ON THE METRIC DIMENSION OF DIRECTED AND UNDIRECTED CIRCULANT GRAPHS

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Abstract

The undirected circulant graph \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) consists of vertices \( v_0, v_1, \ldots, v_{n-1} \) and undirected edges \( v_i v_{i+j} \), where \( 0 \leq i \leq n-1, 1 \leq j \leq t \) \( (2 \leq t \leq \frac{n}{2}) \), and the directed circulant graph \( C_n(1, t) \) consists of vertices \( v_0, v_1, \ldots, v_{n-1} \) and directed edges \( v_i v_{i+1}, v_i v_{i+t} \), where \( 0 \leq i \leq n-1 \) \( (2 \leq t \leq n-1) \), the indices are taken modulo \( n \). Results on the metric dimension of undirected circulant graphs \( C_n(\pm 1, \pm t) \) are available only for special values of \( t \). We give a complete solution of this problem for directed graphs \( C_n(1, t) \) for every \( t \geq 2 \) if \( n \geq 2t^2 \). Grigorious et al. [On the metric dimension of circulant and Harary graphs, Appl. Math. Comput. 248 (2014) 47–54] presented a conjecture saying that \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) = t + p - 1 \) for \( n = 2tk + t + p \), where \( 3 \leq p \leq t + 1 \). We disprove it by showing that \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p+1}{2} \) for \( n = 2tk + t + p \), where \( t \geq 4 \) is even, \( p \) is odd, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \).

Keywords: metric dimension, resolving set, circulant graph, distance.

2010 Mathematics Subject Classification: 05C35, 05C12.

1. Introduction

Let \( V(G) \) be vertex set of a connected (undirected or directed) graph \( G \). The distance \( d(u, v) \) between two vertices \( u, v \) in an undirected graph is the number of edges in a shortest path between \( u \) and \( v \). In a directed graph \( G \) the distance \( d(u, v) \) from a vertex \( u \in V(G) \) to a vertex \( v \in V(G) \) is the length of a shortest directed path from \( u \) to \( v \).
A vertex \( w \) resolves two vertices \( u \) and \( v \) if \( d(u, w) \neq d(v, w) \). For an ordered set of vertices \( W = \{ w_1, w_2, \ldots, w_z \} \), the representation of distances of \( v \) with respect to \( W \) is the ordered \( z \)-tuple

\[
r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).
\]

A set \( W \subset V(G) \) is a resolving set of \( G \) if every two distinct vertices of \( G \) have different representations of distances with respect to \( W \) (if every two vertices of \( G \) are resolved by some vertex in \( W \)). The metric dimension of \( G \) is the number of vertices in a smallest resolving set and it is denoted by \( \dim(G) \). The \( i \)-th coordinate in \( r(v|W) \) is 0 if and only if \( v = w_i \). Thus in order to prove that \( W \) is a resolving set of \( G \), it suffices to show that \( r(u|W) \neq r(v|W) \) for every two different vertices \( u, v \in V(G) \setminus W \).

The metric dimension is an invariant, which has applications in robot navigation [9], pharmaceutical chemistry [2], pattern recognition and image processing [10]. It has been extensively studied. For example, Imran [5] studied barycentric subdivisions of Cayley graphs and Saputro et al. [12] gave bounds on the metric dimension of the lexicographic product of graphs.

Let \( n, m \) and \( a_1, a_2, \ldots, a_m \) be positive integers such that \( 1 \leq a_1 < a_2 < \cdots < a_m \leq \frac{n}{2} \). The undirected circulant graph \( C_n(\pm a_1, \pm a_2, \ldots, \pm a_m) \) consists of the vertices \( v_0, v_1, \ldots, v_{n-1} \) and undirected edges \( v_i v_{i+a_j} \), where \( 0 \leq i \leq n-1, 1 \leq j \leq m \); the indices are taken modulo \( n \).

For generators \( a_1, a_2, \ldots, a_m \) such that \( 1 \leq a_1 < a_2 < \cdots < a_m \leq n-1 \), the directed circulant graph \( C_n(\pm a_1, \pm a_2, \ldots, \pm a_m) \) consists of the vertices \( v_0, v_1, \ldots, v_{n-1} \) and directed edges \( v_i v_{i+a_j} \), where \( 0 \leq i \leq n-1, 1 \leq j \leq m \); the indices are taken modulo \( n \). The directed circulant graph \( C_n(\pm a_1, \pm a_2, \ldots, \pm a_m) \) contains the directed edges \( v_i v_{i-a_j} \).

Circulant graphs form an important family of Cayley graphs. The metric dimension of undirected circulant graphs \( C_n(\pm 1, \pm t) \) was studied for special values of \( t \). Javaid, Rahim and Ali [8] proved that if \( n \equiv 0, 2, 3 \pmod{4} \), then \( \dim(C_n(\pm 1, \pm 2)) = 3 \). Borchert and Gosselin [1] showed that if \( n \equiv 1 \pmod{4} \), then \( \dim(C_n(\pm 1, \pm 2)) = 4 \). The undirected circulant graphs \( C_n(\pm 1, \pm 3) \) were considered in [7] and the graphs \( C_n(\pm 1, \pm \frac{n}{2}) \) for even \( n \) were investigated in [11]. We study the metric dimension for directed circulant graphs with 2 generators. We give a complete solution of this problem for directed graphs \( C_n(1, t) \) for every \( t \geq 2 \) if \( n \geq 2t^2 \).

Exact values of the metric dimension of undirected graphs \( C_n(\pm 1, \pm 2, \pm 3) \) were given in [1] and [6]. Grigorious et al. [4] showed that \( t+1 \) vertices \( v_0, v_1, \ldots, v_t \) resolve the graph \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) if \( n \equiv r \pmod{2t} \), where \( 2 \leq r \leq t+2 \) and they gave the bound \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq r - 1 \) if \( n \equiv r \pmod{2t} \), where \( t+3 \leq r \leq 2t + 1 \). They presented a conjecture saying that \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) = t + p - 1 \) for \( n = 2tk + t + p \), where
3 \leq p \leq t + 1. We disprove it for even \( t \geq 4 \) and odd \( p \geq 5 \) by showing that 
\[ \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p+1}{2} \] 
for \( n = 2tk+t+p \) where \( t \geq 4 \) is even, \( p \) is odd, 
\( 1 \leq p \leq t + 1 \) and \( k \geq 1 \). Note that Chau and Gosselin [3] recently proved that 
\[ \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) = t + \frac{p+1}{2} \] 
for large \( n \), which implies that the metric dimension of the graphs \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) 
is completely determined by the congruence class of \( n \) modulo \( 2t \).

2. Directed Circulant Graphs

We study the metric dimension of directed circulant graphs \( C_n(1, t) \). It is easy to 
see that the graph \( C_n(1, t) \) is isomorphic to the graph \( C_n(-1, -t) \) for \( 2 \leq t \leq n-1 \). 
We present Theorems 1 and 2 for the graph \( C_n(-1, -t) \), because it is easier to 
express distances from vertices in a graph to vertices in chosen resolving sets if 
we consider \( C_n(-1, -t) \) (especially in the proof of Theorem 2).

The distance from the vertex \( v_j \) to the vertex \( v_i \) in \( C_n(-1, -t) \), where \( i, j \in \{0, 1, \ldots, n - 1\} \), is

\[
\begin{align*}
    d(v_j, v_i) &= \left\{ \begin{array}{ll}
        \left\lfloor \frac{j-i}{t} \right\rfloor + p, & p \equiv (j-i) \pmod{t}, \quad \text{if } j \geq i, \\
        \left\lfloor \frac{n+j-i}{t} \right\rfloor + p, & p \equiv (n+j-i) \pmod{t}, \quad \text{if } j < i,
    \end{array} \right.
\end{align*}
\]

where \( 0 \leq p \leq t - 1 \).

**Theorem 1.** Let \( t \geq 2 \) and \( n \geq 2t^2 \). Then 
\[ \dim(C_n(-1, -t)) \geq t. \]

**Proof.** We prove the result by contradiction. Assume that 
\( \dim(C_n(-1, -t)) \leq t - 1 \). Let \( W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{t-1}}\} \) 
be a resolving set of \( C_n(-1, -t) \), where \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_{t-1} \). Since we have at most \( t - 1 \) different vertices in \( W \) and 
the graph \( C_n(-1, -t) \) has at least \( 2t^2 \) vertices, \( C_n(-1, -t) \) contains a set of \( 2t \) 
consecutive vertices \( V' = \{v_j, v_{j+1}, \ldots, v_{j+2t-1}\} \), where \( 0 \leq j \leq n - 1 \), such that 
no vertex of \( W \) is in \( V' \). Without loss of generality we can assume that \( j = n - 2t \), 
which means that \( V' = \{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-1}\} \) and \( i_{t-1} < n - 2t \).

Since \( |W| \leq t - 1 \), there is a \( k \in \{0, 1, \ldots, t - 1\} \), such that no vertex \( v_i \in W \) 
satisfies \( i \equiv k \pmod{t} \). So we can write any vertex of \( W \) in the form \( v_{tr+s} \), where 
\( 0 \leq s \leq t - 1 \), \( s \neq k \) and \( r \geq 0 \).

Let \( v_l \) be any vertex in the set of \( t \) vertices \( \{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-t-1}\} \), such 
that \( l \equiv k \pmod{t} \). Then we can write \( l = tx+k \), where \( 0 \leq k \leq t - 1 \). We 
show that the vertices \( v_{lx+k}, v_{lx+k+t-1} \in V' \) are not resolved by \( W \). Note that
tx + k > tr + s. By (1) we have

\[ d(v_{tx+k}, v_{tr+s}) = \begin{cases} 
\left\lfloor \frac{tx+k-(tr+s)}{t} \right\rfloor + k - s = x - r + \left\lfloor \frac{k-s}{t} \right\rfloor + k - s & \text{if } k > s, \\
x - r + k - s & \text{if } k < s,
\end{cases} \]

\[ d(v_{tx+k}, v_{tr+s}) = \begin{cases} 
\left\lfloor \frac{tx+k-1-(tr+s)}{t} \right\rfloor + k - 1 - s & \text{if } k > s, \\
x + 1 - r + \left\lfloor \frac{k-1-s}{t} \right\rfloor + k - 1 - s + t & \text{if } k < s.
\end{cases} \]

Since \( d(v_{tx+k}, v_{tr+s}) = d(v_{tx+k+t-1}, v_{tr+s}) \) for any vertex \( v_{tr+s} \in W \), the graph \( C_n(-1, -t) \) is not resolved by \( W \). A contradiction.

Let us present an upper bound on the metric dimension of directed circulant graphs with 2 generators.

**Theorem 2.** Let \( 2 \leq t < n \). Then \( \dim(C_n(-1, -t)) \leq t \).

**Proof.** We prove that \( W = \{v_0, v_1, \ldots, v_{n-1}\} \) is a resolving set of \( C_n(-1, -t) \). First we find all vertices \( v_j \) (\( 1 \leq j \leq n-1 \)) of \( C_n(-1, -t) \) such that \( d(v_j, v_0) = x \) for any \( x \geq 1 \). We can write \( j = tr + p \) where \( r \geq 0 \) and \( 0 \leq p \leq t - 1 \). Since by (1), \( d(v_{tr+p}, v_0) = r + p \), we have \( r + p = x \). Thus \( r = x - p \) (\( \geq 0 \)) and then \( v_{t(x-p)+p} \) for \( 0 \leq p \leq t - 1 \) and \( 1 \leq t(x-p)+p \leq n-1 \) are the vertices of \( C_n(1, t) \) such that \( d(v_{t(x-p)+p}, v_0) = x \).

It remains to show that these vertices are resolved by \( v_i \), \( i = 1, 2, \ldots, t - 1 \). It suffices to consider only those vertices \( v_{t(x-p)+p} \) which are not in \( W \), so we can assume that \( t(x-p)+p > i \). For \( i = 1, 2, \ldots, t - 1 \), by (1),

\[ d(v_{t(x-p)+p}, v_i) = \begin{cases} 
-\frac{p+i}{t} + p - i = x - i & \text{if } p \geq i, \\
-x - p + \left\lfloor \frac{p-i}{t} \right\rfloor + p - i + t = x + t - 1 - i & \text{if } p < i.
\end{cases} \]

We know that the first entry of \( r(v_{t(x-p)+p}|W) \) is \( x \). From (3) it follows that the next \( p \) entries (where \( 0 \leq p \leq t - 1 \)) are \( x - i \) and the last \( t - 1 - p \) entries of \( r(v_{t(x-p)+p}|W) \) are \( x + t - 1 - i \).

So if \( p = 0 \) (and if \( v_{tx} \) exists), the first entry of \( r(v_{tx}|W) \) is \( x \) and the other entries are \( x + t - 1 - i \) which means that \( r(v_{tx}|W) = (x, x + t - 2, x + t - 3, \ldots, x + t - 1 - (t - 1)) \). If \( p = 1 \), the first entry of \( r(v_{t(x-1)+1}|W) \) is \( x \), the second entry is \( x - 1 \) and the other entries are \( x + t - 1 - i \), so \( r(v_{t(x-1)+1}|W) = \).
(x, x - 1, x + t - 3, x + t - 4, \ldots, x + t - 1 - (t - 1)). Similarly \(r(v_i(x-2)+2|W) = (x, x - 1, x - 2, x + t - 4, \ldots, x + t - 1 - (t - 1)), \ldots, r(v_i(x-(t-1))+(t-1)|W) = (x, x - 1, x - 2, \ldots, x - (t - 1)).

Since all vertices \(v_j, 1 \leq j \leq n - 1, \) such that \(d(v_j, v_0) = x\) are resolved by \(W,\) we have \(\dim(C_n(-1, -t)) \leq |W| = t.\)

From Theorems 1 and 2 we obtain Corollary 3.

**Corollary 3.** Let \(t \geq 2\) and \(n \geq 2t^2.\) Then \(\dim(C_n(-1, -t)) = t.\)

Since the graphs \(C_n(-1, -t)\) and \(C_n(1, t)\) are isomorphic, we get the following corollary.

**Corollary 4.** Let \(t \geq 2\) and \(n \geq 2t^2.\) Then \(\dim(C_n(1, t)) = t.\)

3. Undirected Circulant Graphs

We give an upper bound on the metric dimension of undirected circulant graphs \(C_n(\pm 1, \pm 2, \ldots, \pm t)\) for \(n \equiv r \mod 2t,\) where \(r = 1\) and \(r = t + 1, t + 3, \ldots, 2t - 1.\)

The distance between two vertices \(v_i\) and \(v_j\) in \(C_n(\pm 1, \pm 2, \ldots, \pm t),\) where \(0 \leq i < j < n,\) is

\[
d(v_i, v_j) = \min\left\{\left\lfloor \frac{j - i}{t} \right\rfloor, \left\lfloor \frac{n - (j - i)}{t} \right\rfloor \right\}.
\]

This equation can be simplified as

\[
d(v_i, v_j) = \begin{cases} 
\left\lfloor \frac{j - i}{t} \right\rfloor & \text{if } 0 \leq j - i \leq \frac{n}{2}, \\
\left\lfloor \frac{n - (j - i)}{t} \right\rfloor & \text{if } \frac{n}{2} < j - i < n.
\end{cases}
\]

**Theorem 5.** Let \(n = 2tk + t + p\) where \(t \geq 4, p \text{ is odd, } 1 \leq p \leq t + 1,\) and \(k \geq 1.\) Then

\[
\dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p + 1}{2}.
\]

**Proof.** Let \(n = 2tk + t + p\) where \(k \geq 1, t \geq 4, p \text{ is even, } 1 \leq p \leq t + 1.\) Let

\[
W_1 = \{v_0, v_2, \ldots, v_{t-2}\}, \quad W_2 = \{v_{t-1}, v_{t+1}, \ldots, v_{2t-3}\}, \\
W_3 = \{v_{tk+t-1}, v_{tk+t+1}, \ldots, v_{tk+t+p-2}\}.
\]

We have \(|W_1| = |W_2| = \frac{t}{2} \text{ and } |W_3| = \frac{p+1}{2}.\) Let us prove that \(W = W_1 \cup W_2 \cup W_3\) is a resolving set of the graph \(C_n(1, 2, \ldots, t).\)
We divide the vertex set of $C_n(\pm 1, \pm 2, \ldots, \pm t)$ into four disjoint sets:

$V_1 = \{v_0, v_1, \ldots, v_t\}$, $V_2 = \{v_{t+1}, v_{t+2}, \ldots, v_{tk+t}\}$, $V_3 = \{v_{tk+t+1}, v_{tk+t+2}, \ldots, v_{tk+t+p-1}\}$, $V_4 = \{v_{tk+t+p}, v_{tk+t+p+1}, \ldots, v_{n-1}\}$.

First we prove that any two vertices of $V_2$ have different representations of distances with respect to $W$. For $x = 1, 2, \ldots, k - 1$; $j = 1, 2, \ldots, t$; $i = 0, 2, \ldots, t - 2$, we have $v_i \in W_1$ and by (5),

$$d(v_{tx+j}, v_i) = x + \left\lceil \frac{j - i}{t} \right\rceil = \begin{cases} x + 1 & \text{if } i < j, \\ x & \text{if } i \geq j, \end{cases}$$

and if $x = k$; $j = 1, 2, \ldots, t$, by (4), we get

$$d(v_{tk+j}, v_i) = \min\left\{ \left\lceil \frac{(tk + j) - i}{t} \right\rceil, \frac{n - [(tk + j) - i]}{t} \right\} = \left\lfloor \frac{k + 1}{t} \right\rfloor = \begin{cases} k + 1 & \text{if } i < j, \\ k & \text{if } i \geq j. \end{cases}$$

Since $j$ (where $1 \leq j \leq t$) is greater than $\left\lfloor \frac{1}{2} \right\rfloor$ elements from the set $\{0, 2, \ldots, t-2\}$, the first $\left\lfloor \frac{1}{2} \right\rfloor$ entries of $r(v_{tx+j}|W_1)$ for $x = 1, 2, \ldots, k$ are equal to $x + 1$ and the other $\frac{n}{2} - \left\lfloor \frac{1}{2} \right\rfloor$ entries are $x$; $r(v_{tx+j}|W_1) = (x + 1, \ldots, x + 1, x, \ldots, x)$. Therefore the only vertices in $V_2$ with the same representations of distances with respect to $W_1$ are the pairs $(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \ldots, (v_{tk+t-1}, v_{tk+t})$. But since for $x = 1, 2, \ldots, k$ and $j = 1, 3, \ldots, t - 3$, we obtain $v_{t+j} \in W_2$ and by (5),

$$d(v_{tx+j}, v_{t+j}) = x - 1, \quad d(v_{tx+j+1}, v_{t+j}) = x - 1 + \left\lceil \frac{1}{t} \right\rceil = x,$$

and for $v_{t-1} \in W_2$, we have

$$d(v_{tx+t-1}, v_{t-1}) = x, \quad d(v_{tx+t}, v_{t-1}) = x + \left\lceil \frac{1}{t} \right\rceil = x + 1,$$

vertices in $W_2$ resolve the pairs $(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \ldots, (v_{tk+t-1}, v_{tk+t})$. Thus no two vertices in $V_2$ have the same representations of distances with respect to $W$.

Let us study representations of distances of the vertices in $V_4$. For $x = 1, 2, \ldots, k - 1$; $j = 0, 1, \ldots, t - 1$; $i = 0, 2, \ldots, t - 2$; we have $v_i \in W_1$ and by (6),

$$d(v_{n-tx+j}, v_i) = \left\lfloor \frac{n - [(n - tx + j) - i]}{t} \right\rfloor = x + \left\lceil \frac{i - j}{t} \right\rceil = \begin{cases} x & \text{if } i \leq j, \\ x + 1 & \text{if } i > j, \end{cases}$$
and if \( x = k \), we get
\[
d(v_{n-tk+j}, v_i) = \min \left\{ \left\lceil \frac{(n-tk+j-i)}{t} \right\rceil, \left\lceil \frac{(n-((n-tk+j)-i)}{t} \right\rceil \right\}
= \min \left\{ k + 1 + \left\lceil \frac{p+j-i}{t} \right\rceil, k + \left\lceil \frac{i-j}{t} \right\rceil \right\} = \begin{cases} k & \text{if } i \leq j, \\ k+1 & \text{if } i > j. \end{cases}
\]

Since \( j \) (where \( 0 \leq j \leq t-1 \)) is greater than or equal to \( \left\lceil \frac{2}{t} \right\rceil + 1 \) elements from the set \( \{0, 2, \ldots, t-2\} \), the first \( \left\lceil \frac{2}{t} \right\rceil + 1 \) entries of \( r(v_{n-tx+j}|W_1) \) (for \( x = 1, 2, \ldots, k \)) are equal to \( x \) and the other entries are \( x+1 \). The only vertices in \( V_4 \) with the same representations of distances with respect to \( W_1 \) are the pairs \((v_{n-tk}, v_{n-(tk+1)}), (v_{n-tk+2}, v_{n-(tk+3)}), \ldots, (v_{n-2}, v_{n-1})\). We show that most of these pairs are resolved by vertices in \( W_2 \). For \( x = 1, 2, \ldots, k-1 \) and \( j = 1, 3, \ldots, t-3 \), we have \( v_{t+j} \in W_2 \) and by (6),
\[
d(v_{n-tx+j}, v_{t+j}) = x + 1, d(v_{n-tx+1-j}, v_{t+j}) = x + 1 + \left\lceil \frac{1}{t} \right\rceil = x + 2,
\]
and for \( v_{t-1} \in W_2, x = 1, 2, \ldots, k, \) by (6),
\[
d(v_{n-tx+t-1}, v_{t-1}) = x, d(v_{n-tx+t-2}, v_{t-1}) = x + \left\lceil \frac{1}{t} \right\rceil = x + 1,
\]
so vertices of \( W_2 \) resolve all pairs of vertices \((v_{n-tk+t-2}, v_{n-(tk+t-1)}), (v_{n-tk+t}, v_{n-(tk+t+1)}), \ldots, (v_{n-2}, v_{n-1})\), which are the pairs \((v_{tk+2t+p-2}, v_{tk+2t+p-1}), (v_{tk+2t+p}, v_{tk+2t+p+1}), \ldots, (v_{n-2}, v_{n-1})\). It remains to resolve the pairs \((v_{tk+t+p}, v_{tk+t+p+1}), (v_{tk+t+p+2}, v_{tk+t+p+3}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})\).

For \( j = 0, 2, \ldots, t-p-3 \), we have \( v_{t+p+j} \in W_2 \) and by (5),
\[
d(v_{tk+t+p+j}, v_{t+p+j}) = k, \quad d(v_{tk+t+p+j+1}, v_{t+p+j}) = k + \left\lceil \frac{1}{t} \right\rceil = k+1,
\]
so the pairs \((v_{tk+t+p}, v_{tk+t+p+1}), \ldots, (v_{tk+2t-3}, v_{tk+2t-2})\) are resolved by \( W_2 \).

For \( j = t-p-1, t-p+1, \ldots, t-4 \), we have \( v_{tk+p+j} \in W_3 \) and by (5),
\[
d(v_{tk+t+p+j}, v_{tk+p+j}) = 1, \quad d(v_{tk+t+p+j+1}, v_{tk+p+j}) = 1 + \left\lceil \frac{1}{t} \right\rceil = 2,
\]
so the pairs \((v_{tk+2t-1}, v_{tk+2t}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})\) are resolved by \( W_3 \). Thus all pairs of vertices in \( V_4 \) are resolved by \( W \).

A vertex \( v \in V_2 \) and a vertex in \( V_1 \) can have the same representations of distances with respect to \( W_1 \) only if all entries of \( r(v|W_1) \) are the same numbers. For \( x = 1, 2, \ldots, k \), we have \( v_{tx+t-1}, v_{tx+t} \in V_2 \) and \( r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = (x+1, \ldots, x+1) \). For \( v_{n-tx+t-2}, v_{n-tx+t-1} \in V_4 \) we have \( r(v_{n-tx+t-2}|W_1) = r(v_{n-tx+t-1}|W_1) = (x, \ldots, x) \), which implies that for \( x = 1, 2, \ldots, k-1 \), we obtain \( r(v_{n-tx+1}|W_1) = r(v_{tx+t}|W_1) = r(v_{n-tx-2}|W_1) = r(v_{n-tx-1}|W_1) \). Since for \( v_{2t-3} \in W_2 \), by (5),
\[
d(v_{tx+t-1}, v_{2t-3}) = x - 1 + \left\lceil \frac{2}{t} \right\rceil = x, \quad d(v_{tx+t}, v_{2t-3}) = x - 1 + \left\lceil \frac{2}{t} \right\rceil = x,
\]
and by (6),
\[
d(v_{n-tx-2}, v_{2t-3}) = x + 2 + \left\lceil \frac{-1}{2} \right\rceil = x + 2, \quad d(v_{n-tx-1}, v_{2t-3}) = x + 2 + \left\lceil \frac{-2}{2} \right\rceil = x + 2,
\]
any vertex in \( V_2 \) and any vertex in \( V_4 \) have different representations of distances with respect to \( W \).

We study representations of the vertices in \( V_3 \). For \( j = 1, 2, \ldots, p - 1 \) and \( i = 0, 2, \ldots, t - 2 \), we have \( v_i \in W_1 \) and by (4),
\[
d(v_{tk+t+j}, v_i) = \min \left\{ k + 1 + \left\lceil \frac{j-i}{t} \right\rceil, k + \left\lceil \frac{p+j-2}{t} \right\rceil \right\} = k + 1,
\]
thus \( r(v_{tk+t+j}|W_1) = (k + 1, \ldots, k + 1) \). The only vertices in \( V_2 \cup V_4 \) with the same representations of distances with respect to \( W_1 \) are \( v_{tk+t} \).

Let us prove that any two vertices in \( V_3 \cup \{ v_{tk+t-1}, v_{tk+t} \} \) have different representations with respect to \( W \). It suffices to consider the vertices in \( V' = (V_3 \cup \{ v_{tk+t-1}, v_{tk+t} \}) \setminus W_3 = \{ v_{tk+t}, v_{tk+t+2}, \ldots, v_{tk+t+p-1} \} \). For \( j = 0, 2, \ldots, p - 1 \) and \( i = 1, 3, \ldots, t - 3 \), we have \( v_{t+i} \in W_2 \) and by (5)
\[
d(v_{tk+t+j}, v_{t+i}) = k + \left\lceil \frac{j-i}{2} \right\rceil = \begin{cases} k & \text{if } i \geq j, \\ k + 1 & \text{if } i < j. \end{cases}
\]
Since \( j \) (for \( j \leq t - 2 \)) is greater than \( \frac{t}{2} \) elements from the set \( \{1, 3, \ldots, t - 3\} \), the first \( \frac{t}{2} \) entries of \( r(v_{tk+t+j}|W_2') \) where \( W_2' = W_2 \setminus \{ v_{t-1} \} \) are equal to \( k + 1 \) and the other \( \frac{t}{2} - \frac{t}{2} - 1 \) entries are \( k \). If \( p = t + 1 \) and \( j = t \), we obtain
\[
r(v_{tk+t+j}|W_2') = r(v_{tk+2t}|W_2') = (k + 1, \ldots, k + 1).
\]
It follows that the only vertices of \( V' \) having the same representations of distances with respect to \( W_2' \) are \( v_{tk+2t} \) and \( v_{tk+2t-2} \) if \( p = t + 1 \). These vertices are resolved by \( v_{tk+t-1} \in W_3 \), since by (5),
\[
d(v_{tk+2t}, v_{tk+t-1}) = 1 + \left\lceil \frac{1}{2} \right\rceil = 2 \quad \text{and} \quad d(v_{tk+2t-2}, v_{tk+t-1}) = 1 + \left\lceil \frac{-1}{2} \right\rceil = 1.
\]
Thus all vertices of \( V_3 \) are resolved by \( W \).

We consider the vertices in \( V_1 \). For \( j = 1, 3, \ldots, t - 1 \) and \( t; i = 0, 2, \ldots, t - 2 \), we have \( v_i \in W_1 \) and \( d(v_j, v_i) = \left\lceil \frac{j-i}{t} \right\rceil = 1 \), thus \( r(v_j|W_1) = (1, \ldots, 1) \) for \( v_j \in V_1 \setminus W_1 \). From the previous part of this proof we know that the only vertices in \( V_2 \cup V_3 \cup V_4 \) having the representation with respect to \( W_1 \) equal to \( (1, \ldots, 1) \) are \( v_{n-2} \) and \( v_{n-1} \). Since \( v_{t-1} \in W_2 \), it remains to resolve all pairs of vertices in the set \( V'' = \{ v_1, v_3, \ldots, v_{t-3}; v_t, v_{n-2}, v_{n-1} \} \).

We study their representations with respect to \( W_2 \). For \( j = 1, 3, \ldots, t - 3 \) and \( i = -1, 1, \ldots, t - 3 \), we have \( v_{t+i} \in W_2 \) and by (5),
\[
d(v_j, v_{t+i}) = 1 + \left\lceil \frac{j-i}{2} \right\rceil = \begin{cases} 1 & \text{if } i \leq j, \\ 2 & \text{if } i > j. \end{cases}
\]
Since \( j \) is greater than or equal to \( \frac{t+3}{2} \) elements from the set \( \{-1, 1, \ldots, t - 3\} \), the first \( \frac{i+3}{2} \) entries of \( r(v_j|W_2) \) are equal to 1 and the other \( \frac{t}{2} - \frac{i+3}{2} \) entries
are 2. The first two entries of \( r(v_j|W_3) \) are always 1. For \( v_t \) and any \( v_{t+i} \in W_2 \), 
\[
d(v_t, v_{t+i}) = \left\lceil \frac{|i|}{t} \right\rceil = 1,
\]
therefore \( r(v_t|W_2) = (1, \ldots, 1) \).

For \( i = -1, 1, \ldots, t-3 \), by (6),
\[
d(v_{n-1}, v_{t+i}) = 1 + \left\lceil \frac{|i+1|}{t} \right\rceil = \begin{cases} 1 & \text{if } i = -1, \\ 2 & \text{if } i \geq 1, \end{cases}
\]
so \( r(v_{n-1}|W_2) = (1, 2, \ldots, 2) \). We have \( d(v_{n-2}, v_{t+i}) = 1 + \left\lceil \frac{|i+2|}{t} \right\rceil = 2 \), thus \( r(v_{n-2}|W_2) = (2, \ldots, 2) \).

The only pair of vertices in \( V'' \) having the same representations with respect to \( W_2 \) is \( (v_{t-3}, v_t) \), which is resolved by \( v_{tk+t-1} \in W_3 \), since by (5) we have 
\[
d(v_{t-3}, v_{tk+t-1}) = k + \left\lceil \frac{2}{t} \right\rceil = k + 1 \text{ and } d(v_t, v_{tk+t-1}) = k + \left\lceil \frac{1}{t} \right\rceil = k.
\]

Every two distinct vertices of the graph \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) have different representations of distances with respect to \( W \), thus \( W \) is a resolving set of \( C_n(\pm 1, \pm 2, \ldots, \pm t) \). Hence \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq |W| = t + \frac{p+1}{2} \).

\[\Box\]

4. Conclusion

We studied the metric dimension of undirected and directed circulant graphs. Results on the metric dimension of undirected circulant graphs \( C_n(\pm 1, \pm t) \) are available only for special values of \( t \). In Section 2 we found exact values of the metric dimension for directed circulant graphs \( C_n(1, t) \) by showing that if \( t \geq 2 \) and \( n \geq 2t^2 \), then \( \dim(C_n(1, t)) = t \).

In Section 3 we presented a bound on the metric dimension of undirected circulant graphs. We proved that for \( n = 2tk + t + p \), where \( t \geq 4 \) is even, \( p \) is odd, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \), \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p+1}{2} \). Note that by [13], \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p}{2} \) if \( t \) and \( p \) are even, \( 2 \leq p \leq t \), thus we have \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \left\lceil \frac{p}{2} \right\rceil \) for \( n = 2tk + t + p \), where \( t \geq 4 \) is even, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \).

Acknowledgments

This work has been supported by the National Research Foundation of South Africa; grant numbers: 112122, 90793.

References


Received 25 May 2017
Revised 23 January 2018
Accepted 23 January 2018