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THE CROSSING NUMBER OF CARTESIAN PRODUCT OF 5-WHEEL WITH ANY TREE

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Abstract

In this paper, we establish the crossing number of join product of 5-wheel with n isolated vertices. In addition, the exact values for the crossing numbers of Cartesian products of the wheels of order at most five with any tree T are given.

Keywords: drawing, crossing number, join product, Cartesian product.

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1. Introduction

Given a graph G, let $V(G)$ and $E(G)$ be, respectively, its vertex and edge set. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. All drawings considered herein are good drawings meaning that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, no more than two edges cross at a point of the plane, and no edge meets a vertex, which is not its endpoint. We denote the number of crossings in a good drawing D of the graph G by $\mathrm{cr}_D(G)$. A good drawing is said to be optimal if it minimizes the number of crossings. The crossing number $\mathrm{cr}(G)$ of a graph G is the number of crossings in any optimal drawing of G in the plane. Let $G^{'}$ be a subgraph of the graph G. Then we easily get

$$cr(G^{'}) \leq cr(G).$$

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For a graph G, let E_i, E_j and E_k be edge-disjoint subsets of $E(G)$. We denote by $\mathrm{cr}_D(E_i, E_j)$ the number of crossings between edges of E_i and edges of E_j in D, and by $\mathrm{cr}_D(E_i)$ the number of crossings among edges of E_i in D. It is easy to see that

$$cr_D(E_i \cup E_j) = cr_D(E_i) + cr_D(E_j) + cr_D(E_i, E_j), \tag{1.1}$$

and

$$cr_D(E_i \cup E_j, E_k) = cr_D(E_i, E_k) + cr_D(E_j, E_k). \tag{1.2}$$

In our paper, we use $\langle E' \rangle$ to represent the edge-induced subgraph of G, where $E' \subseteq E(G)$. For more graph theory terminology and the theory of crossing number, see [1, 5].

The investigation on the crossing numbers of graphs is a classical but very difficult problem. In fact, computing the crossing number of a graph is NP-complete [6]. The exact values of crossing numbers are known only for few specific families of graphs. The complete bipartite graph $K_{m,n}$ is one of them. Zarankiewicz [19] conjectured that the crossing number of $K_{m,n}$ equals $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. This conjecture has been verified by Kleitman [10] for $\min\{m,n\} \le 6$, or, equivalently,

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \min\{m,n\} \le 6.$$

For convenience, the number $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m,n)$ in our paper. Several authors have been researching the crossing numbers of complete multipartite graphs. For the graph $K_{1,4,n}$, the crossing number was established independently in [7, 8]. The crossing number of the graph $K_{1,5,n}$ was given in [18].

The join product of two graphs is also what we are interested in. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. Particularly, let G_2 be a graph on n isolated vertices. The join product of two graphs G_1 and G_2 is also referred as the suspension of order n of the graph G_1 (denoted by G_1^n), where n isolated vertices are called the apices of G_1^n. Let C_n be the cycle of length n, P_n be the path on n vertices, and S_n the star $K_{1,n}$. The first results on crossing numbers of join of paths and cycles as well as of two cycles appeared in [12]. Moreover, the exact values for crossing numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most four were given in [12, 14]. Subsequently, several authors have studied the crossing numbers for join of paths and cycles with some connected graphs of order five [15]. Recently, in [4], the crossing number of join of a disconnected 5-vertex graph Q (see Figure 1) with

n isolated vertices was given. However, there are only few results concerning crossing numbers of join products of discrete graphs, paths and cycles with some graphs on six vertices, see [13]. Let W_n be the wheel on $n+1$ ($n \geq 3$) vertices. In the paper, we present the crossing number of join of a connected 6-vertex graph W_5 (see Figure 1) with n isolated vertices.

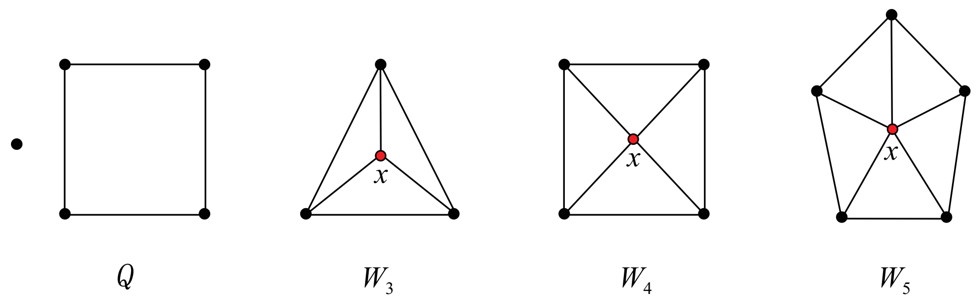


Figure 1. A disconnected 5-vertex graph Q and the wheels W_j ($j = 3, 4, 5$).

The richness of repetitive patterns in Cartesian products of graphs reflects in their drawing and makes Cartesian product one of the few graph classes, for which exact crossing number results are known. We denote the Cartesian product of two graphs G_1 and G_2 by $G_1 \Box G_2$. Klešč [11] established the crossing numbers of the products of all 4-vertex graphs with paths and stars except the crossing number of $K_{1,3} \Box P_n$, which was earlier determined by Jendroľ and Ščerbová [9], who conjectured that $cr(S_m \Box P_n) = (n-1) \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ for $m, n \geq 1$. This conjecture was proved by Bokal in [2]. In [3], the exact value of crossing number for $W_3 \Box T$ was given, where T is a tree with maximum degree at most three. In our paper, we extend the result by giving the crossing numbers of Cartesian products of the wheels W_j ($j = 3, 4, 5$) with any tree T.

In this paper, let nK_1 be the graph on n isolated vertices, we establish the crossing number of the graph $W_5 + nK_1$ (W_5^n) in Section 2. Our method is to construct the relationship between the crossing number of the graph $W_5 + nK_1$ and the crossing number of the graph $Q + (n+1)K_1$ (the graph Q is displayed in Figure 1). Using the zip product operation, in Section 3, we find the exact values of the crossing numbers of Cartesian products of the wheels W_j ($j = 3, 4, 5$) with any tree T. The result complements a recent result [16] by Klešč, who established the crossing number of any wheel with a tree of maximum degree at most five. At present, the Zarankiewicz conjecture on the crossing number of $K_{m,n}$ [19] has not been proved for $\min\{m, n\} \geq 7$, which obstruct our research with respect to the crossing number of $W_j \Box T$ for $j \geq 6$. The main results in this paper are the following theorems.

Theorem 1. *$cr(W_5 + nK_1) = Z(6,n) + n + 3\left\lfloor \frac{n}{2}\right\rfloor$ for $n \geq 1$.*

Theorem 2. *Let T be a tree with maximum degree $\Delta(T)$ and let n_i be the number of vertices of degree i in T. Then*

$$cr(W_j \,\square\, T) = \begin{cases} \sum_{i=1}^{\Delta(T)} n_i(Z(4,i)+i), & j=3, \\ \sum_{i=1}^{\Delta(T)} n_i\left(Z(5,i)+i+\left\lfloor \frac{i}{2}\right\rfloor\right), & j=4, \\ \sum_{i=1}^{\Delta(T)} n_i\left(Z(6,i)+i+3\left\lfloor \frac{i}{2}\right\rfloor\right), & j=5. \end{cases}$$

2. THE PROOF OF THEOREM 1

Let H be the graph $W_5 + nK_1$ with the edge set E and the vertex tripartition (X, Y, Z), where $X = \{x\}$ is the center of the wheel, $Y = \{y_1, y_2, \ldots, y_5\}$ are the rim vertices of the wheel, and $Z = \{z_1, z_2, \ldots, z_n\}$ are the apices added in the join product. It is easy to see that

$$E = E_{XY} \cup E_{XZ} \cup E_{YZ} \cup E_{YY}, \tag{2.1}$$

and

$$\bigcup_{i=1}^{5} \widetilde{E}_{y_i} = E_{XY} \cup E_{YZ}, \tag{2.2}$$

where

$$\begin{aligned} E_{XY} &= \{xy_i \mid i = 1,2,\ldots,5\}, \\ E_{XZ} &= \{xz_j \mid j = 1,2,\ldots,n\}, \\ E_{YZ} &= \{y_iz_j \mid i = 1,2,\ldots,5 \text{ and } j = 1,2,\ldots,n\}, \\ E_{YY} &= \{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1\}, \end{aligned}$$

and $\widetilde{E}_{y_i}$ is a set of all but the edges y_iy_{i-1} and y_iy_{i+1} of the edges incident with the vertex y_i.

Lemma 3 [16]**.** *If ϕ is an optimal drawing of H, then $cr_\phi(E_{YY}) = 0$.*

Lemma 4. *If ϕ is a good drawing of H, then*

$$\begin{aligned} \sum_{i=1}^{5} cr_\phi\left(E\backslash\widetilde{E}_{y_i}\right) &= 4cr_\phi(E) + \sum_{i=1}^{5} cr_\phi\left(E_{XZ}, \widetilde{E}_{y_i}\right) + cr_\phi(E_{XZ}, E_{YY}) \\ &+ cr_\phi(E_{YY}) - cr_\phi\left(E_{XY} \cup E_{XZ} \cup E_{YZ}\right). \end{aligned}$$

Proof. Let $(A \cup B)\backslash\widetilde{E}_{y_i}$ be the edge set that arise from the edge set $A \cup B$ by deleting the edges of $(A \cup B) \cap \widetilde{E}_{y_i}$, where A and B are edge subsets of E. Using (1.1),(1.2) and (2.1), we have

$$\begin{aligned}
\sum_{i=1}^{5} cr_\phi(E\backslash\widetilde{E}_{y_i}) &= \sum_{i=1}^{5} cr_\phi\Big(((E_{XY} \cup E_{XZ}) \cup E_{YZ} \cup E_{YY})\backslash\widetilde{E}_{y_i}\Big) \\
&= \sum_{i=1}^{5} cr_\phi\left((E_{XY} \cup E_{XZ})\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}\right) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{YY}\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) \qquad (2.3) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{XZ}\backslash\widetilde{E}_{y_i}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YY}\backslash\widetilde{E}_{y_i}\right) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{XZ}\backslash\widetilde{E}_{y_i}, E_{YY}\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}, E_{YY}\backslash\widetilde{E}_{y_i}\right).
\end{aligned}$$

Since the edge-induced subgraph $\langle(E_{XY} \cup E_{XZ})\backslash\widetilde{E}_{y_i}\rangle$ is isomorphic to the star S_{n+4}, $cr_\phi\big((E_{XY} \cup E_{XZ})\backslash\widetilde{E}_{y_i}\big) = 0$. This, together with (2.3) and the definition of the edge set $\widetilde{E}_{y_i}$, implies that

$$\begin{aligned}
\sum_{i=1}^{5} cr_\phi\left(E\backslash\widetilde{E}_{y_i}\right) &= \sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi(E_{YY}) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) + \sum_{i=1}^{5} cr_\phi\left(E_{XZ}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YY}\right) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{YY}) \qquad (2.4) \\
&+ \sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}, E_{YY}\right).
\end{aligned}$$

It is easy to verify that each crossing involving two edges of E_{YZ} is calculated three times in $\sum_{i=1}^{5} cr_\phi(E_{YZ}\backslash\widetilde{E}_{y_i})$. Thus,

$$\sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}\right) = 3cr_\phi(E_{YZ}).$$

Similarly, we have

$$\sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) = 3cr_\phi(E_{XY}, E_{YZ}),$$
$$\sum_{i=1}^{5} cr_\phi\left(E_{XZ}, E_{YZ}\backslash\widetilde{E}_{y_i}\right) = 4cr_\phi(E_{XZ}, E_{YZ}),$$
$$\sum_{i=1}^{5} cr_\phi\left(E_{XY}\backslash\widetilde{E}_{y_i}, E_{YY}\right) = 4cr_\phi(E_{XY}, E_{YY}),$$

and

$$\sum_{i=1}^{5} cr_\phi\left(E_{YZ}\backslash\widetilde{E}_{y_i}, E_{YY}\right) = 4cr_\phi(E_{YZ}, E_{YY}),$$

which together with (2.4) and (2.1) gives

$$\begin{aligned}
&\sum_{i=1}^{5} cr_\phi(E\backslash\widetilde{E}_{y_i}) \\
&= \ 3cr_\phi(E_{YZ}) + 5cr_\phi(E_{YY}) + 3cr_\phi(E_{XY}, E_{YZ}) + 4cr_\phi(E_{XZ}, E_{YZ}) \\
&+ 4cr_\phi(E_{XY}, E_{YY}) + 5cr_\phi(E_{XZ}, E_{YY}) + 4cr_\phi(E_{YZ}, E_{YY}) \\
&= \ 4cr_\phi(E) + cr_\phi(E_{XZ}, E_{YZ}) + cr_\phi(E_{XZ}, E_{YY}) + cr_\phi(E_{YY}) \\
&- cr_\phi(E_{XY} \cup E_{XZ} \cup E_{YZ}) \\
&\quad \left(\text{since } \langle E_{XY} \cup E_{XZ}\rangle \text{ is isomorphic to } S_{n+5}\right) \\
&= \ 4cr_\phi(E) + cr_\phi(E_{XZ}, E_{XY} \cup E_{YZ}) + cr_\phi(E_{XZ}, E_{YY}) \\
&+ cr_\phi(E_{YY}) - cr_\phi(E_{XY} \cup E_{XZ} \cup E_{YZ}) \\
&\quad \left(\text{since } \phi \text{ is a good drawing, } cr_\phi(E_{XZ}, E_{XY}) = 0\right) \\
&= \ 4cr_\phi(E) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, \widetilde{E}_{y_i}) + cr_\phi(E_{XZ}, E_{YY}) + cr_\phi(E_{YY}) \\
&- cr_\phi(E_{XY} \cup E_{XZ} \cup E_{YZ}) \qquad (\text{using (2.2)}).
\end{aligned}$$

This completes the proof. ■

Lemma 5. *If ϕ is a good drawing of H with $cr_\phi(H) = Z(6,n) + n + 3\left\lfloor\frac{n}{2}\right\rfloor - a$ for some $a \geq 1$, then*

$$\sum_{i=1}^{5} cr_\phi\left(E_{XZ}, \widetilde{E}_{y_i}\right) \geq \begin{cases} \frac{n^2-n}{2} + 4a - cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}), & n \text{ is even}, \\ \frac{n^2-n-6}{2} + 4a - cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}), & n \text{ is odd}. \end{cases}$$

Proof. Let e_i be the edge xy_i, $i = 1, 2, 3, 4, 5$ and f_j be the edge xz_j, $j = 1, 2, \ldots, n$. Without loss of generality, assume that in the drawing ϕ, the clockwise order of these five images $\phi(e_i)$ around $\phi(x)$ is

$$\phi(e_1) \longrightarrow \phi(e_2) \longrightarrow \phi(e_3) \longrightarrow \phi(e_4) \longrightarrow \phi(e_5).$$

We denote a set of all those images $\phi(f_j)$ by A_i, each of which lies in the angle α_i formed between $\phi(e_i)$ and $\phi(e_{i+1})$, where the indices are read modulo 5 (see Figure 3). Obviously, $\sum_{i=1}^{5} |A_i| = n$. In the plane R^2, there exists a circular neighborhood

$$N\big(\phi(x), \epsilon\big) = \{s \in R^2 : \|s - \phi(x)\| < \epsilon\}$$

with ϵ being such a sufficiently small positive number that for any edge e of $E_{XY} \cup E_{XZ}$, there is no crossing appearing on the segment $\phi(e) \cap N\big(\phi(x), \epsilon\big)$ (see Figure 3), where the circuit C denotes the boundary of $N\big(\phi(x), \epsilon\big)$.

In the following, we shall produce the graph $H_i^{'}$ together with its good drawing $\phi_i^{'}$ for each $i \in \{1, 2, 3, 4, 5\}$, by the following steps (see Figure 2).

Figure 2. The subdrawing of W_5 induced by $\phi(H)$ and the subdrawing of W_4 induced by $\phi_i'(H_i)$.

Step 1. In the drawing $\phi(H)$, remove the vertex $y_i \in Y$ and all the edges incident with the vertex y_i.

Step 2. Connect y_{i-1} to y_{i+1} along the original section of the path $\phi(y_{i-1}y_iy_{i+1})$, i module 5.

Step 3. Successively, remove any crossing that involves edges having a common end or pairs of crossings involving the same two edges to make sure that the obtained drawing is a good drawing.

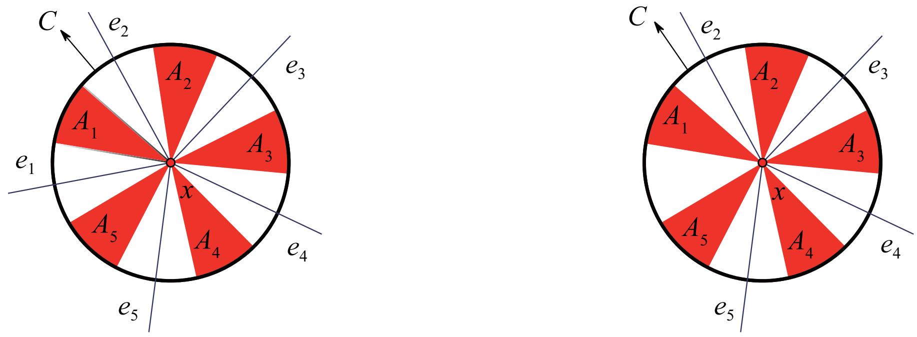


Figure 3. The local situations of $\phi(H)$ and $\phi_1'(H_1')$.

Therefore, we obtain five graphs $H_1', H_2', \ldots, H_5'$ with their good drawings and the local situation of $\phi_1'(H_1')$ is displayed in Figure 3. As $\widetilde{E}_{y_i}$ is a set of all but the edges y_iy_{i-1} and y_iy_{i+1} of the edges incident with the vertex y_i, it is not difficult to verify that for all $i = 1, 2, 3, 4, 5$,

$$cr_{\phi_i'}(H_i') \leq cr_\phi \left(E \backslash \widetilde{E}_{y_i}\right). \tag{2.5}$$

Taking sum for i on the two sides of (2.5), we have

$$\sum_{i=1}^{5} cr_{\phi_i'}(H_i') \leq \sum_{i=1}^{5} cr_\phi \left(E \backslash \widetilde{E}_{y_i}\right). \tag{2.6}$$

For each $i \in \{1, 2, \ldots, 5\}$, in the drawing $\phi_i'(H_i')$, we shall produce the graph H_i'', together with its drawing ϕ_i''. Next, we shall only illustrate how to obtain the graph H_1'' with its drawing ϕ_1'' (see Figure 4).

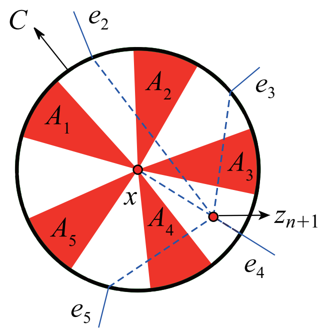
Step 1. In the drawing $\phi_1'(H_1')$, for each $i \in \{2, 3, 4, 5\}$, delete the segment $\phi_1'(e_i) \cap N\big(\phi_1'(x), \epsilon\big)$.

Step 2. Add a new vertex z_{n+1} to a suitable location on the segment $\phi_1'(e_4) \cap N(\phi_1'(x), \epsilon)$.

Step 3. Connect z_{n+1} to each vertex in $\{\phi_1'(x), \phi_1'(y_2), \phi_1'(y_3), \phi_1'(y_4), \phi_1'(y_5)\}$ in such a way as described in Figure 4. To be specific, the drawn edge $z_{n+1}y_i$, $i \in \{2, 3, 4, 5\}$, consists of a curve (dotted line) in the interior of $N\big(\phi_1'(x), \epsilon\big)$ and a segment of $\phi_1'(e_i)$ (solid line) in the outside of $N\big(\phi_1'(x), \epsilon\big)$, and for the drawn edge $z_{n+1}x$, connect z_{n+1} to $\phi_1'(x)$ along the section of $\phi_1'(e_4) \cap N(\phi_1'(x), \epsilon)$.

This implies that

$$cr_{\phi_1''}(H_1'') = cr_{\phi_1'}(H_1') + |A_4| + 2|A_3| + |A_2|. \tag{2.7}$$



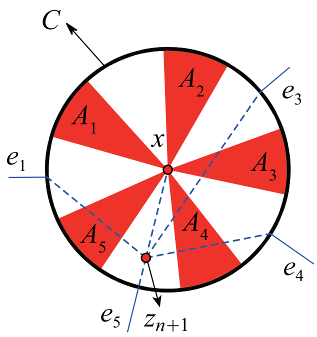


Figure 4. The local situations of $\phi_1''(H_1'')$ and $\phi_2''(H_2'')$.

We can analogously obtain the graphs H_2'', H_3'', H_4'' and H_5'' with their drawings, for example, the local situation of $\phi_2''(H_2'')$ is displayed in Figure 4. Similarly, we respectively obtain

$$cr_{\phi_2''}(H_2'') = cr_{\phi_2'}(H_2') + |A_3| + 2|A_4| + |A_5|, \tag{2.8}$$

$$cr_{\phi_3''}(H_3'') = cr_{\phi_3'}(H_3') + |A_1| + 2|A_5| + |A_4|, \tag{2.9}$$

$$cr_{\phi_4''}(H_4'') = cr_{\phi_4'}(H_4') + |A_2| + 2|A_1| + |A_5|, \tag{2.10}$$

and

$$cr_{\phi_5''}(H_5'') = cr_{\phi_5'}(H_5') + |A_1| + 2|A_2| + |A_3|. \tag{2.11}$$

Taking sum for (2.7)–(2.11) and using (2.6) and Lemma 4, we have

$$\begin{aligned}\sum_{i=1}^{5} cr_{\phi_i''}(H_i'') &= \sum_{i=1}^{5} cr_{\phi_i'}(H_i') + 4\sum_{i=1}^{5}|A_i| \le \sum_{i=1}^{5} cr_\phi\left(E\backslash\widetilde{E}_{y_i}\right) + 4n\\ &= 4cr_\phi(E) + \sum_{i=1}^{5} cr_\phi\left(E_{XZ}, \widetilde{E}_{y_i}\right) + cr_\phi(E_{XZ}, E_{YY})\\ &\quad + cr_\phi(E_{YY}) - cr_\phi\big(E_{XY}\cup E_{XZ}\cup E_{YZ}\big) + 4n.\end{aligned}$$

Since the edge-induced subgraph $\langle E_{XY}\cup E_{XZ}\cup E_{YZ}\rangle$ of H is isomorphic to the complete tripartite graph $K_{1,5,n}$ with $cr(K_{1,5,n}) = Z(6,n) + 4\left\lfloor \frac{n}{2}\right\rfloor$ (see [18]), we have

$$\begin{aligned}\sum_{i=1}^{5} cr_{\phi_i''}(H_i'') \le 4cr_\phi(E) &+ \sum_{i=1}^{5} cr_\phi\left(E_{XZ}, \widetilde{E}_{y_i}\right) + cr_\phi(E_{XZ}, E_{YY})\\ &+ cr_\phi(E_{YY}) - Z(6,n) - 4\left\lfloor \frac{n}{2}\right\rfloor + 4n.\end{aligned} \tag{2.12}$$

It is straightforward to check that, for each $i \in \{1, 2, \ldots, 5\}$, the graph H_i'' is isomorphic to the graph $Q + (n+1)K_1$ (the graph Q is displayed in Figure 1). Using the result from [4], we have

$$cr_{\phi_i''}(H_i'') \geq cr(Q + (n+1)K_1) = Z(5, n+1) + \left\lfloor \frac{n+1}{2} \right\rfloor,$$

which together with (2.12) and $cr_\phi(E) = Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor - a$, gives

$$\begin{aligned}
\sum_{i=1}^{5} cr_\phi\left(E_{XZ}, \widetilde{E}_{y_i}\right) &\geq \sum_{i=1}^{5} cr_{\phi_i''}(H_i'') - 4cr_\phi(E) + Z(6,n) + 4\left\lfloor \frac{n}{2} \right\rfloor - 4n \\
&- cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}) \\
&\geq 5\left(Z(5, n+1) + \left\lfloor \frac{n+1}{2} \right\rfloor\right) - 4\left(Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor - a\right) \\
&+ Z(6,n) + 4\left\lfloor \frac{n}{2} \right\rfloor - 4n - cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}) \\
&= \begin{cases} \frac{n^2-n}{2} + 4a - cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}), & n \text{ is even}, \\ \frac{n^2-n-6}{2} + 4a - cr_\phi(E_{XZ}, E_{YY}) - cr_\phi(E_{YY}), & n \text{ is odd}. \end{cases}
\end{aligned}$$

This completes the proof. ■

2.1. Proof of Theorem 1

We can easily find a good drawing ϕ_0 with

$$cr_{\phi_0}(H) = Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

The required drawing ϕ_0 (see Figure 5(1)) is given as follows.

(1) $\phi_0(x) = (\epsilon, -\epsilon)$, where ϵ is a sufficiently small positive number, $\phi_0(y_1) = (0, 3)$, $\phi_0(y_2) = (0, 2)$, $\phi_0(y_3) = (0, 1)$, $\phi_0(y_4) = (0, -1)$ and $\phi_0(y_5) = (0, -2)$;

(2) $\phi_0(z_j) = ((-1)^j \left\lfloor \frac{j+1}{2} \right\rfloor, 0)$ $(j \in \{1, 2, \ldots, n\})$;

(3) the image of the edge y_1y_5 is an arc and the images of other edges are straight line segments.

Hence, the drawing shows that $cr(H) \leq Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor$. In order to show Theorem 1, we only need to prove that $cr(H) \geq Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor$. For $n = 1$, since the graph $W_5 + K_1$ contains a subdivision of the complete bipartite graph $K_{3,3}$ with $cr(K_{3,3}) = 1$, $cr(W_5 + K_1) \geq cr(K_{3,3}) = 1 = Z(6,1) + 1 + 3\left\lfloor \frac{1}{2} \right\rfloor$. In the following, we shall prove the inequality for $n > 1$. Suppose that there is an optimal drawing D of H with $cr_D(H) < Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor$, that is,

$$cr_D(H) = Z(6,n) + n + 3\left\lfloor \frac{n}{2} \right\rfloor - a, \quad \text{for some } a \geq 1. \tag{2.13}$$

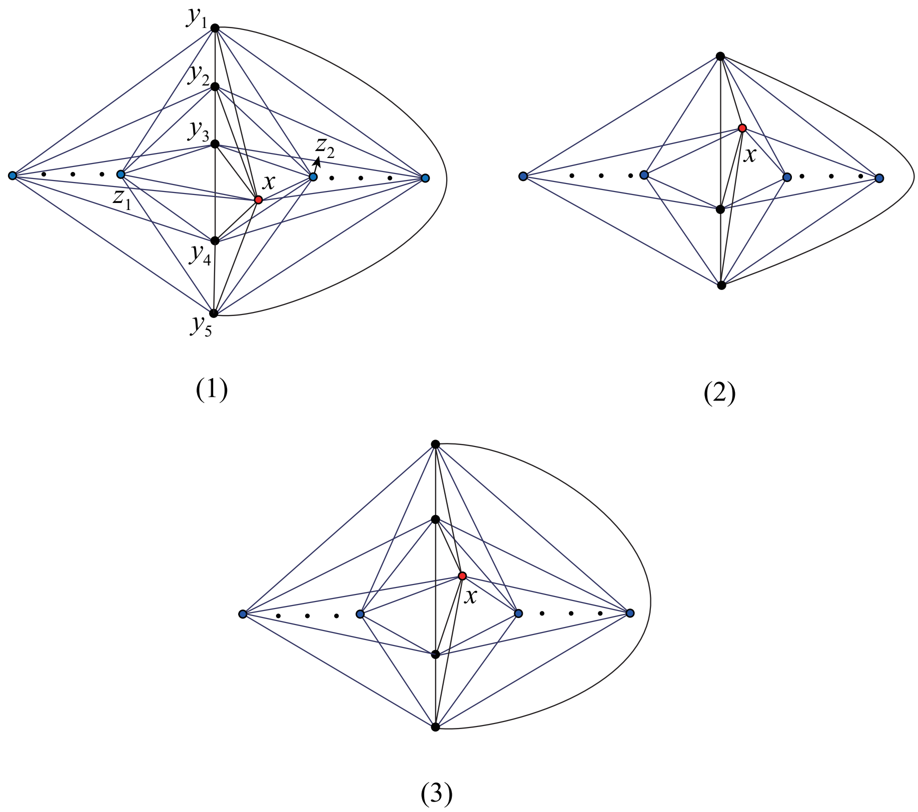


Figure 5. (1) The drawing ϕ_0 of the graph $W_5 + nK_1$; (2) The drawing of the graph $W_3 + nK_1$; (3) The drawing of the graph $W_4 + nK_1$.

Using (1.1), (1.2), (2.1) and (2.2), we get that

$$\begin{aligned} cr_D(H) &= cr_D\big(E_{XZ} \cup (E_{XY} \cup E_{YZ} \cup E_{YY})\big) \\ &= cr_D(E_{XZ}) + cr_D(E_{XY} \cup E_{YZ} \cup E_{YY}) + cr_D(E_{XZ}, E_{YY}) \\ &\quad + cr_D(E_{XZ}, E_{XY} \cup E_{YZ}) \qquad (2.14) \\ &= 0 + cr_D(E_{XY} \cup E_{YZ} \cup E_{YY}) + cr_D(E_{XZ}, E_{YY}) + \sum_{i=1}^{5} cr_D\Big(E_{XZ}, \widetilde{E}_{y_i}\Big). \end{aligned}$$

Since the edge-induced subgraph $\langle E_{XY} \cup E_{YZ} \cup E_{YY} \rangle$ of H contains the complete bipartite graph $K_{5,n+1}$ with $cr(K_{5,n+1}) = Z(5, n+1)$ as a subgraph, $cr\big(\langle E_{XY} \cup E_{YZ} \cup E_{YY}\rangle\big) \geq Z(5, n+1)$. On the other hand, the drawing in Figure 6 shows that $cr\big(\langle E_{XY} \cup E_{YZ} \cup E_{YY}\rangle\big) \leq Z(5, n+1)$. Thus,

$$cr\big(\langle E_{XY} \cup E_{YZ} \cup E_{YY}\rangle\big) = Z(5, n+1),$$

which together with (2.13) and (2.14) yields

$$\sum_{i=1}^{5} cr_D\left(E_{XZ},\widetilde{E}_{y_i}\right) \le Z(6,n)+n+3\left\lfloor\frac{n}{2}\right\rfloor - a - Z(5,n+1) - cr_D(E_{XZ},E_{YY})$$

$$= \begin{cases} \frac{n^2-n}{2} - a - cr_D(E_{XZ},E_{YY}), & n \text{ is even,} \\ \frac{n^2-n+2}{2} - a - cr_D(E_{XZ},E_{YY}), & n \text{ is odd.} \end{cases} \tag{2.15}$$

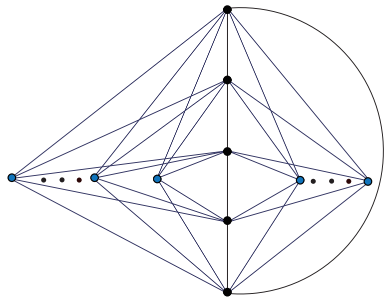


Figure 6. The drawing of the graph $\langle E_{XY}\cup E_{YZ}\cup E_{YY}\rangle$.

By our hypothesis, D is an optimal drawing of H with $cr_D(H) = Z(6,n)+n+3\left\lfloor\frac{n}{2}\right\rfloor - a$ for some $a \ge 1$. So, by combing Lemma 3 and Lemma 5, we have

$$\sum_{i=1}^{5} cr_D\left(E_{XZ},\widetilde{E}_{y_i}\right) \ge \begin{cases} \frac{n^2-n}{2} + 4a - cr_D(E_{XZ},E_{YY}), & n \text{ is even,} \\ \frac{n^2-n-6}{2} + 4a - cr_D(E_{XZ},E_{YY}), & n \text{ is odd.} \end{cases} \tag{2.16}$$

According to n is even or odd, we consider the following two cases.

Case 1. When n is even, by (2.15) and (2.16), we obtain

$$\frac{n^2-n}{2} + 4a - cr_D(E_{XZ},E_{YY}) \le \frac{n^2-n}{2} - a - cr_D(E_{XZ},E_{YY}).$$

Case 2. When n is odd, by (2.15) and (2.16), we get

$$\frac{n^2-n-6}{2} + 4a - cr_D(E_{XZ},E_{YY}) \le \frac{n^2-n+2}{2} - a - cr_D(E_{XZ},E_{YY}).$$

Hence, by simple calculation, we have that for Case 1, $a \le 0$ and for Case 2, $a \le \frac{4}{5}$. This contradiction completes the proof.

In [17], the crossing number of the graph $W_5 + nK_1$ was proved as a case analysis of a significant number of cases. However, in our paper, we construct the relationship between the crossing number of the graph H and the crossing number of the graph $Q + (n+1)K_1$, hence we make this claim more accessible.

3. THE PROOF OF THEOREM 2

We prove Theorem 2 with the help of some definitions and results in [3]. Next, we briefly introduce two concepts that are used in the proof.

For a multiset $L \subseteq V(G_2)$, we denote by $G_1 \Box_L G_2$ the capped Cartesian product of graphs G_1 and G_2, that is, the graph obtained by adding a distinct vertex $v^{'}$ to $G_1 \Box G_2$ for each copy of a vertex $v \in L$ and joining $v^{'}$ to all the vertices of $G_1 \Box \{v\}$. We call each $v^{'}$ a cap of v. For $v \in V(G_2)$, let $\chi_L(v)$ denote the multiplicity of v in L and let $l(v) = \deg_{G_2}(v) + \chi_L(v)$. Let $F \subseteq E(G)$ is a subset of edges of G and π is a permutation of F. A π-subdivision G^π of G is the graph, obtained from G by subdividing every edge $e \in F$ with the vertex v_e and adding the edges $\left\{v_e v_{\pi(e)} | e \in F\right\}$.

3.1. Proof of Theorem 2

A subtree $T^{'}$ of T is obtained by deleting all leaf vertices in T, and let L be the multiset containing each vertex v of $T^{'}$ with $\chi_L(v) = \deg_T(v) - \deg_{T'}(v)$. Thus,

$$l(v) = \deg_{T'}(v) + \deg_T(v) - \deg_{T'}(v) = \deg_T(v), \quad v \in V\left(T^{'}\right).$$

It is easy to verify that $2 \leq l(v) \leq \Delta(T)$ ($\Delta(T)$ is the maximum degree of T) for every vertex $v \in V(T^{'})$. The drawings in Figure 5 show that the wheels W_j ($j = 3, 4, 5$) have all apex-homogeneous drawings (the detailed definition of all apex-homogeneous drawings see [3]) such that each of them is optimal. Note that the central vertex of W_j is a dominating vertex, i.e., a vertex adjacent to all other vertices of the graph. Thus, by Theorem 10 in [3], we have

$$\begin{aligned} cr(W_j \Box_L T^{'}) &= \sum_{v \in V(T^{'})} cr\left(W_j^{l(v)}\right) \\ &= \sum_{i=2}^{\Delta(T)} n_i cr\left(W_j^i\right) \qquad (2 \leq l(v) \leq \Delta(T),\ j \in \{3,4,5\}), \end{aligned} \tag{3.1}$$

where n_i is the number of vertices of degree i in T.

The graph $W_j \Box_L T^{'}$ ($j \in \{3,4,5\}$) is obtained from the Cartesian product $W_j \Box T^{'}$ ($j \in \{3,4,5\}$) by adding r_v caps to $W_j \Box \{v\}$ ($j \in \{3,4,5\}$) for every vertex v of $T^{'}$ (r_v is the number of T-leaves adjacent to v). Actually, the graph

$W_j\Box_L T^{'}$ $(j \in \{3,4,5\})$ has n_1 caps (n_1 is the number of leaf vertices of T). This consistency in combination with Theorem 19 in [3] also implies that a properly chosen π-subdivision of edges connecting a cap of $W_j\Box_L T^{'}$ $(j \in \{3,4,5\})$ with the corresponding rim increases the crossing number by precisely one. To obtain $W_j\Box T$ from $W_j\Box_L T^{'}$ $(j \in \{3,4,5\})$, we need one such π-subdivision for each T-leaf vertex (the number of T-leaf vertices is n_1). Together with (3.1), we get

$$\begin{aligned}
cr(W_j\Box T) &= n_1 + cr\left(W_j\Box_L T^{'}\right) \\
&= n_1\left(W_j^1\right) + \sum_{i=2}^{\Delta(T)} n_i cr\left(W_j^i\right) \quad \left(\text{by } cr\big(W_j^1\big) = cr\big(W_j + K_1\big) = 1\right) \\
&= \sum_{i=1}^{\Delta(T)} n_i cr\left(W_j^i\right) \\
&= \begin{cases} \sum_{i=1}^{\Delta(T)} n_i(Z(4,i)+i), & j = 3 \text{ (by Theorem 3.1 in [14])}, \\ \sum_{i=1}^{\Delta(T)} n_i\left(Z(5,i)+i+\left\lfloor \frac{i}{2}\right\rfloor\right), & j = 4 \text{ (by Theorem 10 in [4])}, \\ \sum_{i=1}^{\Delta(T)} n_i\left(Z(6,i)+i+3\left\lfloor \frac{i}{2}\right\rfloor\right), & j = 5 \text{ (by Theorem 1)}. \end{cases}
\end{aligned}$$

This finishes the proof of Theorem 2.

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