THE PROPER DIAMETER OF A GRAPH

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Abstract

A proper edge-coloring of a graph is a coloring in which adjacent edges receive distinct colors. A path is properly colored if consecutive edges have distinct colors, and an edge-colored graph is properly connected if there exists a properly colored path between every pair of vertices. In such a graph, we introduce the notion of the graph’s proper diameter—which is a function of both the graph and the coloring—and define it to be the maximum length of a shortest properly colored path between any two vertices in the graph. We consider various families of graphs to find bounds on the gap between the diameter and possible proper diameters, paying singular attention to 2-colorings.

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1. Introduction

In a long distance wireless communications network, each node in the network simultaneously sends and receives signals. To avoid interference, the carrier waves for each of these incoming and outgoing signals must be operating at different frequencies. By representing each node by a vertex, each connection by an edge, and each frequency assignment by a color, the problem of finding a path over which a signal can travel without interference is translated to a question about the edge-coloring of graphs. More specifically, it is required that no two consecutive edges in the transmission path have the same color in order to maintain the integrity of a signal from start to finish. The design of such networks is therefore reliant on the ability to color a graph “properly”. More formally, an edge-colored graph is said to be properly colored if no two adjacent edges share a color. An edge-colored connected graph $G$ is called properly connected if between every pair of distinct vertices in $G$, there exists a path that is properly colored. For simplicity, an edge-coloring $c$ of $G$ is called properly connected if it makes $G$ properly connected. Note that while a properly colored graph is necessarily properly connected, the converse is not true. See Figure 1.

Figure 1. A properly connected but not properly colored graph.

The proper connection number of a connected graph $G$, $pc(G)$, defined in [2] and also studied in [1, 5, 6], is the minimum number of colors needed to color
the edges of $G$ to make it properly connected. One finds that if an edge-colored graph is sufficiently dense, then the existence of a properly colored path is assured. For example, using the connectivity $\kappa(G)$ of $G$, Borozan et al. [2] show that if $\kappa(G) \geq 3$, then $pc(G) = 2$ and if $\kappa(G) \geq 2$, then $pc(G) \leq 3$. We refer the reader to [8] for a dynamic survey on the topic of proper connection.

The notion of a properly colored graph and concepts such as proper connection number are obviously related to rainbow colored graphs and analogous concepts introduced by Chartrand et al. in [3]. In fact, the communications application noted at the outset is a generalization of a security protocol devised by Chartrand et al. to address communications between disparate governmental and legal agencies in the wake of the terrorist attack of Sept. 11, 2001 [4].

However, despite a spate of recent papers on the proper and rainbow connection numbers of a graph (inclusive of the basic work in [3] and [2]) the study of the spectrum of possible lengths of certain properly (respectively, rainbow) colored paths in a properly (respectively, rainbow) connected graph has not yet been investigated. In fact, the only known results relating properly colored paths to distance in the graph has been forcing the properly colored paths to be geodesics, shortest paths, as studied in [1, 6, 7]. Here we begin to address this omission by introducing the notion of the proper diameter of a properly connected graph.

Throughout this paper, we assume that all graphs are properly connected. The proper distance between two vertices $u$ and $v$ is the minimum length of a properly colored path from $u$ to $v$. Observe that such a distance function does not define a metric. See Figure 2 for an example of a properly connected graph where the triangle inequality fails.

![Figure 2. The proper distance between $u$ and $v$ is greater than the sum of the proper distances between $u$ and $x$ and between $x$ and $v$.](image)

Generalizing the notion of the diameter of a graph, $\diam(G)$, we formally define proper diameter.

**Definition.** The proper diameter of a graph $G$ with a properly connected $k$-coloring $c$ is defined to be the maximum proper distance between any two vertices in $G$ and will be denoted as $\text{pdim}_{k}(G, c)$.

We will consider all colorings $c$ where $G$ is properly connected and discuss $\text{pdim}_{k}(G)$ which is the maximum of $\text{pdim}_{k}(G, c)$ where $c$ makes $G$ properly connected. Since coloring the edges of a graph cannot shorten the lengths of paths, it follows that the diameter of a graph is a trivial lower bound for the proper diameter. Note that the proper diameter of a graph is a function of
both the graph and the coloring, but for some classes of graphs, such as trees and complete graphs, the diameter and proper diameter are necessarily equal—regardless of the coloring.

We now ask two questions which drive our current investigation.

- How large can the gap be between \( \text{diam}(G) \) and \( \text{pdiam}_2(G) \)?
- What is the spectrum of possible values for \( \text{pdiam}_2(G, c) \)?

As a first step in answering these questions, we address—and completely resolve—the two driving questions for the following graph classes: cycles (Section 2.1), fans (Section 2.2), and complete multipartite graphs (Section 2.3). After the fashion of the result of Borozan et al. [2] noted above, the connectivity of a graph can also be used to give a tight upper bound of \( n - \kappa(G) + 1 \) for \( \text{pdiam}_2(G) \). We prove this result in Section 3. We conclude with two lemmas which provide additional bounds on the proper diameter. The first provides a general lower bound for graphs with odd girth, and the second gives an upper bound on the proper diameter of a graph formed by adding a vertex of degree 2 to a properly connected graph under certain conditions. Note that throughout the proofs of this paper we use \( c(uv) = 1 \) or \( c(uv) = 2 \) to denote an edge \( uv \) of color 1 or color 2, respectively. In the figures, color 1 is represented by a solid red line and color 2 is represented by a dashed blue line.

2. General Classes of Graphs

2.1. Cycles

We begin by considering the straightforward case of even and odd cycles. This class of graphs illustrates that using only two colors is sufficient to produce a colored graph whose proper diameter is quite large—nearly double the diameter.

In order for an odd cycle \( C_{2m+1} \) to be properly connected with two colors, there must be exactly one pair of consecutive edges with the same color. The path between the vertices at opposite ends of these two edges yields proper diameter \( 2m - 1 \).

There are two possible properly connected 2-colorings of an even cycle \( C_{2m} \). If the cycle is properly colored, the proper diameter is certainly equal to the diameter \( m \). Otherwise, exactly one set of three consecutive edges must have the same color while all other consecutive pairs are different. Suppose \( v_1v_2, v_2v_3 \) and \( v_3v_4 \) all have color 1. Then the path between vertices \( v_1 \) and \( v_3 \) gives a proper diameter of \( 2m - 2 \).

**Observation 1.** Given any properly connected 2-coloring \( c \) of a cycle graph, \( \text{pdiam}_2(C_{2m+1}, c) \) is \( 2m - 1 \) and \( \text{pdiam}_2(C_{2m}, c) \) is either \( m \) or \( 2m - 2 \) where both values are attainable.
2.2. Fans

Next, we consider fan graphs $F_{1,n} \cong K_1 \vee P_n$, where $\vee$ denotes graph join, to find tight bounds on the attainable values of $\text{pdim}_2(F_{1,n}, c)$. This class of graphs illustrates that two colors are sufficient to produce a very large difference between the diameter and the proper diameter of a graph. In particular, $F_{1,2k+1}$ can be 2-colored so that its proper diameter is $k$ times its diameter. See Example 2.

Example 2. A fan graph $F_{1,n}$ has diameter 2 and can be 2-colored so that the proper diameter is $n - 1$. Color all edges incident to the vertex $u$ with degree $n$ using color 1. Color the edges of the path that remain by alternating colors. The proper diameter will be the length of this path, $n - 1$. See Figure 3 for this coloring on $F_{1,8}$.

Example 3. A fan graph $F_{1,n}$ can be given a properly connected 2-coloring $c$ so that the proper diameter is 3. Alternate colors of edges incident to the vertex $u$ with degree $n$ beginning with $c(uv_1) = 2$. Color the edges of the path by alternating colors beginning with $c(v_1v_2) = 2$. The proper diameter is 3. If $i$ and $j$ have opposite parity, there is properly colored path $v_iuv_j$ of length 2. Otherwise, there is a properly colored path $v_iuv_{i+1}uv_j$ of length 3. See Figure 4 for this coloring on $F_{1,8}$.
**Theorem 4.** For a properly connected 2-coloring \( c \) of \( F_{1,n} \), if \( n \geq 7 \), then \( \text{pdiam}_2(F_{1,n},c) \in [3,n-1] \). These bounds are tight. If \( 3 \leq n \leq 6 \), then a lower bound of 2 is attainable.

**Proof.** First suppose \( n \geq 7 \). Let \( u \) be the vertex of degree \( n \) and label the path vertices as \( \{v_1, v_2, \ldots, v_n\} \). Suppose that there is a properly connected 2-coloring of \( F_{1,n} \) with proper diameter equal to the diameter, 2. Without loss of generality, suppose \( c(uv_1) = 1 \). Regardless of the coloring of the edges of the path, \( c(uv_4) = 2 \) since the only path of length 2 from \( v_1 \) to \( v_4 \) passes through \( u \). Similarly, all edges from \( u \) to \( \{v_5, v_6, v_7\} \) must have color 2. Since the only path of length 2 from \( v_4 \) to \( v_7 \) passes through \( u \), there is no properly colored path of length 2 from \( v_4 \) to \( v_7 \). As a result, every properly connected 2-coloring of \( F_{1,n} \) with \( n \geq 7 \) has a proper diameter at least 3. Example 3 shows that 3 is a tight lower bound for \( F_{1,n} \) when \( n \geq 7 \).

If \( 3 \leq n \leq 6 \), then a proper diameter of 2 can be achieved by coloring the first \( \left\lceil \frac{n}{2} \right\rceil \) edges from \( u \) to the path with color 1 and the remaining edges from \( u \) to the path with color 2. Alternate coloring edges of the path with colors 1 and 2.

For the upper bound, suppose there is some properly connected 2-coloring \( c \) of \( F_{1,n} \) with proper diameter \( n \). This implies that there is a properly colored path of length \( n \), say \( H \), between some pair of vertices and this is a shortest properly colored path. Note that \( H \) is a Hamiltonian path. The endpoints of \( H \) cannot be nonadjacent vertices \( v_i \) and \( v_j \) with \( 1 < i < j < n \) since any such path would miss at least one vertex and thus would not be Hamiltonian. Also, the vertex \( u \) is adjacent to every vertex on the path and so it cannot be an endpoint of \( H \).

The observations in the previous two sentences imply that either \( v_1 \) or \( v_n \) must be an endpoint of \( H \).

Suppose first that the endpoints of \( H \) are \( v_i \) and \( v_j \) for some \( 2 < j < n \). The only Hamiltonian path from \( v_1 \) to \( v_j \) is \( v_1v_2 \cdots v_{j-1}uv_{n-i+1}v_{n-i} \cdots v_{j} \) which must alternate between color 1 and color 2. This implies that \( uv_{j-1} \) and \( uv_n \) must be opposite in color, say color 1 and color 2, respectively. In order to avoid a shorter properly colored path, the edge \( uv_j \) must also have color 1. If \( c(uv_1) = 1 \), then we have a properly colored path \( v_1uv_nv_{n-1} \cdots v_j \) which has length at most \( n-1 \). If \( c(uv_1) = 2 \), then \( v_1uv_j \) is a properly colored path of length 2. A similar argument can be made if we assume the endpoints of \( H \) are \( v_n \) and \( v_1 \) for some \( 1 < i < n-1 \).

Thus, we have eliminated all possible endpoints of \( H \) except for a path beginning at \( v_1 \) and ending at \( v_n \). In order for \( H \) to contain \( u \), the path must use edges of the form \( uv_iu \) and \( uv_{i+1} \) of opposite color.

**Case 1.** Let \( 1 < i < n-1 \). Without loss of generality, assume that \( c(uv_i) = 1 \), \( c(uv_{i+1}) = 2 \), and the remaining edges on the path alternate color. This implies that \( uv_n \) must be colored 1 or else there is a shorter properly colored path.
v_1v_2\cdots v_iuv_n. If c(uv_1) = 2, then v_1uv_n is a properly colored path of length 2. If c(uv_1) = 1, then v_1uv_{i+1}v_{i+2}\cdots v_n is a properly colored path of length at most n-1, which is a contradiction.

Case 2. Let v_1 = v_i with c(uv_1) = 1 and c(uv_2) = 2. As in the previous case, uv_n must be colored 1. Since c(v_2v_3) = 1, the edge v_1v_2 must also be colored 1 to avoid a properly colored path without u. But, now v_1v_2uv_n is a properly colored path of length 3, which is again a contradiction.

Case 3. Let v_{n-1} = v_i with c(uv_{n-1}) = 1 and c(uv_n) = 2. If uv_1 is colored 1, then v_1uv_n is a properly colored path of length 2. In a fashion similar to the second case, c(v_{n-2}v_{n-1}) = c(v_{n-1}v_n) = 2 to avoid a properly colored path without u. If c(uv_1) is color 2, then v_1uv_{n-1}v_n is a properly colored path of length 3, giving a contradiction.

Example 2 demonstrates the sharpness of the upper bound.

Theorem 4 establishes tight bounds on the range of values of \( \text{pdiam}_2(F_{1,n}, c) \). The next theorem establishes that all values in this range are attainable.

**Theorem 5.** If \( n \geq 7 \), then every value in the interval \([3, n-1]\) is an attainable value of \( \text{pdiam}_2(F_{1,n}, c) \).

**Proof.** By Theorem 4, we know \( \text{pdiam}_2(F_{1,n}, c) \) is within the range of values \([3, n-1]\). Also, as described in the proof of Theorem 4, the endpoints of this range are achieved as in Examples 3 and 2, respectively. We will use these colorings to create a properly connected 2-coloring of \( F_{1,n} \) where \( \text{pdiam}_2(F_{1,n}, c) = n-k \) for \( 1 \leq k \leq n-3 \) and \( n \geq 7 \). Thus, we provide a properly connected 2-coloring \( c \) for every value within the given range.

As before, let \( u \) be the vertex of degree \( n \) and label the path vertices as \( v_1 \) through \( v_n \). Let \( L \) consist of the set of vertices \( v_i \) where \( 1 \leq i \leq n-(k-1) \) and let \( R \) consist of the set of vertices \( v_i \) where \( n-(k-1)+1 \leq i \leq n \). Note that when \( k = 1 \), the set \( R \) is empty and the coloring we describe is as in Example 2.

Color the edges from \( u \) to the vertices of \( L \) with color 1. If \( |R| \) is odd, then color the edges from \( u \) to the vertices of \( R \) by alternating in color starting with color 2. If \( |R| \) is even, then color the edges from \( u \) to the vertices of \( R \) by alternating in color starting with color 2 up to and including \( v_{n-1} \). For reasons evident in the following paragraph, let \( c(uv_n) = 2 \).

Now, we color the edges of the path vertices \( v_1 \) through \( v_{n-(k-1)} \) by alternating in color starting with color 1. Color the edge from \( v_{n-(k-1)} \) to \( v_{n-(k-1)+1} \) with color 2. Then continue coloring the path vertices \( v_{n-(k-1)+1} \) through \( v_n \) by alternating in color starting with color 2 to eliminate shorter properly colored paths. This coloring is depicted in Figure 5. Clearly, \( u \) is adjacent to \( v_i \) for all \( i \) and thus has a path of length 1 to these vertices. Since the subgraph induced on \( L \cup \{u\} \) is colored as in Example 2, \( \text{pdiam}_2(F_{1,n}[L \cup \{u\}], c) = n-(k-1)-1 = n-k \). The
subgraph induced on $R \cup \{u\}$ is colored as in Example 3, but with $c(uv_n) = 2$ in order to have a properly colored path from $v_1$ to $v_n$. If $1 \leq k \leq 4$, then $\text{pdiam}_2(F_{1,n}[R \cup \{u\}], c) \leq 2$. If $k \geq 5$, then $\text{pdiam}_2(F_{1,n}[R \cup \{u\}], c) = 3$. We will now show that the path length between any vertex in $L$ and any vertex in $R$ is at most 3. For every vertex $v_i$ in $L$, $c(uv_i) = 1$, and so we can show this for a single vertex in $L$, say $v_1$, without loss of generality. Since $c(uv_1) = 1$, $v_1$ can reach any vertex $v_j$ in $R$ with $c(uv_j) = 2$ via the path $v_1uv_j$ of length 2. Otherwise, if $v_j$ in $R$ has $c(uv_j) = 1$ with $j < n$, then $v_1uv_j+1v_j$ is a path of length 3 from $v_1$ to $v_j$. Since the proper distance between $v_1$ and $v_{n-(k-1)}$ is $n - k \geq 3$ and since the proper distance between any other pair of vertices is either 2 or 3, we see that $\text{pdiam}(G, c) = n - k$, as desired.

![Figure 5](image-url)

Figure 5. A properly connected fan with $\text{pdiam}_2(F_{1,n}, c) = n - k$ for $1 \leq k \leq n - 3$ and $n \geq 7$.

### 2.3. Complete multipartite graphs

Finally, we consider properly connected 2-colorings $c$ of complete multipartite graphs $K_{n_1,n_2,\ldots,n_\ell}$ with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$. If $\ell \geq 3$, then $\text{pdiam}_2(K_{n_1,n_2,\ldots,n_\ell}, c) \in [2,4]$. These bounds are tight and all values are attainable. If $\ell = 2$, then the graph is a complete bipartite graph $K_{n,m}$. Since no pair of vertices in the same partite set can have a proper distance of 3, the only possible values of $\text{pdiam}_2(K_{n,m}, c)$ are 2 and 4. Both values are attainable.

**Theorem 6.** Let $K_{n_1,n_2,\ldots,n_\ell}$ be a properly connected 2-colored, complete $\ell$-partite graph with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, where $\ell \geq 2$. The proper diameter of $K_{n_1,n_2,\ldots,n_\ell}$ falls in the interval $[2,4]$.

**Proof.** Consider any properly connected 2-coloring $c$ of a complete $\ell$-partite graph with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, where $\ell \geq 2$. Since there is a path of length 1 between vertices in distinct partite sets, we only need to consider the lengths of properly colored paths between vertices in the same partite set. Let
a1 and a2 be any two distinct vertices in a partite set A. Let P = a1x1x2 · · · xja2 be a shortest properly colored path between a1 and a2. Assume for the moment that the proper distance between a1 and a2 is at least 5 so that j ≥ 4. For some h between 2 and j − 1, there is an xh on P such that xh is not in A. Since P is properly colored, we may assume that c(xh−1xh) = 1 and c(xhxh+1) = 2. Then c(a1xh) = 2 as otherwise a1xhxh+1 · · · xja2 is a shorter properly colored path than P. Similarly, c(xh,a2) = 1 as otherwise a1 · · · xhxja2 is a shorter properly colored path than P. However, since c(a1xh) = 2 and c(xh,a2) = 1, the path a1xh,a2 is a properly colored path of length 2 between a1 and a2, which is a contradiction.

In Theorems 7, 8 and 9, we respectively classify when a proper diameter of 2, 3, or 4 is attainable for complete multipartite graphs. Focusing on complete bipartite graphs, a proper diameter of 2 is attainable except for those Kn,m where one partite set is much larger than the other \( \max\{n, m\} > 2\min\{n, m\} \). On the other hand, besides those Kn,m with very small partite sets \( \min\{n, m\} = 1 \) or \( \max\{n, m\} ≤ 2 \) a proper diameter of 4 can always be achieved.

**Theorem 7.** Let \( K_{n_1,n_2,...,n_{\ell}} \) be an \( \ell \)-partite graph of order \( n \) where \( n_\ell ≥ 2 \) and \( 1 \leq n_1 ≤ n_2 ≤ \cdots ≤ n_\ell \). There exists a properly connected 2-coloring of \( K_{n_1,n_2,...,n_{\ell}} \) with proper diameter 2 if and only if \( n_\ell ≤ 2^{n−n_\ell} \).

**Proof.** Let \( G = K_{n_1,n_2,...,n_{\ell}} \). Label the partite sets of \( G \) as \( A_1, A_2, \ldots , A_\ell \) where \( A_i = \{a^i_1, a^i_2, \ldots , a^i_{n_i}\} \), so that partite set \( A_i \) has size \( n_i \). We let \( B \) refer to the \( n−n_\ell \) vertices not in \( A_\ell \), and for clarity, we label these vertices as \( b_1, b_2, \ldots , b_{n−n_\ell} \) when otherwise distinguishing between the distinct partite sets of the vertices of \( B \) would overcomplicate the proof.

In any properly connected 2-coloring \( c \) of \( G \), each vertex \( a \in A_\ell \) has a coloring tuple with \( B \), say \( (c_1, c_2, \ldots , c_{n−n_\ell}) \) which indicates that the edge \( ab_i \) has color \( c_i \in \{1, 2\} \) where \( 1 ≤ i ≤ n − n_\ell \). Thus, there are \( 2^{n−n_\ell} \) possible coloring tuples and so \( 2^{n−n_\ell} \) ways to color the edges between any vertex \( a \in A_\ell \) and the \( n−n_\ell \) vertices of \( B \).

We prove the forward direction by contrapositive. Let \( n_\ell > 2^{n−n_\ell} \) and consider any properly connected 2-coloring \( c \) of \( G \). We argue that \( G \) does not have proper diameter 2. Since \( n_\ell > 2^{n−n_\ell} \), the number of vertices in \( A_\ell \) is greater than the number of possible coloring tuples, and thus, at least two vertices in \( A_\ell \), say \( a^\ell_i \) and \( a^\ell_j \), have the same coloring tuple with \( B \). This means that for every vertex \( b \in B \), \( c(a^\ell_ib) = c(a^\ell_jb) \) for all \( b \in B \). As a result, there does not exist a properly colored path of length 2 between \( a^\ell_i \) and \( a^\ell_j \) and so the proper diameter of \( G \) is greater than 2.

For the backwards direction, we construct a coloring with proper diameter 2. The first step of this coloring process is described in this paragraph. For
each $A_i$ where $1 \leq i \leq \ell - 1$, do the following. Consider the edges in the set 
$\{a_1^i a_1^i, a_2^i a_2^i, \ldots, a_n^i a_n^i\}$. This set of edges is a matching between $A_i$ and the first 
n_i vertices of $A_\ell$. Color these edges with color 1. Then color all remaining edges 
between $A_i$ and the first $n_{\ell-1}$ vertices of $A_\ell$, namely, $a_1^i, a_2^i, \ldots, a_{n_{\ell-1}}^i$, with color 
2. This portion of the coloring ensures that there is a properly colored path of 
length 2 between any pair of vertices in $A_i$, as desired. To see this, for any 2 
distinct vertices $a_j^i$ and $a_j^i$ in $A_i$, note that the path $a_j^i a_j^i a_j^i$ is a properly colored 
path of length 2.

Upon completing the first step of our coloring construction, described in the 
previous paragraph, observe that a color has been assigned to each of the edges 
incident to the first $n_{\ell-1}$ vertices of $A_\ell$ and to none of the edges incident to the 
final $n_\ell - n_{\ell-1}$ vertices of $A_\ell$. Furthermore, by choice of the coloring in step 1, each 
of the initial $n_{\ell-1}$ vertices of $A_\ell$ has a distinct coloring tuple $(c_1, c_2, \ldots , c_{n-n_\ell})$ 
with the $n - n_\ell$ vertices of $B$. This is due to how we colored the edges incident 
to vertices in $A_{\ell-1}$, since for all vertices $a_\ell^i \in A_\ell$ where $1 \leq i \leq n_{\ell-1}$, $a_\ell^i$ is the 
only vertex connected to $a_\ell^{i-1}$ by an edge colored by $c$ with color 1. Therefore, 
when done step 1, we used $n_{\ell-1}$ of the $2^{n-n_\ell}$ possible coloring tuples with $B$ and 
so there are $2^{n-n_\ell} - n_{\ell-1}$ that remain.

Finally, we discuss how to color the edges incident to the remaining $n_\ell - n_{\ell-1}$ 
vertices in $A_\ell$. Since $n_\ell \leq 2^{n-n_\ell}$, we see that $n_{\ell-1} - n_{\ell-1} \leq 2^{n-n_\ell} - n_{\ell-1}$ which 
means that the number of vertices connected to $A_\ell$ with uncolored edges is no 
larger than the number of unused coloring tuples. Thus, there are enough coloring 
tuples to ensure that we can color the edges between the remaining vertices of $A_\ell$ 
and $B$ so that each remaining vertex $a_\ell^j \in A_\ell$ has a distinct coloring tuple with $B$. 
After doing so, all vertices in $A_\ell$ have a distinct coloring tuple with $B$.

It remains to show that there is a path of length 2 between any pair of 
vertices in $A_\ell$. Let $a_\ell^i$ and $a_\ell^j$ be distinct vertices in $A_\ell$ with coloring tuples 
$(c_1, c_2, \ldots , c_{n-n_\ell})$ and $(\tilde{c}_1, \tilde{c}_2, \ldots , \tilde{c}_{n-n_\ell})$, respectively. By construction, these 
tuples are distinct and so $c_l \neq \tilde{c}_l$ for some $l$. Then $a_\ell^i b_l a_\ell^j$ is a properly colored path 
of length 2.

Finally, color the edges between the partite sets $A_1, A_2, \ldots , A_{\ell-1}$ in any fashion. 
The proper diameter of the constructed 2-coloring of $G$ is 2, as desired. □

**Theorem 8.** For $\ell \geq 3$, there exists a properly connected 2-coloring with proper 
diameter 3 of any complete $\ell$-partite graph $K_{n_1,n_2,\ldots,n_\ell}$ where $1 \leq n_1 \leq n_2 \leq \cdots \leq 
n_\ell$ and $n_\ell \geq 2$.

**Proof.** Let $A$, $B$ and $C$ be three partite sets, where $A$ is a largest partite set. 
First, let us suppose that there are at least four vertices in $A$. Let $A_1$ be all but 
any two vertices of $A$ with all edges of color 1 to $B$ and $C$. Let $A_2$ be a single 
vertex of $A$ with all edges of color 1 to $B$ and all edges of color 2 to $C$. Let $A_3$ be 
a single vertex of $A$ with all edges of color 2 to $B$ and all edges of color 1 to $C$. 

Color the edges between $B$ and $C$ with color 2. Then there is a properly colored path containing a vertex in $B$ and a vertex in $C$ between any pair of vertices in $A_1$, giving a shortest properly colored path of length 3. Similarly, there is a properly colored path of length 3 between any existing pairs of vertices in $B$ or in $C$ using the vertex in $A_3$ or $A_2$, respectively. Since vertices in different partite sets are adjacent, and the vertices in $A_1$, $A_2$ and $A_3$ that are not both in $A_1$ are connected by a properly colored path of length 2, the proper diameter of this coloring is 3. See Figure 6.

Figure 6. This figure depicts a properly connected 2-coloring of a complete 3-partite graph with proper diameter 3. A thick edge indicates that all edges between the sets are the same color.

To generalize the above coloring for any number of partite sets, color the edges from any additional partite set $D$ to every existing vertex in the graph using color 1. Between each pair of vertices in $D$, there is a properly colored path of length 3 using an edge of color 1 from $D$ to $B$, an edge of color 2 from $B$ to $C$, and an edge of color 1 from $C$ to $D$.

If $A$ has exactly 3 vertices and at least one of $B$ or $C$ has 2 or 3 vertices, then each set $A_1$, $A_2$, and $A_3$ consists of a single vertex with the same coloring described above. Since either $B$ or $C$ has 2 vertices, the proper diameter of 3 is achieved by the properly colored path between the pair of vertices in that set. If either $A$ has 3 vertices and both $B$ and $C$ have a single vertex or if each partite set has at most 2 vertices, color a single edge between $B$ and $C$ with color 2, and color all remaining edges in the graph with color 1. Then any existing pairs of vertices in $B$ or in $C$ have proper distance 2, and the shortest properly colored path between each pair of vertices in $A$ requires an edge of color 1 from $A$ to $B$, the edge of color 2 from $B$ to $C$, and an edge of color 1 from $C$ to $A$. Since $A$ must have at least 2 vertices, this gives a proper diameter of 3. In any of these cases, the same generalization as above also holds for any additional partite set $D$ by coloring the edges between $D$ and all other partite sets with color 1.
Theorem 9. There exists a properly connected 2-coloring of $K_{n,m}$ with proper diameter 4 if and only if $\min\{n,m\} \geq 2$ and $\max\{n,m\} \geq 3$. Furthermore, for $\ell \geq 3$, there exists a properly connected 2-coloring of any complete $\ell$-partite graph $K_{n_1,n_2,\ldots,n_\ell}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, with proper diameter 4 if and only if the largest partite set contains at least three vertices.

Proof. Let $G = K_{n,m}$ be a complete bipartite graph with a properly connected 2-coloring $c$. If $\min\{n,m\} = 1$ or $\max\{n,m\} \leq 2$, then $G \cong K_{1,\max\{n,m\}}$ or $G \cong C_4$, none of which can be colored so as to have proper diameter 4. Thus, if $\text{pdiam}_2(K_{n,m}, c) = 4$, then $\min\{n,m\} \geq 2$ and $\max\{n,m\} \geq 3$.

Now, consider $K_{n,m}$ where the smaller partite set has at least 2 vertices and the larger partite set has at least 3. Let $A$ be a set of any two vertices from the larger partite set and let $A'$ be a set containing the remaining vertices of $A$. Also, let $B$ be a set containing any one vertex from the smaller partite set and let $B'$ contain the remaining vertices of $B$. Color all edges between $A$ and $B$ and between $A'$ and $B'$ with color 1. Color all other edges with color 2. The proper distance between the two vertices in $A$ is 4 and so the proper diameter of this coloring is 4 as well. See Figure 7 for an example involving $K_{3,5}$. A similar coloring to Figure 7 appears in [2].

![Figure 7. A properly connected 2-coloring of $K_{3,5}$ with proper diameter 4.](image)

For $\ell \geq 3$, suppose that a largest partite set of a complete $\ell$-partite graph $K_{n_1,n_2,\ldots,n_\ell}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, contains at least 3 vertices. As in the bipartite case, let $A$ be a set of any two vertices from a largest partite set and let $A'$ contain the remaining vertices. Let $B$ and $C$ denote additional partite sets. Color all edges between $A$ and $B$, between $A'$ and $C$, and between $B$ and $C$ with color 1. Color all edges between $A$ and $C$ and between $A'$ and $B$ with color 2. Then the proper distance between the pair of vertices in $A$ is 4, and the proper distance between any other pair of vertices is at most 4. Hence, the proper diameter is 4. For any additional partite set $D$, color all edges between $D$ and the other partite sets with color 1. Then the proper distance is 3 for any existing pairs of vertices in $D$, and so the proper diameter remains 4.
Finally, assume a complete \(\ell\)-partite graph \(K_{n_1,n_2,\ldots,n_\ell}\), where \(1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell\), has proper diameter 4 and each partite set contains at most 2 vertices. Then there must be a partite set \(A\) with exactly 2 vertices, say \(a_1\) and \(a_2\). Suppose \(a_1x_1x_2x_3a_2\) is a shortest properly colored path from \(a_1\) to \(a_2\) of length 4 with \(c(x_1x_2) = 1\) and \(c(x_2x_3) = 2\). Since \(x_2\) is not in \(A\), there is an edge \(a_1x_2\), which must have color 2 otherwise \(a_1x_2x_3a_2\) is a shorter properly colored path. Similarly, edge \(x_2a_2\) must have color 1. This leads to an alternating path \(a_1x_2a_2\) of length 2, giving a contradiction. Therefore, if no partite set contains at least 3 vertices, then the proper diameter must be less than 4.

We conclude with Corollary 10, which follows from Theorem 6, Theorem 7, and Theorem 9, and summarizes the facts about complete bipartite graphs.

**Corollary 10.** Let \(m \geq n\) and \(m \geq 2\). The proper diameter of any properly connected 2-coloring of \(K_{n,m}\) is 2 or 4. Moreover,

(i) A proper diameter of 2 is attainable if and only if \(m \leq 2n\).

(ii) A proper diameter of 4 is attainable if and only if \(n \geq 2\) and \(m \geq 3\).

### 3. General Bounds

Whereas the results of previous sections focus on specific graph families, in this section we provide preliminary bounds on the maximum value of \(\text{pdiam}_2(G, c)\), which is \(\text{pdiam}_2(G)\) for general graphs \(G\). Analyzing the connectivity of \(G\) is a natural starting point, since a large value of \(\kappa(G)\) provides density and flexibility in routing from a given vertex to another. Theorem 12 provides a worst-case bound on the proper diameter of a 2-coloring of \(G\) and shows that as the connectivity increases for a fixed value of \(n\), the maximum proper diameter for 2-colorings decreases. The proof is by contradiction and relies on Lemma 11. To guide the reader through the proof of Lemma 11, we first assume that a 3-connected graph has a shortest properly colored Hamiltonian path between \(v_1\) and \(v_t\). Then we find a shorter properly colored path by starting at \(v_1\) and iterating through edges of the form \(v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}\) for all \(j\) needed to reach \(v_t\). For example, in Figure 8, a shorter properly colored path is \(v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8\).

**Lemma 11.** In any properly connected graph \(G\) on \(t\) vertices with \(\kappa(G) \geq 3\), \(\text{pdiam}_2(G) < t - 1\).

**Proof.** Assume \(G\) is a 3-connected graph on \(t\) vertices with a properly connected 2-coloring \(c\), where \(\text{pdiam}_2(G, c) = t - 1\). Then there exists a shortest properly colored path \(P = v_1 v_2 \cdots v_t\) between some vertices \(v_1\) and \(v_t\). The edges of \(P\)
alternate between 2 colors, so suppose \(c(v_{2i−1}v_{2i}) = 1\) and \(c(v_{2i}v_{2i+1}) = 2\) for each integer \(i\) with \(1 ≤ i ≤ \left\lfloor \frac{t−1}{2} \right\rfloor\). Note that \(c(v_{t−1}, v_t) = 1\) if \(t\) is even.

We now construct a specific sequence of vertices \(v_i, v_{i+1}, \ldots, v_m\) in order to find a shorter properly colored path from \(v_1\) to \(v_t\) than \(P\). Since \(G\) is 3-connected, \(v_1\) must have 2 neighbors besides \(v_2\). This corresponds to two edges of the form \(v_1v_i\) for \(i > 2\). Let \(i_1\) be the largest index such that \(v_1v_{i_1} \in E(G)\). In particular, \(3 < i_1 < t\). Following similar logic as that just used to define \(v_{i_1}\), define \(v_j\) for \(j ≥ 2\) reiteratively as below. See Figure 8 for an illustration.

- If \(v_t\) is adjacent to a vertex of \(\{v_1, v_3, v_4, \ldots, v_{i_1−1}\}\), then let \(i_j = t\) and \(m = j\).
- Otherwise, there must be an edge from \(L_{i_j} = \{v_1, v_3, v_4, \ldots, v_{i_j−1}\}\) to \(R_{i_j} = \{v_{i_j+1}, \ldots, v_t\}\). If not, then \(\{v_2, v_{i_j−1}\}\) is a cut set. From all the edges that exist between \(L_{i_j}\) and \(R_{i_j}\), let \(v_j\) be the vertex with the largest index from \(R_{i_j}\) that is incident to such an edge. Define \(e_j\) as an edge from \(L_{i_j}\) to \(R_{i_j}\) with endpoints \(v_j\) and \(v_{j−1}\), respectively. Note that \(e_j\) cannot be incident to vertices \(v_2\) or \(v_{i_j−1}\), because they are excluded from \(L_{i_j}∪R_{i_j}\). We exclude \(v_2\) from \(L_{i_j}\) in order to build paths from \(v_1\) to \(v_t\) that skip \(v_2\). Note that \(i_j < i_{j+1}\) by construction.

![Figure 8. A strategic sequence](image)

We proceed by induction on \(j\) to show that for \(1 ≤ j ≤ m−1\), we have \(c(e_{j+1}) = c(v_{i_{j+1}}, v_{i_{j+1}+1})\), and additionally, there is a properly colored path \(P_{i_{j+1}}\) from \(v_1\) to \(v_{i_{j+1}}\) of length at most \(i_j − 1\) with terminal edge \(v_{i_{j+1}}, v_{i_{j+1}+1}\).

When \(j = 1\), we have \(v_{i_{j+1}} = v_{i_2}\). If \(i_2 = t\), then depending on whether the color of edge \(e_2 = v_{i_2}v_{i_2} = v_{i_2}v_{i_2}\) differs from \(c(v_{i_2−1}v_{i_2})\) or \(c(v_{i_2}v_{i_2}+1)\), either \(v_{i_2}v_{i_2}−1v_{i_2}\) or \(v_{i_2}v_{i_2}v_{i_2}+1\), respectively, would be a properly colored path from \(v_1\) to \(v_{i_2}\) that is shorter than \(P\), as these paths miss \(v_{i_1}\) and \(v_2\), respectively. Thus, we may assume that \(i_2 < t\). In this case, if \(c(e_2) = c(v_{i_2}v_{i_2}) = c(v_{i_2−1}v_{i_2})\), then one of \(v_1v_{i_2}v_{i_2}v_{i_2}v_{i_2}+1\) or \(v_1v_{i_1}v_{i_1−1}v_{i_2}v_{i_2}v_{i_2}+1\), would be a shorter properly colored path from \(v_1\) to \(v_2\). Therefore, we must have \(c(e_2) = c(v_{i_2}v_{i_2}) = c(v_{i_2}v_{i_2}+1)\). Additionally, if \(c(v_{i_2−1}v_{i_2}) = c(v_{i_2}v_{i_2})\), there is a properly colored path \(P_{i_2}\) of length at most \(i_1−1\), given by \(v_1v_{i_1}v_{i_1−1}v_{i_2}\). Note that this path is properly colored only if \(c(v_1v_{i_1}) \neq c(v_{i_1}v_{i_1+1})\), which we may assume is true since otherwise \(c(v_1v_{i_1}) \neq c(v_{i_1}v_{i_1+1})\) and then \(v_1v_{i_1}v_{i_1+1} \cdots v_{i_2}\)
is a properly colored path from \( v_1 \) to \( v_t \) that is shorter than \( P \). If \( c( v_{i_2} v_{i_2} ) \neq c( v_{i_2} v_{i_2} ) \), then \( v_1 v_2 \cdots v_{i_2-1} v_{i_2} v_{i_2} \) is a properly colored path of length at most \( i_1 - 1 \), giving the desired \( P_{i_2} \).

Assume \( c( v_{i_1+j} v_{i_1+j} ) = c( v_{i_1+j} v_{i_1+j} ) \) for all \( j \) such that \( 1 \leq j < \ell \) for some \( \ell \leq m - 1 \). Additionally, assume that there is a properly colored path \( P_{i_1+j} \) from \( v_1 \) to \( v_{i_1+j} \) of length at most \( i_1 - 1 \) with terminal edge \( v_{i_1+j} v_{i_1+j} \). Suppose that \( c( v_{i_1+j} v_{i_1+j} ) \neq c( v_{i_1+j} v_{i_1+j} ) \). Now if \( c( v_{i_1+j} v_{i_1+j} ) \neq c( v_{i_1+j} v_{i_1+j} ) \), then \( P' = v_1 v_2 \cdots v_{i_1+j} v_{i_1+j} v_{i_1+j+1} \cdots v_t \) is a shorter properly colored path from \( v_1 \) to \( v_t \) that skips \( v_i \), which is a contradiction. Otherwise, we know that \( c( v_{i_1+j} v_{i_1+j} ) = c( v_{i_1+j} v_{i_1+j} ) \).

By induction, there is a properly colored path \( P_{i_1} \) from \( v_1 \) to \( v_{i_1} \) of length at most \( i_{i_1-1} - 1 \) with terminal edge \( v_{i_1} v_{i_1} \) such that \( c( v_{i_1} v_{i_1} ) = c( v_{i_1} v_{i_1} ) \). Thus, \( P_{i_1} v_{i_1+1} \cdots v_{i_1+1} v_{i_1+1} v_{i_1+2} \cdots v_t \) is a shorter properly colored path from \( v_1 \) to \( v_t \), which is a contradiction. This means that our supposition is incorrect and thus \( c( v_{i_1} v_{i_1} ) = c( v_{i_1} v_{i_1} ) \). Moreover, the path \( P_{i_1} v_{i_1+1} \cdots v_{i_1+1} v_{i_1+1} \) is the desired proper path of length at most \( i_{i_1-1} - 1 \) with terminal edge \( v_{i_1} v_{i_1} \). When \( \ell = m - 1 \) this gives a properly colored path from \( v_1 \) to \( v_{i_1} = v_t \) of length at most \( i_{m-1} - 1 \) yielding a contradiction.

**Theorem 12.** For any properly connected 2-colored graph \( G \) of order \( n \geq 2 \),

\[
(1) \quad pdiam_2(G) \leq n - \kappa(G) + 1.
\]

The bound in (1) is tight.

**Proof.** Let \( c \) be a properly connected 2-coloring of a graph \( G \) of order \( n \geq 2 \). First we will show that the bound is tight by the following construction. Let \( \ell \) be a positive integer and start with an alternating path \( P_{2\ell+1} = v_1 v_2 \cdots v_{2\ell+1} \) in which \( c(v_{2i-1} v_{2i}) = 1 \) and \( c(v_{2i} v_{2i+1}) = 2 \) for all \( i \). To this, we add the edges \( v_1 v_{2\ell} \) and \( v_2 v_{2\ell+1} \) coloring the former with color 2 and the latter with color 1. This graph is properly connected, the underlying uncolored graph is 2-connected, and the proper diameter is \( 2\ell \). Finally, we add \( k - 2 \) vertices, coloring all possible edges between each vertex and the vertices of \( P_{2\ell+1} \) with color 1, thus inducing a complete graph in color 1. See Figure 9. The resulting graph is still properly connected, the underlying uncolored graph is \( k \)-connected, and the proper diameter is still \( 2\ell = n - k + 1 \). Note that we achieve the desired bound if \( k = \kappa(G) \) and also if \( n - k \) is even using a similar construction.

Suppose that the proper diameter exceeds \( n - \kappa(G) + 1 \). Then there exists a pair of vertices in \( G \), say \( u \) and \( v \), such that the proper distance between \( u \) and \( v \)
is at least $n - \kappa(G) + 2$. Let $P$ be a shortest properly colored path on $t$ vertices from $u$ to $v$. Note that $t$ is one more than the proper distance between $u$ and $v$. Let $R$ be the remaining vertices and let $G' = G[P]$. Since $|R| \leq \kappa(G) - 3$, we see that $\kappa(G') \geq 3$. By Lemma 11, $\text{pdiam}_2(G', c) < t - 1$, and so there exists a properly colored path between $u$ and $v$ in $G'$ with length less than $P$, which is a contradiction.

Figure 9. This figure illustrates a properly connected graph $G$ with proper diameter $n - k + 1$. The thick solid red edge depicts that all edges between the two sets of vertices are color 1.

Next, we look at Lemma 13, which gives a general lower bound using girth. In Section 2.1 we argue that any properly colored odd cycle on $2m + 1$ vertices has proper diameter $2m - 1$. This observation generalizes to any graph whose girth is given by an odd cycle and also demonstrates that the bound in Lemma 13 is tight.

**Lemma 13.** If $G$ is a graph with properly connected 2-coloring $c$ and girth $2m + 1$ for $m \geq 2$, then $\text{pdiam}_2(G, c) \geq 2m - 1$.

**Proof.** Let $v_1, v_2, \ldots, v_{2m+1}$ denote the vertices of a $(2m + 1)$-cycle in $G$, with $m \geq 2$. However the edges of the cycle are colored, since there are two colors and an odd number of edges, some pair of adjacent edges $v_iv_{i+1}$ and $v_{i+2}v_{i+3}$ must share a color. Thus, the shortest properly colored path from $v_i$ to $v_{i+2}$ on this cycle must have length $2m - 1$. Suppose there is a properly colored path of length less than $2m - 1$ from $v_i$ to $v_{i+2}$. Let $v_iw_1w_2\cdots w_{i+2}$ denote such a path of minimum length in $G$. Then $v_iw_1w_2\cdots w_{i+1}v_{i+2}$ is a cycle of length less than $2m + 1$ since there is no shorter path between any two vertices on this path. Therefore, $\text{pdiam}_2(G, c) \geq 2m - 1$ if $G$ contains an odd cycle with length $2m + 1$ and no smaller cycle.
We conclude this section by considering how a small modification to a properly connected graph can affect the proper diameter. In general, removing a vertex from a properly connected graph does not necessarily result in a new graph that is still properly connected since the deleted vertex may have been required to obtain a properly colored path between some pair of vertices. Additionally, adding vertices and edges to a graph does not always yield a properly connected graph, depending on how the additional edges are colored. In the following lemma, given a non-complete graph \( G \) with a properly connected 2-coloring \( c \), we look at one way to form a new graph \( G' \) with a properly connected 2-coloring \( c' \) by adding a degree-2 vertex and two edges to \( G \), and we consider how the proper diameter of \( G' \) under \( c' \) is related to that of \( G \) under \( c \). By Lemma 14, under certain conditions, the addition of a single vertex of degree 2 to \( G \) at worst doubles the proper diameter of the original edge-colored graph.

**Lemma 14.** Let \( G \) be a graph with a properly connected 2-coloring \( c \) such that \( \text{pdiam}_2(G, c) = d > 1 \). Let \( u_1, u_{d+1} \in V(G) \) such that the proper distance between \( u_1 \) and \( u_{d+1} \) is \( d \), and let \( u_1u_2 \cdots u_d u_{d+1} \) be a shortest properly colored path from \( u_1 \) to \( u_{d+1} \). Let \( G' \) be a graph formed by adding a vertex \( v \) to \( G \) and edges \( vu_1 \) and \( vu_{d+1} \) with edge-coloring \( c' : E(G') \to \{1, 2\} \) such that

\[
c'(e) = \begin{cases} 
  c(e) & \text{if } e \in E(G), \\
  3 - c(u_1u_2) & \text{if } e = u_1v, \\
  3 - c(u_d u_{d+1}) & \text{if } e = u_{d+1}v.
\end{cases}
\]

Then \( \text{pdiam}_2(G', c') \leq 2d \).

Figure 10. A properly connected \( P_5 \) and \( P_6 \) and a degree-2 vertex \( v \) in the cases where \( d = 4 \) and \( d = 5 \).

**Proof.** The only edges that do not exist in both \( G \) and \( G' \) are those incident with \( v \). Since \( c \) and \( c' \) assign the same colors to all other edges, it suffices to show that for any vertex \( w \in V(G') \), there exists a properly colored path of length of at most \( 2d \) between \( w \) and \( v \).

Observe that by definition, \( c'(u_1v) \neq c'(u_1u_2) \) and \( c'(u_{d+1}v) \neq c'(u_d u_{d+1}) \). Hence, paths \( P_1 = vu_1u_2 \cdots u_d u_{d+1} \) and \( P_2 = u_1u_2 \cdots u_d u_{d+1}v \) are both properly colored in \( G' \). Since \( P_1 \) and \( P_2 \) each has length \( d+1 \), the proper distance between \( v \) and \( u_i \) for \( 1 \leq i \leq d + 1 \) is therefore at most \( d + 1 \), as desired.
Let us now consider any vertex $w_1 \neq v$ where $w_1$ is not on the path $P = u_1u_2\cdots u_{d+1}$. Let $Q = w_1w_2\cdots w_n$ be a shortest properly colored path in $G$ between $w_1$ and any vertex on $P$. Then the endpoint of $Q$ is a vertex on $P$, so $w_n = u_j$ for some $1 \leq j \leq d + 1$. Also, the vertices $w_1, w_2, \ldots, w_{n-1}$ are not on $P$ as otherwise there exists a properly colored path that is shorter than $Q$ between $w_1$ and some vertex on $P$. Finally, the path $Q$ has length at most $d$ since $\text{pdim}_2(G, c) = d$.

The path $Q$ exists in $G'$ and its edges are colored the same by $c'$ as $c$. Since the color of the final edge of $Q$ must be distinct from either $c'(u_{j-1}, u_j)$ or $c'(u_j, u_{j+1})$, either the path $QP_1'$ formed by concatenating $Q$ with $P_1' = u_ju_{j-1}\cdots u_1v$ or the path $QP_2'$ formed by concatenating $Q$ with $P_2' = u_ju_{j+1}\cdots u_{d+1}v$ is a properly colored path from $w_1$ to $v$. Note that $P_1'$ and $P_2'$ have length at most $d$ if $Q$ ends at an internal vertex of $P$ but one of $P_1'$ or $P_2'$ has length exactly $d + 1$ if $Q$ ends at an endpoint of $P$. Hence, if $Q$ ends at an internal vertex of $P$ or if $Q$ has length strictly less than $d$, then both $QP_1'$ and $QP_2'$ have length at most $2d$, and so no matter which of $QP_1'$ and $QP_2'$ is properly colored, the vertices $w_1$ and $v$ have a proper distance of at most $2d$, as desired.

Otherwise the path $Q$ has length exactly $d$ and ends at an endpoint of $P$. Since $Q$ by definition is a shortest properly colored path in $G$ between $w_1$ and any vertex on $P$ and since $\text{pdim}_2(G, c) = d$, this means that a shortest properly colored path $Q'$ between $w_1$ and $u_2$ exists and is also of length exactly $d$. Say $Q'$ has an internal vertex $u_k \in P = u_1u_2\cdots u_{d+1}$. Then the subpath on $Q'$ from $w_1$ to $u_k$ yields a properly colored path shorter than $d$ from $w_1$ to $P$, giving a contradiction. Hence, only the endpoint of $Q'$ is on $P$. As before, either concatenating the path $Q'$ with $u_2u_3\cdots u_{d+1}v$ or $u_2u_1v$ yields a properly colored path from $w_1$ to $v$. Also, both of these concatenated paths have length at most $2d$ since $u_2$ is an internal vertex of $P$. Thus, the claim holds.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\text{diam}(G)$</th>
<th>$\text{pdim}_2(G, c)$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>path, $P$</td>
<td>$\text{diam}(P)$</td>
<td>$\text{diam}(P)$</td>
<td>If the path has $n$ vertices, then $\text{diam}(P) = \text{pdim}_2(P, c) = n - 1$ holds for all colorings $c$.</td>
</tr>
<tr>
<td>$K_n$</td>
<td>1</td>
<td>1</td>
<td>This is the only possibility.</td>
</tr>
<tr>
<td>$C_{2m}$</td>
<td>$m$</td>
<td>${m, 2m - 2}$</td>
<td>These are the only possibilities. See Observation 1.</td>
</tr>
<tr>
<td>$C_{2m+1}$</td>
<td>$m$</td>
<td>$2m - 1$</td>
<td>This is the only possibility. See Observation 1.</td>
</tr>
<tr>
<td>$F_{l,n}, n \geq 7$</td>
<td>2</td>
<td>$[3, n - 1]$</td>
<td>Bounds are tight and all values are attainable. See Theorems 4 and 5.</td>
</tr>
<tr>
<td>$K_{n,m}$</td>
<td>2</td>
<td>${2, 4}$</td>
<td>See Corollary 10.</td>
</tr>
<tr>
<td>$K_{m_1,m_2,\ldots,m_l}$</td>
<td>2</td>
<td>$[2, 4]$</td>
<td>Bounds are tight and all values are attainable. See Theorems 6 through 9.</td>
</tr>
</tbody>
</table>

Table 1. Diameter vs. proper diameter.
4. Conclusion

Table 1 compares the proper diameter for the graph families discussed in this paper to their diameter. For odd cycles and sufficiently large fan graphs, the proper diameter cannot equal the diameter of the graph using two colors. We also see that fan graphs and complete multipartite graphs (with at least three partite sets) are two families of graphs that can attain any proper diameter value between the upper and lower bounds with two colors.

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