(v) \geq 8 \), then \( f_3(v) \leq \left\lceil \frac{3d(v)}{4} \right\rceil \) by Lemma 3, and it follows that \( w'(v) \geq \frac{1}{2} = \frac{2(4d(v) - 8)}{16} \geq 0 \).

Hence, we complete the proof of Lemma 5.

\[ \text{Theorem 6.} \quad G \text{ is edge-} k \text{-choosable, where } k = \max\{7, \Delta(G) + 1\}. \]

\[ \text{Proof.} \quad \text{Let } G \text{ be a minimal counterexample to the theorem. Then there is an edge assignment } L \text{ with } |L(e)| \geq k \text{ for all } e \in E(G), \text{ where } k = \max\{7, \Delta(G) + 1\}, \text{ such that } G \text{ is not edge-} L \text{-colorable. By Lemma 5, we consider three cases as follows.} \]

\[ \text{Case 1.} \quad G \text{ contains an edge } uv \text{ with } d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}. \text{ Let } G' = G - uv. \text{ Then } G' \text{ has an edge-} L \text{-coloring } \psi. \text{ Since there exist at most } \max\{6, \Delta(G)\} \text{ edges adjacent to } uv \text{ and } |L(uv)| \geq \max\{7, \Delta(G) + 1\}, \text{ we can color } uv \text{ with some color from } L(uv) \text{ that was not used by } \psi \text{ on the edges adjacent to } uv. \text{ It is easy to show that any edge-} L \text{-coloring of } G' \text{ can be extended to an edge-} L \text{-coloring of } G. \text{ This contradicts the choice of the graph } G. \]

\[ \text{Case 2.} \quad G \text{ contains an even cycle } C = v_1v_2 \cdots v_{2n} \text{ with } d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3. \text{ Let } G' \text{ be the subgraph of } G \text{ obtained by deleting the edges of } C. \text{ Then } G' \text{ has an edge-} L \text{-coloring } \psi. \text{ We define an edge assignment } L' \text{ of } C \text{ such that } L'(e) = L(e) \setminus \{ \psi(e') \mid e' \in E(G') \text{ is adjacent to } e \text{ in } G \} \text{ for each } e \in E(C). \text{ It is easy to see that } L'(e) \geq 2 \text{ for each } e \in E(C). \text{ It is showed in [3] that any even cycle is edge-2-choosable. So } C \text{ is edge-} L' \text{-colorable and it follows that } G \text{ is edge-} L \text{-colorable, a contradiction.} \]

\[ \text{Case 3.} \quad G \text{ has a 6-vertex } u \text{ with five neighbors } v, w, x, y, z \text{ such that } d(v) = d(y) = 3 \text{ and } vw, xy, yz \in E(G). \text{ Let } v' \in N(v) \setminus \{ u, w \}. \text{ According to Case 1, we assume that } d(v_1) + d(v_2) = d(x) = d(z) = d(v') = 6. \text{ Without loss of generality, we consider the worst case that } |L(e)| = 7 \text{ for all } e \in E(G). \text{ By minimality of } G, \text{ } G' = G - \{ y, v \} \text{ has an edge-} L \text{-coloring } \psi. \text{ For each } e \in E(G), \text{ let } L'(e) = L(e) \setminus \{ \psi(e') \mid e' \in E(G') \text{ is adjacent to } e \text{ in } G \}. \]

If \( |L'(xy)| \geq 3 \), then we can color \( vv', vw, vu, yu, yz \) and \( xy \) successively to obtain an edge-\( L \)-coloring of \( G \), a contradiction. So \( |L'(xy)| = 2 \). By the same argument, we have \( |L'(yz)| = |L'(vw)| = |L'(vv')| = 2 \). If \( |L'(uy)| \geq 4 \), then we can color \( vv', vw, vu, xy, yz \) and \( uy \) successively, a contradiction. So \( |L'(uy)| = 3 \). By the same argument, we have \( |L'(uv)| = |L'(vy)| = |L'(vw)| = |L'(vv')| = 2 \) and \( |L'(uy)| = |L'(uv)| = 3 \).
If \( L'(xy) \neq L'(yz) \), without loss of generality, we assume that there is a color 
\( a \in L'(xy) \setminus L'(yz) \), then we color \( xy \) with \( a \) firstly, and then color \( vv', vv, vu, yu \) 
and \( yz \) successively, a contradiction. So \( L'(xy) = L'(yz) \). By the same argument, we have 
\( L'(vw) = L'(vu') \).

Without loss of generality, we assume that \( \psi(ux) = 1, \psi(uz) = 2, \psi(uw) = 3, \)
\( L'(xy) = L'(yz) = \{ \alpha, \beta \} \). Then \( 1 \in L(xy) \) and \( 2 \in L(yz) \) for otherwise \( |L'(xy)| \geq 3 \) or \( |L'(yz)| \geq 3 \). Thus the colors \( 1, 2, \alpha, \beta \) are all distinct. At the same time, 
we have that \( L'(ux) \subseteq \{ 1, 2, 3 \} \) for otherwise we can recolor \( ux \) with a color in 
\( L'(ux) \setminus \{ 1, 2, 3 \} \), color \( xy \) with 1, and color \( vv', vv, vu, yu \) and \( yz \) successively to obtain an edge-\( L \)-coloring of \( G \), a contradiction. By the same argument, we have 
\( L'(uz) \subseteq \{ 1, 2, 3 \} \) and \( L'(uw) \subseteq \{ 1, 2, 3 \} \). So \( L'(ux) \cup L'(uz) \cup L'(uw) = \{ 1, 2, 3 \} \).

Now if \( 1 \in L'(uz) \) and \( 2 \in L'(ux) \), that is, \( \{ 1, 2 \} \subseteq L'(uz) \cap L'(ux) \), then we recolor \( ux \) with 2, and \( uz \) with 1 to obtain a contradiction. So \( \{ 1, 2 \} \not\subseteq L'(uz) \cap L'(ux) \). Similarly, we have \( \{ 1, 3 \} \not\subseteq L'(uz) \cap L'(uw) \) and \( \{ 2, 3 \} \not\subseteq L'(uz) \cap L'(uw) \). These three results imply that \( |L'(ux)| = |L'(uz)| = |L'(uw)| = 2 \). Let \( a \in L'(ux) \setminus \{ 1 \} \), \( b \in L'(uz) \setminus \{ 2 \} \) and \( c \in L'(uw) \setminus \{ 3 \} \). Then \( \{ a, b, c \} = \{ 1, 2, 3 \} \). Thus we recolor \( ux \) with \( a \), \( uz \) with \( b \) and \( uw \) with \( c \) to obtain a final contradiction.

This completes the proof of Theorem 6.

According to the theorem, it is easy to obtain the following corollary.

**Corollary 7.** If \( \Delta(G) \geq 6 \), then \( \chi'_{\text{list}}(G) \leq \Delta(G) + 1 \).

The following result is about edge-\( \Delta \)-choosable of embedded planar graphs without 6-cycles with two chords.

**Theorem 8.** \( G \) is edge-\( k \)-choosable if \( k = \max\{9, \Delta(G)\} \).

This theorem implies that if \( G \) is a planar graph \( G \) with \( \Delta(G) \geq 9 \) and every 
6-cycle of \( G \) contains at most one chord, then \( G \) is edge-\( \Delta \)-choosable.

**Proof.** Suppose that there is an edge assignment \( L \) with \( |L(e)| \geq k \) for all \( e \in E(G) \) such that \( G \) is not edge-\( L \)-colorable, but all subgraphs of \( G \) are edge-\( L \)-colorable.

**Lemma 9** [4]. The graph \( G \) has the following properties.

1. \( G \) is connected and \( \delta(G) \geq 2 \).
2. \( G \) contains no edges \( uv \) with \( d(u) + d(v) \leq 10 \).
3. \( G \) contains no 2-alternating cycles, that is, \( G \) does not contain an even cycle \( C = v_1 v_2 \cdots v_{2n} v_1 \) with \( d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2 \).

Suppose \( G_2 \) be the subgraph induced by the edges incident with the 2-vertices of \( G \). By Lemma 9(2), any two 2-vertices are not adjacent in \( G \), so \( G_2 \) does not contain any odd cycle. By Lemma 9(3), \( G_2 \) contains no even cycle. So \( G_2 \) is a
forest. It follows that $G_2$ contains a matching $M$ such that all 2-vertices in $G_2$ are saturated. If $uv \in M$ and $d(u) = 2$, then $v$ is called the 2-master of $u$. It is easy to see that each 2-vertex has one exactly 2-master and each 9$^+$-vertex can be the 2-master of at most one 2-vertex.

**Lemma 10** [21]. Let $X = \{ x \in V(G) \mid d_G(x) \leq 3 \}$ and $Y = \bigcup_{x \in X} N(x)$. If $X \neq \emptyset$, then there exists a bipartite subgraph $M'$ of $G$ with partite sets $X$ and $Y$ such that $d_{M'}(x) = 1$ for any $x \in X$ and $d_{M'}(y) \leq 2$ for any $y \in Y$. Here, we call $w$ the 3-master of $u$ if $uw \in M'$ and $u \in X$.

Now we use the method of redistribution of charge in order to obtain a contradiction. We assign an “initial charge” $c(x)$ to each element $x \in V(G) \cup F(G)$, where $c(x) = 3d(x) - 10$ if $x \in V(G)$ and $c(x) = 2d(x) - 10$ if $x \in F(G)$. Then

$$
\sum_{x \in V(G) \cup F(G)} c(x) = \sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) < 0.
$$

Our discharging rules are defined as follows.

**R1.** Let $v$ be a 2-vertex. If $v$ is incident with a 3-face and a 6$^+$-face $f$, then $v$ receives 2 from $f$ and 2 from its 2-master. Otherwise, $v$ receives 2 from its 2-master and 2 from its 3-master.

**R2.** Every 3-vertex $v$ receives 1 from its 3-master.

**R3.** Let $f$ be a 3-face and $v$ be a 4$^+$-vertex incident with $f$. Then $f$ receives $a$ from $v$, where

$$
a = \begin{cases} 
\frac{1}{7} & \text{if } d(v) = 4, \\
\frac{3}{7} & \text{if } 5 \leq d(v) \leq 6, \\
\frac{4}{7} & \text{if } d(v) = 7, \\
2 & \text{if } d(v) \geq 8.
\end{cases}
$$

**R4.** Let $f$ be a 4-face incident with a 4$^+$-vertex $v$. Then $f$ receives $a$ from $v$, where

$$
a = \begin{cases} 
\frac{1}{7} & \text{if } 4 \leq d(v) \leq 5, \\
\frac{3}{4} & \text{if } 6 \leq d(v) \leq 7, \\
1 & \text{if } 8 \leq d(v).
\end{cases}
$$

Let $c'(x)$ be the final charge on $x \in V(G) \cup F(G)$. Then $\sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) < 0$. In the following, we will check that $c'(x) \geq 0$ for all $x \in V(G) \cup F(G)$ to get a contradiction.

Let $f$ be a face of $G$. If $d(f) \geq 6$, then $f$ is incident with at most $(d(f) - 5)$ 2-vertices each of which is incident with a 3-face, and it follows that $c'(f) \geq c(f) - 2(d(f) - 5) = 0$. If $d(f) = 5$, then $f$ retains its initial charge and we have
Let $v$ be a vertex of $G$. If $d(v) = 2$, then $c'(v) = c(v) + 2 = 2$ by R1. If $d(v) = 3$, then $c'(v) = c(v) + 1 = 0$ by R2. If $d(v) = 4$, then $c'(v) \geq c(v) - \frac{1}{2} \times 4 = 0$ by R3 and R4. Suppose that $d(v) = 5$. Then $c(v) = 15 - 10 = 5$ and $f_3(v) \leq 3$ by Lemma 3. If $f_3(v) = 3$, then $f_4(v) \leq 1$ and it follows from R3 and R4 that $c'(v) \geq c(v) - 3 \times \frac{3}{2} - 1 = 1 \times \frac{3}{2} = 0$. If $f_3(v) = 2$, then $c'(v) \geq c(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{2} \geq 0$ by R3 and R4. If $d(v) = 6$, then $f_3(v) \leq 4$ by Lemma 3 and we have $c'(v) \geq c(v) - 4 \times \frac{3}{2} - 2 = 3 \times \frac{3}{2} > 0$. If $d(v) = 7$, then $f_3(v) \leq 5$ and we have $c'(v) \geq c(v) - 5 \times \frac{7}{4} - 2 = 4 \times \frac{7}{4} > 0$. Suppose that $d(v) = 8$. Then $f_3(v) \leq 6$ by Lemma 3, and it may be the 3-master of two 3-vertices by Lemma 10. If $f_3(v) = 6$, then $f_4(v) = 0$ and it follows that $c'(v) \geq c(v) - 6 \times 2 = 0$. If $f_3(v) = 5$, then $f_4(v) = 1$ and it follows that $c'(v) \geq c(v) - 5 \times 2 - 1 - 2 = 0$. If $f_3(v) \leq 4$, then $c'(v) \geq c(v) - 4 \times 2 - 4 \times 1 - 2 = 0$. By R3 and R4. So $c'(v) \geq 0$ if $d(v) = 8$.

Now we assume that $d(v) \geq 9$. By Lemmas 9 and 10, $v$ may be the 3-master of two 3-vertices and the 2-master of a 2-vertex, that is, $v$ sends at most 5 to its incident 3-vertices. Suppose that $d(v) = 9$. Then $f_3(v) \leq 6$. If $f_3(v) \leq 3$, then $c'(v) \geq c(v) - 3 \times 2 - 6 \times 1 - 5 = 0$. If $f_3(v) = 4$, then $f_4(v) \leq 4$ and $c'(v) \geq c(v) - 4 \times 2 - 4 \times 1 - 5 = 0$. If $f_3(v) = 5$, then $f_4(v) \leq 2$ and $c'(v) \geq c(v) - 5 \times 2 - 2 \times 1 - 5 = 0$. For $f_3(v) = 6$, we have $f_4(v) \leq 1$. If $f_4(v) = 0$, then $c'(v) \geq c(v) - 6 \times 2 - 5 = 0$. Otherwise, $v$ and its neighbors must induce a configuration isomorphic to Figure 4. Thus, if $d(y) = 2$ or $d(x) = 2$, then $f_1$ is a 6'-face. If $d(y) = 2$ or $d(z) = 2$, then $f_2$ is a 6'-face. By R1, $v$ sends at most 2 to its adjacent 2-vertices. By R2, $v$ sends at most 2 to its adjacent 3-vertices. So $c'(v) \geq c(v) - 6 \times 2 - 1 - 4 = 0$.

Suppose that $d(v) = 10$. Then $f_3(v) \leq 7$. If $f_3(v) = 7$, then $f_4(v) \leq 1$ and it follows that $c'(v) \geq c(v) - 7 \times 2 - 1 - 5 = 0$. If $f_3(v) = 6$, then $f_4(v) \leq 2$ and it follows that $c'(v) \geq c(v) - 6 \times 2 - 2 \times 1 - 5 = 0$. If $f_3(v) \leq 5$, then $c'(v) \geq c(v) - 5 \times 2 - 5 \times 1 - 5 = 0$. Suppose that $d(v) = 11$. Then $c(v) = 3 \times 11 - 10 = 22$ and $f_3(v) \leq 8$. If $7 \leq f_3(v) \leq 8$, then $f_4(v) \leq 1$ and it follows that $c'(v) \geq 22 - 8 \times 2 - 1 - 5 = 0$. If $f_3(v) \leq 6$, then $c'(v) \geq 22 - 6 \times 2 - 5 \times 1 - 5 = 0$. If $d(v) \geq 12$, then $c'(v) \geq c(v) - \frac{3d(v)}{4} \times 2 - (d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor) \times 1 - 5 = 2d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor - 15 \geq 0$.

Till now, we have checked that $c'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This contradiction completes the proof of Theorem 8.
Figure 4. \( d(v) = 9, f_3(v) = 6 \) and \( f_4(v) = 1 \).

References


List Edge Coloring of Planar Graphs


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