LIST EDGE COLORING OF PLANAR GRAPHS WITHOUT 6-CYCLES WITH TWO CHORDS

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Abstract

A graph $G$ is edge-$L$-colorable if for a given edge assignment $L = \{L(e) : e \in E(G)\}$, there exists a proper edge-coloring $\varphi$ of $G$ such that $\varphi(e) \in L(e)$ for all $e \in E(G)$. If $G$ is edge-$L$-colorable for every edge assignment $L$ such that $|L(e)| \geq k$ for all $e \in E(G)$, then $G$ is said to be edge-$k$-choosable. In this paper, we prove that if $G$ is a planar graph without 6-cycles with two chords, then $G$ is edge-$k$-choosable, where $k = \max\{7, \Delta(G) + 1\}$, and is edge-$t$-choosable, where $t = \max\{9, \Delta(G)\}$.

Keywords: planar graph, edge choosable, list edge chromatic number, chord.

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1. Introduction

Graphs considered in this paper are finite, simple and undirected. The terminologies and notations used but undefined in this paper can be found in [2]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply $V$, $E$, $\Delta$ and $\delta$) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively. A cycle $C$ of length $k$ is called a $k$-cycle in the graph $G$. If $xy \in E(G) \setminus E(C)$ and $x, y \in V(C)$, $xy$ is called to be a chord of $C$ in the graph $G$. 

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An edge coloring of a graph $G$ is a mapping $\varphi$ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ for some positive integer $k$. An edge coloring is called proper if every two adjacent edges receive different colors. The edge chromatic number $\chi'(G)$ is the smallest integer $k$ such that $G$ has a proper edge-coloring into the set $\{1, 2, \ldots, k\}$.

We say that $L$ is an edge assignment for the graph $G$ if it assigns a list $L(e)$ of possible colors to each edge $e$ of $G$. If $G$ has a proper edge-coloring $\varphi$ such that $\varphi(e) \in L(e)$ for each edge $e$ of $G$, then we say that $G$ is edge-$L$-colorable or $\varphi$ is an edge-$L$-coloring of $G$. The graph $G$ is edge-$k$-choosable if it is edge-$L$-colorable for every edge assignment $L$ satisfying $|L(e)| \geq k$ for all $e \in E(G)$. The list edge chromatic number $\chi'_{\text{list}}(G)$ of $G$ is the smallest $k$ such that $G$ is edge-$k$-choosable.

On the list edge coloring of a graph, there is a celebrated conjecture known as the list edge coloring conjecture, which was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris (see [8, 13]).

**Conjecture 1** [9]. If $G$ is a multigraph, then $\chi'_{\text{list}}(G) = \chi'(G)$.

The conjecture has been proved for a few classes of graphs, such as graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [4], outerplanar graphs [19], bipartite multigraphs [4, 7], complete graphs of odd order [9]. Vizing [15] proposed a weaker conjecture than Conjecture 1.

**Conjecture 2** [9]. Every graph $G$ is edge-$(\Delta(G) + 1)$-choosable.

Harris [10] showed that $\chi'_{\text{list}}(G) \leq 2\Delta(G) - 2$ if $G$ is a graph with $\Delta(G) \geq 3$. This implies Conjecture 2 for the case $\Delta(G) = 3$. Juvan et al. [14] settled the case for $\Delta(G) = 4$ in 1999. And there are some other special cases of Conjecture 2 which have been confirmed, such as complete graphs [8], graphs with girth at least $8\Delta(G)(\ln \Delta(G)(G) + 1.1)$ [15], planar graphs with $\Delta(G) \geq 8$ [1], and planar graphs with $\Delta(G) \neq 5$ and without intersecting 3-cycles [20]. Suppose that $G$ is a planar graph without $k$-cycles for some fixed integer $3 \leq k \leq 6$. Then it was proved that Conjecture 2 holds if $G$ satisfies one of the four following conditions:

(i) either $k = 3$ or $k = 4$ and $\Delta(G) \neq 5$ [22],
(ii) $k = 4$ [17],
(iii) $k = 5$ [20],
(iv) $k = 6$ and $\Delta(G) \neq 5$ [18].

Other related known results on this topic can be found in [5, 11, 12, 16].

Cai [6] proved that if $G$ is a planar graph without chordal 6-cycles, then $G$ is edge-$k$-choosable, where $k = \max\{8, \Delta(G) + 1\}$. In this paper, we will strengthen this result and obtain that if $G$ is a planar graph and each 6-cycle of $G$ contains at most one chord, then $\chi'_{\text{list}}(G) \leq \max\{7, \Delta(G) + 1\}$ and $\chi'_{\text{list}}(G) \leq \max\{9, \Delta(G)\}$. 

Cai [6] proved that if $G$ is a planar graph without chordal 6-cycles, then $G$ is edge-$k$-choosable, where $k = \max\{8, \Delta(G) + 1\}$. In this paper, we will strengthen this result and obtain that if $G$ is a planar graph and each 6-cycle of $G$ contains at most one chord, then $\chi'_{\text{list}}(G) \leq \max\{7, \Delta(G) + 1\}$ and $\chi'_{\text{list}}(G) \leq \max\{9, \Delta(G)\}$.
2. Main Results and Their Proofs

In the section, we always assume that all graphs are planar graphs that have been embedded in the plane and $G$ is a planar graph without 6-cycles with two chords. We use $d_G(x)$, or simply $d(x)$, to denote the degree of a vertex $x$ in $G$. For $f \in F(G)$, if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk, then we write $f = u_1u_2 \cdots u_nu_1$. The degree of a $f$, denoted by $d(f)$, is the number of edges incident with $f$, where each cut-edge is counted twice. We denote by $d(f)$ the minimum degree of vertices incident with the face $f$. A vertex (face) $x$ is called to be a $k$-vertex ($k$-face), $k^+$-vertex ($k^+$-face) and $k^-$-vertex ($k^-$-face), if $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, respectively. $f_i(v)$ is the number of $i$-faces incident with $v$ for each $v \in V(G)$.

First, we give some properties on $G$.

**Lemma 3.** If $v$ is a $5^+$-vertex of $G$, then $f_5(v) \leq \left\lfloor \frac{3}{2}d(v) \right\rfloor$.

**Proof.** Since $G$ contains no 6-cycles with two chords, $v$ is not incident with four consecutive 3-faces. So $f_3(v) \leq \left\lfloor \frac{3}{2}d(v) \right\rfloor$. ■

**Lemma 4.** Let $u$ be a 4-vertex of $G$.

1. If $f_3(u) = 3$, then $f_4(u) = 0$, that is, $u$ is incident with a $5^+$-face.
2. If $f_3(u) = 2$, then $f_4(u) \leq 1$.

**Proof.** Let neighbors of $u$ be $u_1, u_2, u_3, u_4$ and faces incident with $u$ be $f_1, f_2, f_3, f_4$ in the clockwise order, where $f_1$ is incident with $u_1, u_2$.

1. Without loss of generality, we assume that $f_1, f_2, f_3$ are 3-faces. If $f_4$ is a 4-face $uu_1uu_4u$, then the 6-cycle $uu_2uu_3uu_1u$ contains two chords $uu_3$ and $uu_4$, a contradiction. So $d(f_4) \geq 5$, that is, $f_5(u) = 1$.

2. Suppose that two 3-faces incident with $u$ are not adjacent, without loss of generality, we assume that $f_1, f_3$ are 3-faces. If $f_2$ is a 4-face $uu_2uu_3u$, then the 6-cycle $uu_1uu_2uu_3uu_4u$ contains two chords $uu_2$ and $uu_3$, a contradiction. So $d(f_2) \geq 5$. By the same argument, we have $d(f_4) \geq 5$.

Suppose that two 3-faces incident with $u$ are adjacent, without loss of generality, we assume that $f_1, f_2$ are 3-faces. If $f_3$ is a 4-face $uu_3uu_4u$, then we must have $v = u_1$. Since $d(u) = 4$, $d(u_4) \geq 5$. Thus if $f_4$ is a 4-face $uu_1uu_4u$, then we also have $w = u_3$, it is impossible. So $d(f_4) \geq 5$. By the same argument, if $d(f_4) = 4$, then $d(f_5) \geq 5$. Hence $f_4(u) \leq 1$. ■

**Lemma 5.** $G$ satisfies at least one of the following conditions.

1. $G$ has an edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$.
2. $G$ has an even cycle $C = v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$. 

(3) \( G \) has a 6-vertex \( u \) with five neighbors \( v, w, x, y, z \) such that \( d(v) = d(y) = 3 \) and \( vw, xy, yz \in E(G) \) (see Figure 1).

![Figure 1. The subgraph for Lemma 5(3).](image)

**Proof.** Let \( G \) be a minimal counterexample to the lemma. It is easy to check that \( G \) is connected. By the choice of \( G \), we have the following observations.

(P1) For any edge \( uv \), \( d(u) + d(v) \geq \max \{9, \Delta(G) + 3\} \) by (1). Then \( \delta(G) \geq 3 \) and all neighbors of a \( i \)-vertex must be \( (9 - i)^+ \)-vertices, where \( i = 3, 4 \) or 5.

(P2) Let \( G_3 \) be the subgraph induced by the edges incident with 3-vertices of \( G \). Then \( G_3 \) is a forest.

By (P1), every two 3-vertices are not adjacent, and it follows that \( G_3 \) contains no even cycles. So \( G_3 \) is a bipartite subgraph. By (2), \( G_3 \) contains no even cycles. So \( G_3 \) is a forest and (P2) holds. Let \( V_1 \) be the set of 3-vertices of \( G \). Thus for any component of \( G_3 \), we select a vertex \( u \notin V_1 \) as a root of the tree. Then every 3-vertex has exactly two children. If \( uv \in E(G_3) \), \( u \in V_1 \) and \( v \) is a child of \( u \), then \( v \) is called a 3-master of \( u \). Note that each 3-vertex has exactly two 3-masters and each vertex of degree at least 6 can be the 3-master of at most one 3-vertex.

According to the Euler’s formula \(|V(G)| - |E(G)| + |F(G)| = 2\) of a planar graph \( G \), we have

\[
\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -10(|V(G)| - |E(G)| + |F(G)|) = -20 < 0.
\]

Now we define the initial weight function on \( V(G) \cup F(G) \) by letting \( w(x) = 3d(x) - 10 \) for any \( x \in V(G) \) and \( w(x) = 2d(x) - 10 \) for any \( x \in F(G) \). Thus the total sum of weights is the negative number \( -20 \). We use the following rules to redistribute the initial charge that leads to a new charge \( w'(x) \).

**R1.** Every 3-vertex \( v \) receives \( \frac{1}{2} \) from each of its 3-masters.

**R2.** Let \( f = uu'vv' \) be a 4-face in \( G \) with \( d(u) \leq \min\{d(u'), d(v), d(v')\} \). If \( d(u) \geq 4 \), then \( f \) receives \( \frac{1}{2} \) from each of its incident vertices. Otherwise, \( f \) receives nothing from \( u \), receives \( \frac{1}{2} \) from \( v \), \( \frac{3}{4} \) from \( u' \) and \( \frac{3}{4} \) from \( v' \).
R3. Let $f$ be a 3-face incident with a $4^+$-vertex $v$. Then $f$ receives $a$ from $v$.

R3.1. If $d(v) = 4$, then

$$a = \begin{cases} 
\frac{1}{2} & \text{if } f_4^-(v) = 4 \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located in the middle of three consecutive 3-faces incident with } v, \\
\frac{3}{4} & \text{if } f_3(v) = 2 \text{ and } f_4(v) = 1, \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located in one side of three consecutive 3-faces incident with } v, \\
1 & \text{otherwise}.
\end{cases}$$

R3.2. If $d(v) = 5$, then

$$a = \begin{cases} 
\frac{3}{2} & \text{if } f_3(v) = 3 \text{ and one of the following conditions holds:} \\
& (i) f_4(v) = 1, \\
& (ii) f_4(v) = 0 \text{ and } f \text{ is located in the middle of three consecutive 3-faces incident with } v, \\
& (iii) \text{two faces adjacent to } f \text{ at } v \text{ are } 5^+\text{-faces}, \\
\frac{7}{4} & \text{otherwise}.
\end{cases}$$

R3.3. If $d(v) \geq 6$, then

$$a = \begin{cases} 
\frac{3}{2} & \text{if } f \text{ is adjacent to two non-adjacent } (3, 6, 6^+)\text{-faces at } v \\
& \text{and } d(v) = 6, \\
\frac{7}{4} & \text{if } f \text{ is incident with a } 3\text{-vertex}, \\
\text{otherwise}.
\end{cases}$$

In the following, we will check that $w'(x) \geq 0$ for all elements $x \in V(G) \cup F(G)$ to obtain the following obvious contradiction.

$$0 \leq \sum_{v \in V \cup F} w'(x) = \sum_{v \in V \cup F} w(x) = -20.$$  

First, we consider the final charge of any face $f$. If $d(f) \geq 5$, then it retains its initial charge and it follows that $w'(f) = w(f) = 2d(f) - 10 \geq 0$. Suppose that $d(f) = 4$. Then $w(f) = 8 - 10 = -2$. If $\delta(f) = 3$, then $w'(f) = w(f) + \frac{3}{2} + \frac{1}{2} + \frac{3}{4} = 0$ by R2. Otherwise $w'(f) = w(f) + 4 \times \frac{1}{2} = 0$. So $w'(f) \geq 0$ if $d(f) = 4$.

Suppose that $d(f) = 3$. Then $w(f) = 6 - 10 = -4$. If $\delta(f) = 3$, then $f$ is incident with two $6^+\text{-vertices}$ by (P1) and it follows that $w'(f) = w(f) + 2 + 2 = 0$ by R3.3. If $\delta(f) \geq 5$, then $f$ receives at least $\frac{3}{2}$ from each of its incident vertices by R3.2 and R3.3, so $w'(f) \geq w(f) + 3 \times \frac{3}{2} > 0$. In the following, we assume that $\delta(f) = 4$. Let $f$ be a 3-face $uvw$ such that $d(u) = 4$. Then $d(v) \geq 5$ and $d(w) \geq 5$ by (P1). According to R3.1, we consider the following three cases.

**Case 1.** $f$ receives $\frac{1}{2}$ from $u$, that is, $f_4^-(u) = 4$ or $f_3(u) = 3$ and $f$ is located in the middle of three consecutive 3-faces incident with $u$.  

...
It suffices to check that $f$ receives at least $\frac{7}{4}$ from each of $v$ and $w$. Thus

$$w'(f) \geq w(f) + \frac{1}{2} + \frac{3}{2} + \frac{7}{4} = 0,$$

a contradiction.

**Subcase 1.1.** $f_3(u) = 4$, that is, $u$ is incident with four faces of degree at most 4. Then $f_3(u) = 4$ or $f_3(u) = 1$ by Lemma 4. If $f_3(u) = 1$, then all faces adjacent to $f$ are $4^+$-faces, and it follows from R3.2 and R3.3 that $f$ receives at least $\frac{7}{4}$ from $v$, $w$ respectively. If $f_3(u) = 4$, then any 3-face incident with $u$ must be adjacent to a $5^+$-face and it follows from R3.2 and R3.3 that $f$ receives at least $\frac{7}{4}$ from $v$, $w$ respectively.

**Subcase 1.2.** $f_3(u) = 3$ and $f$ is located in the middle of three consecutive 3-faces incident with $u$. If $d(v) \geq 6$, then two faces adjacent to $f$ at $v$ are not $(3, 6, 6^+)$-faces (since $d(u) = 4$ and $uv$ is incident with two $(4, 5^+, 6^+)$-faces) and it follows from R3.3 that $f$ receives at least $\frac{7}{4}$ from $v$. Suppose that $d(v) = 5$. Let five faces incident with $v$ be $f, f_1, \ldots, f_4$ in clockwise order, where $uv$ is incident with $f$ and $f_1$ (see Figure 2). Then $d(f_1) \geq 5$ since $G$ contains no 6-cycles with two chords. If $f_3(v) = 3$, then $f_4(v) = 0$, and $f$ is not located in the middle of three consecutive 3-faces incident with $v$ (since $d(f_4) \geq 5$), and only one face adjacent to $f$ at $v$ is a $5^+$-face (since $d(f_1) = 3$). So $f$ receives at least $\frac{7}{4}$ from $v$ by R3.2. By symmetry, $f$ receives at least $\frac{7}{4}$ from $w$.

**Figure 2.** $d(u) = 4$, $f_3(u) = 3$ and $f$ is located in the middle of three consecutive 3-faces incident with $u$.

**Case 2.** $f$ receives $\frac{3}{2}$ from $u$. Then $f_3(u) = 2$ and $f_4(u) = 1$, or $f_3(v) = 3$ and $f$ is located in the one side of these 3-faces by R3.1. Suppose that $f_3(u) = 2$ and $f_4(u) = 1$. Then the induced subgraph of $u$ and its neighbors must be isomorphic to a configuration as Figure 3, where $w = x$ or $w = y$. If $vx$ is incident with two 3-faces $uxu$ and $vxx'$, then the 6-cycle $xx'vyux$ contains two chords $uv$ and $uy$, a contradiction. If $vx$ is incident with a 4-face $vxx'x''v$, then the 6-cycle $xx'x''vyux$ contains two chords $uv$ and $xv$, a contradiction, too. So $vx$ is incident with a $5^+$-face. By the same argument, $vy$ is incident with a $5^+$-face, too. By R3.2 and R3.3, $f$ receives at least $\frac{7}{4}$ from $v$, at least $\frac{7}{4}$ from $w$. So $w'(f) \geq w(f) + \frac{7}{4} + \frac{7}{4} = 0$ by R3.
Suppose that $u$ is incident with three 3-faces and $f$ is located in the one side of these 3-faces. Then $u$ is incident with a $5^+$-face by Lemma 4. Without loss of generality, we assume that $uv$ is incident with two 3-faces. By the similar arguments with Subcase 1.2, $v$ sends at least $\frac{7}{4}$ to $f$. So $w'(f) \geq w(f) + \frac{3}{4} + \frac{3}{2} + \frac{7}{4} = 0$.

**Case 3.** $f$ receives 1 from $u$. Since $d(v) \geq 5$, $v$ sends at least $\frac{3}{2}$ to $f$ by R3.2 and R3.3. Similarly, $w$ sends at least $\frac{3}{2}$ to $f$. So $w'(f) \geq w(f) + 1 + \frac{3}{2} + \frac{3}{2} = 0$.

Till now, we have checked that $w'(f) \geq 0$ for any face $f \in F(G)$. Next, we begin to check the new charge of all vertices of $(G\setminus F)$ consecutively.

**Subcase 1.** $f_1(v) = 7$. Then $f_3(v) \leq 5$ by Lemma 3. If $f_3(v) = 5$, then $w' \geq w'(1) = w'(v) - 2 \times 1 = 0$. Otherwise, $w'(v) \geq 0$.

**Subcase 2.** $f_1(v) = 5$. Then $w'(v) = 15 - 10 = 5$ and $f_3(v) \leq 3$ by Lemma 3. If $f_3(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times \frac{7}{4} - 3 \times \frac{1}{2} = 0$ by R2 and R3.2. Suppose that $f_3(v) = 3$. If $f_4(v) = 1$, then $f_5(v) \geq 1$ and it follows that $w'(v) \geq w(v) - 3 \times \frac{3}{2} - \frac{1}{2} = 0$ by R2 and R3.2. Otherwise $f_5(v) = 2$ and it follows that $w'(v) \geq w(v) - 2 \times \frac{3}{2} - \frac{3}{2} = 0$ by R3.2.

Suppose that $d(v) = 6$. Then $w'(v) = 18 - 10 = 8$ and $f_3(v) \leq \lfloor \frac{3}{4} \times 6 \rfloor = 4$ by Lemma 3. It follows from $(P2)$ that it may be the 3-master of some 3-vertex $u$, that is, $v$ needs to send at most $\frac{1}{2}$ to its neighbors by R1. If $f_3(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0$ by R1–R3. If $f_3(v) = 3$, then $f_5(v) \geq 1$ and it follows that $w'(v) \geq w(v) - 3 \times 2 - 2 \times \frac{3}{4} - \frac{1}{2} = 0$. Suppose that $f_3(v) = 4$. Then $f_4(v) = 0$. If $v$ is incident with at most two $(3,6,6^+)$-faces, then $w'(v) \geq w(v) - 2 \times 2 - 2 \times \frac{3}{2} - \frac{1}{2} = 0$. Otherwise, $v$ is incident with three $(3,6,6^+)$-faces by $(P2)$ and (3) of the lemma, and $v$ is incident with three consecutive 3-faces in which the middle 3-face is incident with two non-adjacent $(3,6,6^+)$-faces. So $w'(v) \geq w(v) - 3 \times 2 - \frac{3}{2} - \frac{1}{2} = 0$ by R1 and R3.3.

Suppose that $d(v) = 7$. Then $f_3(v) \leq 5$ by Lemma 3. If $f_3(v) = 5$, then
\[ f_4(v) = 0 \text{ and } w'(v) \geq w(v) - 5 \times 2 - \frac{1}{2} > 0. \text{ If } f_3(v) = 4, \text{ then } f_4(v) \leq 1 \text{ and } w'(v) \geq w(v) - 3 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0. \]

If \( d(v) \geq 8 \), then \( f_3(v) \leq \left\lfloor \frac{3d(v)}{4} \right\rfloor \) by Lemma 3, and it follows that \( w'(v) \geq w(v) - 2 \times \left\lfloor \frac{3d(v)}{4} \right\rfloor - \frac{3}{2} \left( d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor \right) - \frac{1}{2} = \frac{21(d(v)-8)}{16} \geq 0. \]

Hence, we complete the proof of Lemma 5.

\section*{Theorem 6.} \( G \) is edge-\( k \)-choosable, where \( k = \max\{7, \Delta(G) + 1\} \).

\section*{Proof.} Let \( G \) be a minimal counterexample to the theorem. Then there is an edge assignment \( L \) with \( |L(e)| \geq k \) for all \( e \in E(G) \), where \( k = \max\{7, \Delta(G) + 1\} \), such that \( G \) is not edge-\( L \)-colorable. By Lemma 5, we consider three cases as follows.

\subsection*{Case 1.} \( G \) contains an edge \( uv \) with \( d(u) + d(v) \leq \max\{8, \Delta(G) + 2\} \). Let \( G' = G - uv \). Then \( G' \) has an edge-\( L \)-coloring \( \psi \). Since there exist at most \( \max\{6, \Delta(G)\} \) edges adjacent to \( uv \) and \( |L(uv)| \geq \max\{7, \Delta(G) + 1\} \), we can color \( uv \) with some color from \( L(uv) \) that was not used by \( \psi \) on the edges adjacent to \( uv \). It is easy to show that any edge-\( L \)-coloring of \( G' \) can be extended to an edge-\( L \)-coloring of \( G \). This contradicts the choice of the graph \( G \).

\subsection*{Case 2.} \( G \) contains an even cycle \( C = v_1v_2 \cdots v_{2n-1}v_1 \) with \( d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3 \). Let \( G' \) be the subgraph of \( G \) obtained by deleting the edges of \( C \). Then \( G' \) has an edge-\( L \)-coloring \( \psi \). We define an edge assignment \( L' \) of \( C \) such that \( L'(e) = L(e) \setminus \{\psi(e')\} \) \( e' \in E(G') \) is adjacent to \( e \in G \) for each \( e \in E(C) \). It is easy to see that \( L'(e) \geq 2 \) for each \( e \in E(C) \). It is shown in [3] that any even cycle is edge-2-choosable. So \( C \) is edge-\( L' \)-colorable and it follows that \( G \) is edge-\( L \)-colorable, a contradiction.

\subsection*{Case 3.} \( G \) has a 6-vertex \( u \) with five neighbors \( v, w, x, y, z \) such that \( d(v) = d(y) = 3 \) and \( vw, xy, yz \in E(G) \). Let \( v' \in N(v) \setminus \{u, w\} \). According to Case 1, we assume that \( d(v_1) + d(v_2) \geq \max\{9, \Delta(G) + 3\} \) for any edge \( v_1v_2 \in E(G) \). Since \( d(u) + d(v) = 6 + 3, \Delta(G) = 6 \) and \( d(w) = d(x) = d(z) = d(v') = 6 \). Without loss of generality, we consider the worst case that \( |L(e)| = 7 \) for all \( e \in E(G) \). By minimality of \( G \), \( G' = G - \{y, v\} \) has an edge-\( L \)-coloring \( \psi \). For each \( e \in E(G) \), let \( L'(e) = L(e) \setminus \{\psi(e')\} e' \in E(G') \) is adjacent to \( e \in G \).

If \( |L'(xy)| \geq 3 \), then we can color \( vv', vv, vu, yz \) and \( xy \) successively to obtain an edge-\( L \)-coloring of \( G \), a contradiction. So \( |L'(xy)| = 2 \). By the same argument, we have \( |L'(yz)| = |L'(vw)| = |L'(vv')| = 2 \). If \( |L'(uy)| \geq 4 \), then we can color \( vv', vv, vu, xy, yz \) and \( uy \) successively, a contradiction. So \( |L'(uy)| = 3 \). By the same argument, we have \( |L'(uv)| = |L'(vz)| = |L'(vw)| = |L'(vv')| = 2 \) and \( |L'(uy)| = |L'(uv)| = 3 \).
If \( L'(xy) \neq L'(yz) \), without loss of generality, we assume that there is a color \( a \in L'(xy) \cap L'(yz) \); then we color \( xy \) with \( a \) firstly, and then color \( vv', vv, vu, yu \) and \( yz \) successively, a contradiction. So \( L'(xy) = L'(yz) \). By the same argument, we have \( L'(vw) = L'(vv') \).

Without loss of generality, we assume that \( \psi(ux) = 1, \psi(uz) = 2, \psi(uw) = 3 \), \( L'(xy) = L'(yz) = \{a, \beta\} \). Then \( 1 \in L(xy) \) and \( 2 \in L(yz) \) for otherwise \( |L'(xy)| \geq 3 \) or \( |L'(yz)| \geq 3 \). Thus the colors \( 1, 2, \alpha, \beta \) are all distinct. At the same time, we have that \( L'(ux) \subseteq \{1, 2, 3\} \) for otherwise we can recolor \( ux \) with a color in \( L'(ux) \setminus \{1, 2, 3\} \), color \( xy \) with 1, and color \( vv', vv, vu, yu \) and \( yz \) successively to obtain an edge-\( L \)-coloring of \( G \), a contradiction. By the same argument, we have \( L'(uz) \subseteq \{1, 2, 3\} \) and \( L'(uw) \subseteq \{1, 2, 3\} \). So \( L'(ux) \cup L'(uz) \cup L'(uw) = \{1, 2, 3\} \).

Now if \( 1 \in L'(uz) \) and \( 2 \in L'(ux) \), that is, \( \{1, 2\} \subseteq L'(uz) \cap L'(ux) \), then we recolor \( uz \) with 2, and \( uv \) with 1 to obtain a final contradiction. So \( \{1, 2\} \nsubseteq L'(uz) \cap L'(ux) \). Similarly, we have \( \{1, 3\} \nsubseteq L'(uz) \cap L'(uw) \) and \( \{2, 3\} \nsubseteq L'(uz) \cap L'(uw) \). These three results imply that \( |L'(ux)| = |L'(oz)| = |L'(uw)| = 2 \). Let \( a \in L'(ux) \setminus \{1\} \), \( b \in L'(oz) \setminus \{2\} \) and \( c \in L'(uw) \setminus \{3\} \). Then \( \{a, b, c\} = \{1, 2, 3\} \). Thus we recolor \( uy \) with \( a \), \( uz \) with \( b \) and \( uv \) with \( c \) to obtain a final contradiction.

This completes the proof of Theorem 6.

According to the theorem, it is easy to obtain the following corollary.

**Corollary 7.** If \( \Delta(G) \geq 6 \), then \( \chi_{list}(G) \leq \Delta(G) + 1 \).

The following result is about edge-\( \Delta \)-choosable of embedded planar graphs without 6-cycles with two chords.

**Theorem 8.** \( G \) is edge-\( k \)-choosable if \( k = \max\{9, \Delta(G)\} \).

This theorem implies that if \( G \) is a planar graph \( G \) with \( \Delta(G) \geq 9 \) and every 6-cycle of \( G \) contains at most one chord, then \( G \) is edge-\( \Delta \)-choosable.

**Proof.** Suppose that there is an edge assignment \( L \) with \( |L(e)| \geq k \) for all \( e \in E(G) \) such that \( G \) is not edge-\( L \)-colorable, but all subgraphs of \( G \) are edge-\( L \)-colorable.

**Lemma 9** [4]. The graph \( G \) has the following properties.

1. \( G \) is connected and \( \delta(G) \geq 2 \).
2. \( G \) contains no edges \( uv \) with \( d(u) + d(v) \leq 10 \).
3. \( G \) contains no 2-alternating cycles, that is, \( G \) does not contain an even cycle \( C = v_1v_2 \cdots v_{2n}v_1 \) with \( d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2 \).

Suppose \( G_2 \) be the subgraph induced by the edges incident with the 2-vertices of \( G \). By Lemma 9(2), any two 2-vertices are not adjacent in \( G \), so \( G_2 \) does not contain any odd cycle. By Lemma 9(3), \( G_2 \) contains no even cycle. So \( G_2 \) is a
forest. It follows that $G_2$ contains a matching $M$ such that all 2-vertices in $G_2$ are saturated. If $uv \in M$ and $d(u) = 2$, then $v$ is called the 2-master of $u$. It is easy to see that each 2-vertex has one exactly 2-master and each 9$^+$-vertex can be the 2-master of at most one 2-vertex.

**Lemma 10** [21]. Let $X = \{x \in V(G) \mid d_G(x) \leq 3\}$ and $Y = \bigcup_{x \in X} N(x)$. If $X \neq \emptyset$, then there exists a bipartite subgraph $M'$ of $G$ with partite sets $X$ and $Y$ such that $d_{M'}(x) = 1$ for any $x \in X$ and $d_{M'}(y) \leq 2$ for any $y \in Y$. Here, we call $w$ the 3-master of $u$ if $uw \in M'$ and $u \in X$.

Now we use the method of redistribution of charge in order to obtain a contradiction. We assign an “initial charge” $c(x)$ to each element $x \in V(G) \cup F(G)$, where $c(x) = 3d(x) - 10$ if $x \in V(G)$ and $c(x) = 2d(x) - 10$ if $x \in F(G)$. Then

\[
\sum_{x \in V(G) \cup F(G)} c(x) = \sum_{x \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) < 0.
\]

Our discharging rules are defined as follows.

**R1.** Let $v$ be a 2-vertex. If $v$ is incident with a 3-face and a 6$^+$-face $f$, then $v$ receives 2 from $f$ and 2 from its 2-master. Otherwise, $v$ receives 2 from its 2-master and 2 from its 3-master.

**R2.** Every 3-vertex $v$ receives 1 from its 3-master.

**R3.** Let $f$ be a 3-face and $v$ be a 4$^+$-vertex incident with $f$. Then $f$ receives $a$ from $v$, where

\[
a = \begin{cases} 
\frac{1}{2} & \text{if } d(v) = 4, \\
\frac{3}{2} & \text{if } 5 \leq d(v) \leq 6, \\
\frac{1}{4} & \text{if } d(v) = 7, \\
2 & \text{if } d(v) \geq 8.
\end{cases}
\]

**R4.** Let $f$ be a 4-face incident with a 4$^+$-vertex $v$. Then $f$ receives $a$ from $v$, where

\[
a = \begin{cases} 
\frac{1}{2} & \text{if } 4 \leq d(v) \leq 5, \\
\frac{3}{4} & \text{if } 6 \leq d(v) \leq 7, \\
1 & \text{if } 8 \leq d(v).
\end{cases}
\]

Let $c'(x)$ be the final charge on $x \in V(G) \cup F(G)$. Then $\sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) < 0$. In the following, we will check that $c'(x) \geq 0$ for all $x \in V(G) \cup F(G)$ to get a contradiction.

Let $f$ be a face of $G$. If $d(f) \geq 6$, then $f$ is incident with at most $(d(f) - 5)$ 2-vertices each of which is incident with a 3-face, and it follows that $c'(f) \geq c(f) - 2(d(f) - 5) = 0$. If $d(f) = 5$, then $f$ retains its initial charge and we have
\[c'(f) = c(f) = 2d(f) - 10 \geq 0.\] Suppose that \(d(f) = 4\). If \(\delta(f) \leq 3\), then \(f\) is incident with at least two \(8^+\)-vertices by Lemma 9.\(2\) and it follows from R4 that \(c'(f) \geq c(f) + 2 \times 1 = 0\). Otherwise \(c'(f) \geq c(f) + 2 \times \frac{3}{2} + 2 \times \frac{3}{4} > 0.\) Suppose that \(d(f) = 3\). If \(\delta(f) \leq 3\), then \(f\) is incident with two \(8^+\)-vertices by Lemma 9.\(2\) and it follows from R3 that \(c'(f) = c(f) + 2 + 2 = 0\). If \(\delta(f) = 4\), then \(f\) is incident with two \(7^+\)-vertices by Lemma 9.\(2\). Note that any \(4\)-vertex sends at least \(\frac{1}{2}\) to each of its incident \(3\)-face. So \(c'(f) \geq c(f) + \frac{1}{2} + 2 \times \frac{3}{4} = 0\). If \(\delta(f) \geq 5\), then \(c'(f) \geq c(f) + 3 \times \frac{3}{2} > 0\). So \(c'(f) \geq 0\) if \(d(f) = 3\).

Let \(v\) be a vertex of \(G\). If \(d(v) = 2\), then \(c'(v) = c(v) + 2 + 2 = 0\) by R1. If \(d(v) = 3\), then \(c'(v) = c(v) + 1 = 0\) by R2. If \(d(v) = 4\), then \(c'(v) \geq c(v) - \frac{1}{2} \times 4 = 0\) by R3 and R4. Suppose that \(d(v) = 5\). Then \(c(v) = 15 - 10 = 5\) and \(f_3(v) \leq 3\) by Lemma 3. If \(f_3(v) = 3\), then \(f_4(v) \leq 1\) and it follows from R3 and R4 that \(c'(v) \geq c(v) - 3 \times \frac{3}{2} - 1 \times \frac{1}{2} = 0\). If \(f_3(v) = 2\), then \(c'(v) \geq c(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{2} \geq 0\) by R3 and R4. If \(d(v) = 6\), then \(f_3(v) \leq 4\) by Lemma 3 and we have \(c'(v) \geq c(v) - 4 \times \frac{3}{2} - 2 \times \frac{3}{4} > 0\). If \(d(v) = 7\), then \(f_3(v) \leq 5\) and we have \(c'(v) \geq c(v) - 5 \times \frac{3}{2} - 2 \times \frac{3}{4} > 0\). Suppose that \(d(v) = 8\). Then \(f_3(v) \leq 6\) by Lemma 3, and it may be the 3-master of two \(3\)-vertices by Lemma 10. If \(f_3(v) = 6\), then \(f_4(v) = 0\) and it follows that \(c'(v) \geq c(v) - 6 \times 2 - 2 = 0\). If \(f_3(v) = 5\), then \(f_4(v) \leq 1\) and it follows that \(c'(v) \geq c(v) - 5 \times 2 - 1 = 2 > 0\). If \(f_3(v) \leq 4\), then \(c'(f) \geq c(v) - 4 \times 2 - 4 \times 1 = 2 = 0\) by R3 and R4. So \(c'(f) \geq 0\) if \(d(v) = 8\).

Now we assume that \(d(v) \geq 9\). By Lemmas 9 and 10, \(v\) may be the 3-master of two \(3\)-vertices and the 2-master of a \(2\)-vertex, that is, \(v\) sends at most 5 to its incident \(3\)-vertices. Suppose that \(d(v) = 9\). Then \(f_3(v) \leq 6\). If \(f_3(v) \leq 3\), then \(c'(v) \geq c(v) - 3 \times 2 - 6 \times 1 = 5 = 0\). If \(f_3(v) = 4\), then \(f_4(v) \leq 4\) and \(c'(v) \geq c(v) - 4 \times 2 - 4 \times 1 = 5 = 0\). If \(f_3(v) = 5\), then \(f_4(v) \leq 2\) and \(c'(v) \geq c(v) - 5 \times 2 - 2 \times 1 = 5 = 0\). For \(f_3(v) = 6\), we have \(f_4(v) \leq 1\). If \(f_4(v) = 0\), then \(c'(v) \geq c(v) - 6 \times 2 = 5 = 0\). Otherwise, \(v\) and its neighbors must induce a configuration isomorphic to Figure 4. Thus, if \(d(v) = 2\) or \(d(x) = 2\), then \(f_4(v) = 1\) and it follows that \(c'(v) \geq c(v) - 5 \times 2 - 5 \times 1 - 5 = 0\). Otherwise, \(v\) and its neighbors must induce a configuration isomorphic to Figure 4. Thus, if \(d(v) = 2\) or \(d(x) = 2\), then \(f_4(v) = 1\) and it follows that \(c'(v) \geq c(v) - 5 \times 2 - 5 \times 1 - 5 = 0\). Suppose that \(d(v) = 11\). Then \(c(v) = 3 \times 11 - 10 = 22\) and \(f_3(v) \leq 8\). If \(7 \leq f_3(v) \leq 8\), then \(f_4(v) \leq 1\) and it follows that \(c'(v) \geq 22 - 8 \times 2 - 1 - 5 = 0\). If \(f_3(v) \leq 6\), then \(c'(v) \geq 22 - 6 \times 2 - 5 \times 1 - 5 = 0\). If \(d(v) \geq 12\), then \(c'(v) \geq c(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor \times 2 \times \left(d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor \right) \times 1 - 5 = 2d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor - 15 \geq 0\).

Till now, we have checked that \(c'(x) \geq 0\) for all \(x \in V(G) \cup F(G)\). This contradiction completes the proof of Theorem 8.
Figure 4. $d(v) = 9$, $f_3(v) = 6$ and $f_4(v) = 1$.

References


List Edge Coloring of Planar Graphs


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