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STAR-CRITICAL RAMSEY NUMBERS FOR CYCLES VERSUS K_4

Chula J. Jayawardene

Department of Mathematics
University of Colombo, Colombo, Sri Lanka

e-mail: c_jayawardene@yahoo.com

David Narváez

Golisano College of Computing and Information Sciences
Rochester Institute of Technology, Rochester, NY 14623, USA

e-mail: den9562@rit.edu

and

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, Rochester, NY 14623, USA

e-mail: spr@cs.rit.edu

Abstract

Given three graphs G, H and K we write $K \rightarrow (G, H)$, if in any red/blue coloring of the edges of K there exists a red copy of G or a blue copy of H. The Ramsey number $r(G, H)$ is defined as the smallest natural number n such that $K_n \rightarrow (G, H)$ and the star-critical Ramsey number $r_*(G, H)$ is defined as the smallest positive integer k such that $K_{n-1} \sqcup K_{1,k} \rightarrow (G, H)$, where n is the Ramsey number $r(G, H)$. When $n \geq 3$, we show that $r_*(C_n, K_4) = 2n$ except for $r_*(C_3, K_4) = 8$ and $r_*(C_4, K_4) = 9$. We also characterize all Ramsey critical $r(C_n, K_4)$ graphs.

Keywords: Ramsey theory, star-critical Ramsey numbers.

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1. Introduction

Let G and H be two finite graphs. If for every 2-coloring (red and blue) of the edges of a complete graph K_n there exists a copy of G in the first color (red) or a copy of H in the second color (blue), we denote this by $K_n \rightarrow (G, H)$. The *Ramsey number* $r(G, H)$ is the smallest positive integer n such that $K_n \rightarrow (G, H)$. The *classical Ramsey number* $r(s, t)$ is defined as $r(K_s, K_t)$. Exact determination of their values, in particular the *diagonal Ramsey numbers* $r(n, n)$, (see [10] for a survey) becomes notoriously difficult for larger parameters. One of the variations of classical Ramsey numbers, namely *star-critical Ramsey numbers*, were introduced by Hook and Isaak in 2010 [6,7]. They deal with finding $r_*(G, H)$, which is defined as the smallest positive integer k such that $K_{n-1} \sqcup K_{1,k} \rightarrow (G, H)$, where $n = r(G, H)$ and $K_{n-1} \sqcup K_{1,k}$ is the graph obtained by identifying the k vertices of degree 1 in $K_{1,k}$ with any k vertices of the complete graph K_{n-1}. One of the goals of the study of star-critical Ramsey numbers can be seen as an enhancement of understanding of classical cases. For $n = r(G, H)$, we know that $K_n \rightarrow (G, H)$ but $K_{n-1} \sqcup K_{1,k} \not\rightarrow (G, H)$ for $k < r_*(G, H)$. Thus, $r_*(G, H)$ is zooming in at what is happening at the classical case. Several authors studied $r_*(G, H)$ for special pairs of graphs, such as for trees versus complete graphs, stripes versus stripes, fans versus complete graphs, and others [5–7, 13].

In 1973, Bondy and Erdős [1] obtained several interesting results related to $r(C_n, K_m)$. Shortly afterwards, it was conjectured by Erdős and others that $r(C_n, K_m) = (n-1)(m-1)+1$ for all $n \geq m \geq 3$, except the case $r(C_3, K_3) = 6$. Over decades, many authors proved parts of this conjecture (see [10] for detailed references), and now it is known to hold for all $n \geq m$ when $m \leq 7$. The problem of determining $r(C_n, K_m)$ becomes much more difficult for fixed n and large m.

The main result of this paper is the determination of $r_*(C_n, K_4)$. Determination of $r_*(C_n, K_m)$ for $n \geq 5$ is an interesting long-term challenge in itself, however at the moment looking hopelessly difficult in light of the comments in the previous paragraph. It is hoped, however, that thorough understanding of critical graphs and of $r_*(C_n, K_m)$ for the cases when $r(C_n, K_m)$ is known, may help in obtaining new results about the still open classical cases for larger parameters.

2. Notation

All graphs $G = (V, E)$ considered in this paper are finite graphs without loops or multiple edges. A set $I \subseteq V(G)$, is said to be an *independent set* if no two vertices of I are connected by an edge in G. That is, in the complement of G the vertices of I form a clique of order $|I|$. The independence number of a graph G, denoted by $\alpha(G)$, is the largest order of an independent set in G. For any subset S of $V(G)$, the *subgraph induced by* S, denoted by $G[S]$, is defined as the subgraph formed by S and all the edges of G connecting pairs of vertices in S. The subgraph $G \setminus S$ is defined as the graph $G[V(G) \setminus S]$. The graph obtained by the *disjoint union of n copies of G* is denoted by nG. The Wagner graph illustrated in Figure 5(b) is denoted by W_8.

The *complete graph* on n vertices is denoted by K_n, a *cycle* of length n is denoted by C_n and a *star* on $n+1$ vertices is the graph $K_{1,n}$. For $p < r(G,H)$ a 2-coloring of K_p that does not contain a red G or a blue H is called a $(G,H;p)$ *good* coloring [3]. For $p = r(G,H)-1$ such good colorings are called *critical*. For a red/blue coloring of a graph G, and vertices $u, v \in V(G)$ such that $\{u,v\} \in E(G)$, we say that u is a red (respectively blue) neighbor of v if $\{v,u\}$ is colored red (respectively blue). The notation $K_n \sqcup K_{1,k}$ indicates the operation of identifying the k vertices of degree 1 in $K_{1,k}$ with k vertices of the complete graph K_n. The notation $K_n \setminus K_{1,k}$ indicates the graph obtained from removing k edges incident to a vertex in K_n. Notice that $K_n \sqcup K_{1,k} = K_{n+1} \setminus K_{1,n-k}$. The *lower size Ramsey number* $l(G,H)$ is the smallest integer l such that there exists a subgraph K of $K_{r(G,H)}$ with $|E(K)| = l$ and $K \to (G,H)$. As observed in [7],

$$l(G,H) - \binom{r(G,H)-1}{2} \leq r_*(G,H) \leq r(G,H) - 1. \tag{1}$$

3. PROPERTIES OF (C_n, K_4) RAMSEY CRITICAL GRAPHS

It is known that $r(C_n, K_4) = 3n-2$ for $n \geq 4$ and $r(C_3, K_4) = 9$ (see [12] for the general case, and [10] for pointers to partial contributions). In this section we characterize all C_n-free graphs without $\overline{K}_4$ on $r(C_n, K_4) - 1$ vertices, i.e., all Ramsey critical graphs for these parameters. We will make use of some external lemmas which we include below for the sake of completeness.

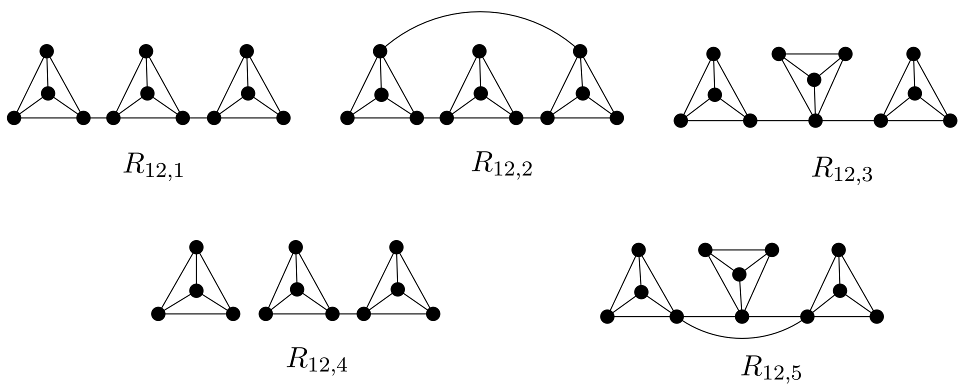


Figure 1. $R_{12,k}$, $1 \leq k \leq 5$.

Lemma 1 ([8], Lemma 4)**.** *Any C_5-free graph of order* 12 *with no independent set of* 4 *vertices is isomorphic to one of the graphs* $R_{12,1}$, $R_{12,2}$, $R_{12,3}$, $R_{12,4}$, $R_{12,5}$ (*Figure* 1) *or* $R_{12,6} \cong 3K_4$.

Lemma 2 ([4], Corollary 1.14(a))**.** *Let* $n \geq 5$. *Then* G *is a* (C_n, C_3)*-critical coloring if and only if* $G_{blue} = K_{n-1,n-1}$ *or* $K_{n-1,n-1} - e$ *for some edge* e.

The next lemma is a direct consequence of a result by Bollobás *et al.* [2].

Lemma 3. *Suppose* G *contains the cycle* $U = (u_1, u_2, \ldots, u_{n-1}, u_1)$ *of length* $n-1$ *but no cycle of length* n. *Let* $X = V(G) \setminus \{u_1, u_2, \ldots, u_{n-1}\}$, $\alpha(G) = m-1$ *where* $m \leq \frac{n+3}{2}$, *and suppose that* $I = \{x_1, x_2, \ldots, x_{m-1}\} \subseteq X$ *is an independent set. Then no member of* I *is adjacent to* $m-2$ *or more vertices in the cycle* U.

Lemma 4. *For* $n \geq 6$, *any* C_n*-free graph of order* $3(n-1)$ *with no independent set of* 4 *vertices contains* $3K_{n-1}$.

Proof. Suppose that G is a C_n-free graph on $3(n-1)$ vertices with no independent set of 4 vertices. Then, as $r(C_{n-1}, K_4) = 3n-5$ (cf. [5, 10]), there exists a cycle $U = (u_1, u_2, \ldots, u_{n-1}, u_1)$ of length $n-1$. Define $H = G \setminus U$ as the induced subgraph of G not containing the vertices of the cycle, so $|V(H)| = 2(n-1)$.

Suppose that there exists an independent set X in H of order 3, hence $\alpha(G) = 3$. From Lemma 3, as $4 \leq \frac{n+3}{2}$, every vertex X is incident to at most one vertex in U. Then, as $n-1 > 3$, we have an independent set of order 4 containing X, which is a contradiction. Hence H contains no independent set of order 3 and H is a C_n-free graph of order $2(n-1)$. By Lemma 2, we conclude that H is equal to $2K_{n-1}$ or $2K_{n-1}+e$ since $n \geq 5$. In the case H contains a $2K_{n-1}+e$, let a and b be the vertices such that $\{a, b\}$ represents the only edge e joining the two K_{n-1}'s. In the case H does not contain a $2K_{n-1}+e$, let a and b be any two vertices of H, belonging to the two disjoint K_{n-1}'s. Now consider any two vertices of U, say u and v, and suppose that $\{u, v\} \notin E(G)$. Since there is no C_n in G, each of the vertices u and v must be adjacent to at most one vertex of each copy of K_{n-1} in H. Therefore, as $n > 3$, we can select vertex x_1 in the first K_{n-1} and vertex x_2 in the second K_{n-1}, distinct from a and b, and such that x_1 and x_2 are not adjacent to u or v. This gives us that $\{u, v, x_1, x_2\}$ is an independent set of order 4, which is a contradiction. Therefore, $\{u, v\} \in E(G)$. Since u, v are arbitrary vertices in U, we can conclude that U induces a K_{n-1} as required. ∎

4. Main Result

Theorem 5. *It holds that*

$$r_*(C_n, K_4) = \begin{cases} 8 & if\ n = 3, \\ 9 & if\ n = 4, \\ 2n & if\ n \geq 5. \end{cases}$$

Proof. We break up the proof into three cases.

Case $n = 3$. Let W_8^* be the graph of order 9 obtained from W_8 (i.e., the Wagner graph) by adding a vertex and connecting it to two non-adjacent vertices in the original graph. Color the edges of $K_8 \sqcup K_{1,7} \cong K_9 - e$ with red and blue, so that the red graph is isomorphic to W_8^*, as indicated in Figure 2. This graph has no red C_3 and has no blue K_4 and thus $K_8 \sqcup K_{1,7} \not\to (C_3, K_4)$. Therefore, $r_*(C_3, K_4) \geq 8$. Using (1), we have $r_*(C_3, K_4) \leq r(C_3, K_4) - 1 = 8$ and thus $r_*(C_3, K_4) = 8$.

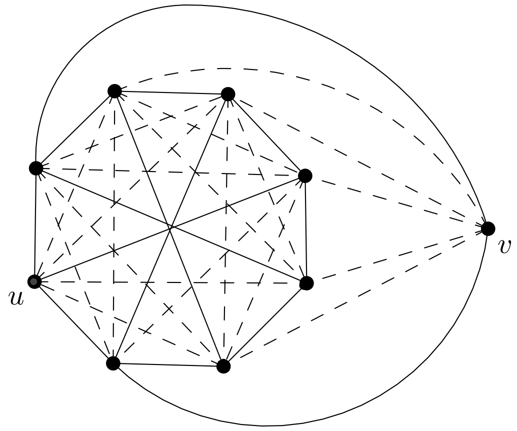


Figure 2. A coloring of $K_9 - e$ which contains no red C_3 and no blue K_4. In this figure the red edges are indicated by solid lines and the blue edges by dashed lines. Notice that the edge between the nodes labeled u and v is missing.

Case $n = 4$. Let x be the vertex in $R_{9,5}$ (see Figure 6) of degree 2 and let y be a vertex adjacent to x in this graph. Let $R_{9,5}^*$ be the graph of order 10 obtained from $R_{9,5}$ by adding a vertex v and connecting it to x and y. Color the edges of $K_9 \sqcup K_{1,8} \cong K_{10} - e$ using red and blue so that the red graph is isomorphic to $R_{9,5}^*$, as indicated in Figure 3. This coloring has no red C_4 and no blue K_4, and thus $K_9 \sqcup K_{1,8} \not\to (C_4, K_4)$. Therefore, $r_*(C_4, K_4) \geq 9$. Using (1), we have $r_*(C_4, K_4) \leq r(C_4, K_4) - 1 = 9$, and thus $r_*(C_4, K_4) = 9$.

Case $n \geq 5$. Color the edges of $K_{3(n-1)+1} \setminus K_{1,n-2}$ using red and blue so that the red graph consists of a $2K_{n-1} \cup (K_{n-1} \sqcup K_{1,1})$ as illustrated in Figure 4.

Therefore, $r_*(C_n, K_4) \geq 2n$. In order to show that $r_*(C_n, K_4) \leq 2n$, assume by contradiction that there exists a red/blue coloring of $G = K_{3(n-1)+1} \setminus K_{1,n-3}$ with no red C_n and no blue K_4. Let v be a vertex in G of degree $2n$ and let H be the graph obtained from G by deleting v.

By Lemmas 1 and 4, we see that H contains a red $3K_{n-1}$. Let us denote the sets of vertices of its three components by V_1, V_2 and V_3. Since there is no red C_n in the coloring, v has at most one red neighbor in each of the three sets V_1,

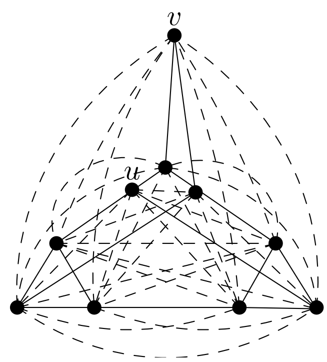


Figure 3. A coloring of $K_{10} - e$ which contains no red C_4 and no blue K_4. In this figure the red edges are indicated by solid lines and the blue edges by dashed lines. Notice that the edge between the nodes labeled u and v is missing.

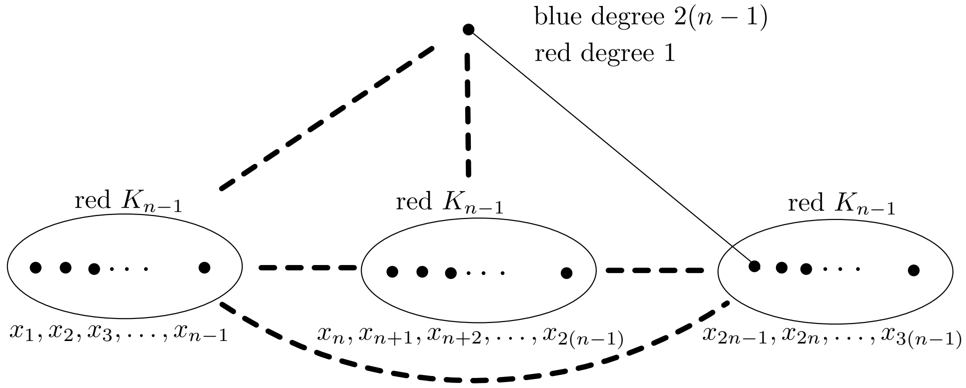


Figure 4. A coloring of $K_{3(n-1)+1} \setminus K_{1,n-2}$ which has no red C_n and no blue K_4.

V_2 and V_3. If v is adjacent to exactly two vertices in some V_i $(1 \leq i \leq 3)$ then, without loss of generality, we may assume that v is adjacent to all the vertices in V_1 and V_2. In particular, v is adjacent to at least 4 vertices in each V_i $(1 \leq i \leq 2)$. Select $v_1 \in V_1$ and $v_2 \in V_2$ such that v_1 has no red neighbors in $G[V_2 \cup V_3 \cup \{v\}]$ and v_2 has no red neighbors in $G[V_1 \cup V_3 \cup \{v\}]$ (this is possible because each V_i can have at most 3 vertices with red neighbors outside V_i). Because there are no red C_n's in the coloring, we can find a $v_3 \in V_3$ such that $\{v, v_3\}$ is colored blue. Then, $\{v, v_1, v_2, v_3\}$ will induce a blue K_4, a contradiction. Therefore, given any $1 \leq i \leq 3$, we get that v must be adjacent to at least 3 of the vertices in V_i. Thus, without loss of generality, we can assume that v is adjacent to at least 4 vertices in V_1, 3 vertices in V_2, and 3 vertices in V_3. Because v can have at most one red neighbor in each of V_2 and V_3, we can select two vertices $v_2 \in V_2$ and $v_3 \in V_3$ such that $\{v, v_2, v_3\}$ induces a blue triangle in G. Next, select $v_1 \in G_1$ such that it has no red neighbors in $G[V_2 \cup V_3 \cup \{v\}]$ (this is possible because V_1 can have at most 3 vertices with red neighbors outside V_1). But then $\{v, v_1, v_2, v_3\}$ will induce a blue K_4, a contradiction. ∎

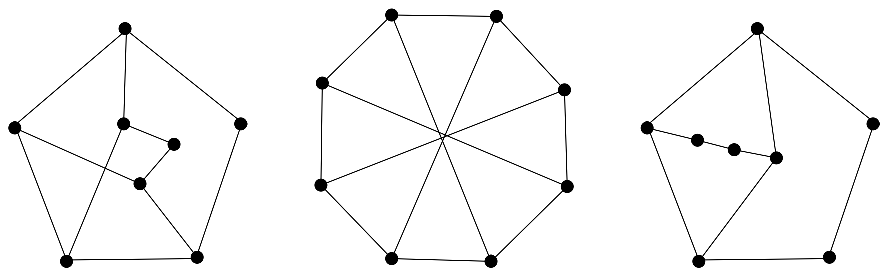
5. ALL (C_n, K_4) RAMSEY CRITICAL GRAPHS

In this section we present characterization of all (C_n, K_4) Ramsey critical graphs obtained without explicit use of computations. This may help in future extensions of the main result of this paper to graphs other than K_4. We also performed computations generating all $(C_n, K_4; v)$ good colorings for $n \leq 7$, obtaining full agreement on the common part. Full understanding of $(C_n, K_m; v)$ good colorings may help in further progress on both classical and star-critical Ramsey numbers for cycles versus K_m. Once again, we make use of an external lemma which we include here for the sake of completeness.

Lemma 6 ([11], Lemma 4)**.** *A C_4-free graph G of order 9 and no independent set of 4 vertices is isomorphic to one of the graphs $R_{9,1}$, $R_{9,2}$, $R_{9,3}$, $R_{9,4}$, $R_{9,5}$, $R_{9,6}$, $R_{9,7}$ (Figure 6) or $R_{9,8} \cong 3K_3$.*

Lemma 7. *The set of $r(C_n, K_4)$-critical graphs consists of:*

- *Three critical graphs for $n = 3$, with the red graphs, of the red/blue coloring, corresponding to $R_{8,1}$ (Figure 5(a)), $R_{8,2} \cong W_8$ (Figure 5(b)) or $R_{8,3}$ (Figure 5(c)).*



(a) $R_{8,1}$ (b) $R_{8,2} \cong W_8$ (c) $R_{8,3}$

Figure 5. Graphs $R_{8,k}$, $1 \leq k \leq 3$.

- *Eight critical graphs for $n = 4$, with the red graphs, of the red/blue coloring, given by $R_{9,1}$, $R_{9,2}$, $R_{9,3}$, $R_{9,4}$, $R_{9,5}$, $R_{9,6}$, $R_{9,7}$ (Figure 6) or $R_{9,8} \cong 3K_3$.*
- *Six critical graphs for $n = 5$, with the red graphs, of the red/blue coloring, corresponding to K_{12}, given by $R_{12,1}$, $R_{12,2}$, $R_{12,3}$, $R_{12,4}$, $R_{12,5}$ (Figure 1) or $R_{12,6} \cong 3K_4$.*

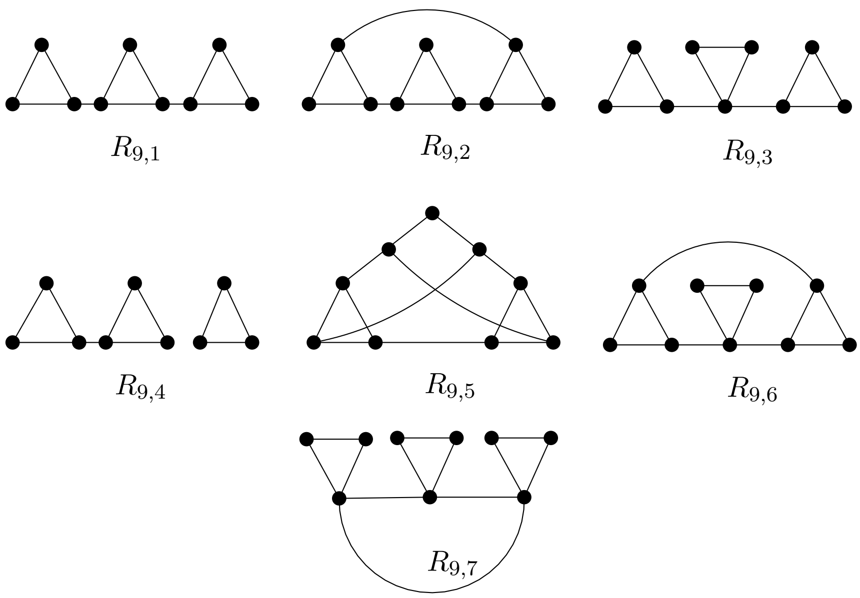


Figure 6. Graphs $R_{9,k}$, $1 \leq k \leq 7$.

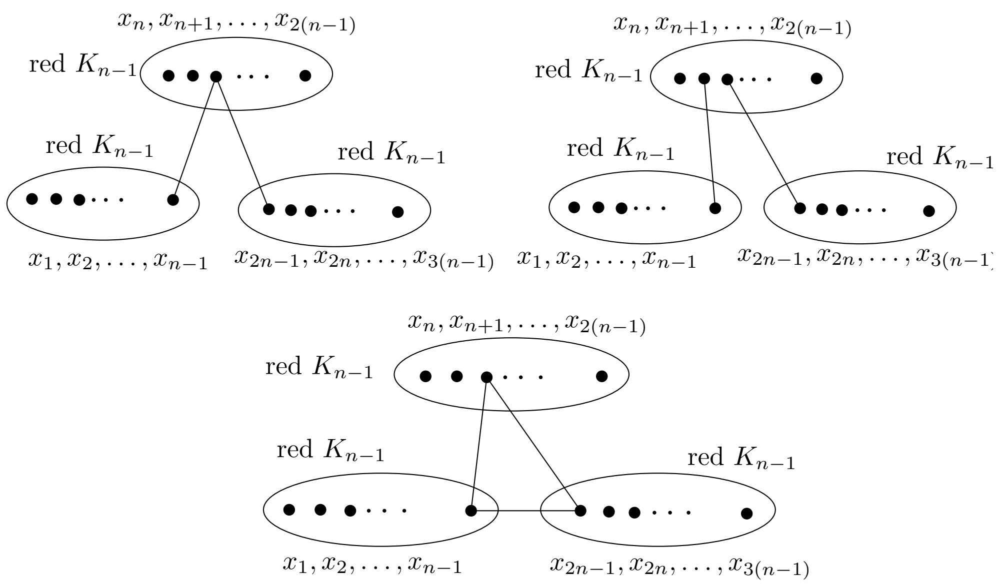


Figure 7. The red graphs $R_{3n-3,1}$, $R_{3n-3,2}$, and $R_{3n-3,3}$.

- *Five critical graphs for $n \geq 6$, with the red graph, of the red/blue coloring, corresponding to $K_{3(n-1)}$, denoted by $R_{3n-3,1}$, $R_{3n-3,2}$, $R_{3n-3,3}$, $R_{3n-3,4}$ or $R_{3n-3,5}$, where $R_{3n-3,4} \cong 3K_{n-1} + e$ and $R_{3n-3,5} \cong 3K_{n-1}$. The other three red graphs, namely $R_{3n-3,1}$, $R_{3n-3,2}$, $R_{3n-3,3}$, are illustrated in Figure 7.*

Proof. There are three $r(C_3, K_4)$ critical graphs which is easily verifiable (one of them is the Wagner graph). For $r(C_4, K_4)$ the result follows from Lemma 6 and the fact that $r(C_4, K_4) = 10$ (cf. [10]). For $r(C_5, K_4)$ the result follows from Lemma 1 and the fact that $r(C_5, K_4) = 13$ [8]. When $n \geq 6$, for $r(C_n, K_4)$ the red graph of the red/blue coloring corresponding to $K_{3(n-1)}$, must contain a $3K_{n-1}$ by Lemma 4. In order to avoid a red C_n, as there can be at most one red edge between any two of the red K_{n-1} graphs, we see that there are only 5 distinct colorings and the corresponding red graphs are given by $R_{3n-3,k}$ for $1 \leq k \leq 5$. ■

$n \backslash v$	6	7	8	9	10	11	12	13	14	15	16	17	18
3	15	9	3										
4	22	30	22	8									
5	44	63	81	73	52	19	6						
6	72	133	198	259	236	192	138	81	22	5			
7	120	302	490	666	868	972	653	463	368	241	127	27	5

Table 1. Number of $(C_n, K_4; v)$ good colorings for $n \in \{3, 4, 5, 6, 7\}$.

Table 1 shows the number of $(C_n, K_4; v)$ good colorings for small values of n. This dataset was generated by exploiting the fact that all $(C_n, K_4; v+1)$ good colorings can be obtained from all the $(C_n, K_4; v)$ good colorings by adding one vertex and connecting it to every vertex in the original coloring, then coloring the new edges avoiding C_n in the first color and K_4 in the second color. The initial set $(C_n, K_4; v)$ for $v = 6$ was generated by enumerating all models of the Boolean formula encoding the non-arrowing property and then keeping one representative from each isomorphism class using `nauty`[1] [9].

As discussed in the Introduction and this section, the determination of the exact values of $r_*(C_n, K_m)$ for all $n \geq 5$ is an interesting and very difficult challenge, and any further partial progress on this problem will be welcome. We also expect that new results on $r_*(C_n, K_m)$ could shed some light on still open classical cases of $r(C_{n'}, K_{m'})$, for $n' > n$ or $m' > m$.

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