REGULAR COLORINGS IN REGULAR GRAPHS

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Abstract

An \((r - 1, 1)\)-coloring of an \(r\)-regular graph \(G\) is an edge coloring (with arbitrarily many colors) such that each vertex is incident to \(r - 1\) edges of one color and 1 edge of a different color. In this paper, we completely characterize all \(4\)-regular pseudographs (graphs that may contain parallel edges and loops) which do not have a \((3, 1)\)-coloring. Also, for each \(r \geq 6\) we construct graphs that are not \((r - 1, 1)\)-colorable and, more generally, are not \((r - t, t)\)-colorable for small \(t\).

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1. Introduction

A graph with no loops or multiple edges is called simple; a graph in which both multiple edges and loops are allowed is called a pseudograph. Unless specified otherwise, the word “graph” in this paper is reserved for pseudographs. All (pseudo)graphs considered here are undirected and finite. Note that we count a loop twice in the degree of a vertex.

The famous Berge-Sauer conjecture asserts that every \(4\)-regular simple graph contains a \(3\)-regular subgraph [6]. This conjecture was settled by Tashkinov in 1982 [12]. In fact, he proved that every connected \(4\)-regular pseudograph with either at most two pairs of multiple edges and no loops or at most one pair of multiple edges and at most one loop contains a \(3\)-regular subgraph. Observe that this cannot hold for all \(4\)-regular pseudographs, because the graph consisting of a single vertex with two loops contains no \(3\)-regular subgraph. The following question remains open.

Question 1. Which \(4\)-regular pseudographs contain \(3\)-regular subgraphs?

Note that in 1988, Tashkinov [13] determined the values of \(t\) and \(r\) for which every \(r\)-regular pseudograph contains a \(t\)-regular subgraph. Beyond finding regular subgraphs in regular graphs, finding factors—that is, regular spanning subgraphs—in regular graphs is also of special interest. As early as 1891, Petersen [10] studied the existence of factors in regular graphs. Since then numerous results on factors have appeared—see, for example, [2, 5, 7, 11]. The concept of factors can be generalized as follows: for any set of integers \(S\), an \(S\)-factor of a graph is a spanning subgraph in which the degree of each vertex is in \(S\) [8]. Several authors [1, 3, 9] have recently studied \(\{a, b\}\)-factors in \(r\)-regular graphs with \(a + b = r\). In particular, Akbari and Kano [1] made the following conjecture:

Conjecture 1. If \(r\) is odd and \(0 \leq t \leq r\), then every \(r\)-regular graph has an \(\{r - t, t\}\)-factor.
However, Axenovich and Rollin [3] disproved this conjecture. The following theorem summarizes what is known about \( \{r-t, t\} \)-factors of \( r \)-regular graphs. (Note that although intended for simple graphs, the result of Petersen [10] applies to pseudographs as well.)

**Theorem 2.** Let \( t \) and \( r \) be positive integers with \( t \leq \frac{r}{2} \).

(a) When \( r \) is even.
- If \( t \) is even, then every \( r \)-regular graph has a \( t \)-factor, and thus has an \( \{r-t, t\} \)-factor (Petersen [10]).
- Every \( r \)-regular graph of even order has an \( \left\{ \frac{r}{2} + 1, \frac{r}{2} - 1 \right\} \)-factor (Lu, Wang, and Yu [9]).
- If \( t \) is odd and \( t \leq \frac{r}{2} - 2 \), then there exists a connected \( r \)-regular graph of even order that has no \( \{r-t, t\} \)-factor [9].
- If \( t \) is odd and \( t = \frac{r}{2} \), then every \( r \)-regular subgraph of even order has an \( \{r-t, t\} \)-factor [9].
- If \( t \) is odd, then trivially, no \( r \)-regular graph of odd order has an \( \{r-t, t\} \)-factor.

(b) When \( r \) is odd and \( r \geq 5 \).
- If \( t \) is even, then every \( r \)-regular graph has an \( \{r-t, t\} \)-factor (Akbari and Kano [1]).
- If \( t \) is odd and \( \frac{r}{3} \leq t \), then every \( r \)-regular graph has an \( \{r-t, t\} \)-factor [1].
- If \( t \) is odd and \( (t+1)(t+2) \leq r \), then there exists an \( r \)-regular graph that has no \( \{r-t, t\} \)-factor (Axenovich and Rollin [3]).

(c) Every \( 3 \)-regular graph has a \( \{2, 1\} \)-factor (Tutte [14]).

An \((r-t, t)\)-coloring of an \( r \)-regular graph \( G \) is an edge-coloring (with at least two colors) such that each vertex is incident to \( r-t \) edges of one color and \( t \) edges of a different color. An ordered \((r-t, t)\)-coloring of \( G \) is an \((r-t, t)\)-coloring using integers as colors such that each vertex is incident to \( r-t \) edges of some color \( i \) and \( t \) edges of some color \( j \) with \( i < j \). Thus, in a graph with an ordered \((r-t, t)\)-coloring, regardless of how many colors are used, the set of edges colored with the minimum integer induces an \((r-t)\)-regular subgraph, and the set of edges colored with the maximum integer induces a \( t \)-regular subgraph.

Bernshteyn [4] introduced \((3, 1)\)-colorings as an approach to answer Question 1. A possible advantage of working with \((3, 1)\)-colorings is that this is a locally-defined notion. Bernshteyn proved the following.

**Theorem 3** (Bernshteyn [4]). A connected \( 4 \)-regular graph contains a \( 3 \)-regular subgraph if and only if it admits an ordered \((3, 1)\)-coloring.

We observe that the notion of an \((r-t, t)\)-coloring of an \( r \)-regular graph generalizes that of an \( \{r-t, t\} \)-factor. Indeed, an \( r \)-regular graph \( G \) has an
A. Bernshteyn et al. (r − t, t)-coloring with two colors if and only if G has an \{r − t, t\}-factor: the two color classes are precisely the \{r − t, t\}-factor and its complement, which is another \{r − t, t\}-factor. Thus, (r − t, t)-colorings provide a common approach to attacking Question 1 as well as any unresolved cases from Conjecture 1, specifically, when r and t are both odd and 3t < r < (t + 1)(t + 2). As an (r − t, t)-coloring with more than two colors can exist when there is no \{r − t, t\}-factor, we consider the following general question.

**Question 2.** For which r and t does every r-regular graph have an (r − t, t)-coloring?

For r ≥ 6, we resolve Question 2 for various values of t, including t = 1 (see Section 3). However, the question remains open for r = 5 and t = 1.

There are trivial examples of 4-regular graphs without (3,1)-colorings, such as a single vertex with two loops. However, Theorem 3 motivates the following weaker version of Question 1.

**Question 3.** Which 4-regular graphs have (3,1)-colorings?

The arrows in Figure 1 indicate the relationships among t-factors, \{r − t, t\}-factors, ordered (r − t, t)-colorings, (r − t, t)-colorings, and t-regular subgraphs of r-regular graphs that hold for arbitrary r and t.

![Figure 1](image-url)

Figure 1. Implications that hold for every r-regular graph G and for all integers 0 < t < r.

Now we are ready to describe our main results. First, in Section 2, we characterize all 4-regular graphs which are not (3,1)-colorable, which settles Question 3. Because the statement of the result requires additional definitions, we postpone it until then (see Theorem 4). Then, in Section 3, we construct relevant examples of r-regular graphs for r ≥ 6 and various t: some with no (r − t, t)-coloring, others with an (r − t, t)-coloring but no \{r − t, t\}-factor.

2. **(3,1)-Colorings in 4-Regular Graphs**

In this section we characterize 4-regular graphs that do not admit (3,1)-colorings. Let us first establish some terminology. Let G_1 and G_2 be vertex-disjoint graphs
with (possibly loop) edges $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. The disjoint union of $X$ and $Y$ is denoted by $X \cup Y$. The edge adhesion of $G_1$ and $G_2$ at $e_1$ and $e_2$ is the graph $G = (G_1, e_1) + (G_2, e_2)$ obtained by subdividing edges $e_1$ and $e_2$ and identifying the two new vertices. (See Figure 2.) That is,

\[ V(G) = V(G_1) \cup V(G_2) \cup \{w\}; \]
\[ E(G) = (E(G_1) \setminus \{e_1\}) \cup (E(G_2) \setminus \{e_2\}) \cup \{u_1w, v_1w, u_2w, v_2w\}. \]

The adhesion of a loop to graph $H$ at edge $e = uv \in E(H)$ is the graph $H' = (H, e) + O$ obtained by subdividing $e$ and adding a loop at the new vertex. (See Figure 3.) That is,

\[ V(H') = V(H) \cup \{x\}; \]
\[ E(H') = (E(H) \setminus \{e\}) \cup \{ux, vx, xx\}. \]

Let $C$ be a cycle, which has $|E(C)| = |V(C)|$ (allowing for a degenerate cycle on 1 or 2 vertices). A double cycle is obtained from $C$ by doubling each edge. We say a double cycle is even (respectively, odd) if it has an even (respectively, odd) number of vertices. (See Figure 4.)

Clearly, double cycles and graphs resulting from edge adhesions of two 4-regular graphs or from the adhesion of a loop to a 4-regular graph are all 4-regular. We are now ready to give the main result of this section.

**Theorem 4.** A connected 4-regular graph is not $(3,1)$-colorable if and only if it can be constructed from odd double cycles via a sequence of edge adhesions.
From Theorem 4 we see that any 4-regular graph that is not (3,1)-colorable has an odd number of vertices. Indeed, any 4-regular graph with an even number of vertices has a \( \{3,1\} \)-factor by Theorem 2 and hence a (3,1)-coloring using two colors.

**Remark 5.** Theorem 4 naturally lends itself to a proof by induction. In particular, an equivalent statement is that a connected 4-regular graph is not (3,1)-colorable if and only if it is an odd double cycle or obtained from two 4-regular, non-(3,1)-colorable graphs by a sequence of edge adhesions.

Before we prove Theorem 4, we need to develop a few lemmas.

**Lemma 6.** A double cycle with \( n \geq 1 \) vertices is (3,1)-colorable if and only if \( n \) is even.

**Proof.** Even double cycles have perfect matchings and are thus (3,1)-colorable.

Assume that there is a (3,1)-coloring \( c \) of an odd double cycle \( G \). Let \( G' \) denote the cycle obtained by removing one of the parallel edges between any two adjacent vertices in \( G \). Color an edge in \( G' \) red if its corresponding parallel edges in \( G \) are of the same color under \( c \) and blue otherwise. Observe that the edges incident to any vertex in \( G' \) are of different colors, since \( c \) is a (3,1)-coloring of \( G \). This is a contradiction since \( G' \) is an odd cycle.

**Lemma 7** (Bernshteyn [4]). If \( G \) is a 4-regular graph and there exists a non-double edge \( uv \) in \( G \) with \( u \neq v \) such that \( G - \{u,v\} \) is connected, then \( G \) is (3,1)-colorable.

**Lemma 8** (Bernshteyn [4]). If \( G \) is a 4-regular graph and \( G' = (G,e) + O \) for some edge \( e \in E(G) \), then either \( G \) or \( G' \) has a 3-regular subgraph.

**Lemma 9.** Let \( G_1 \) and \( G_2 \) be (3,1)-colorable 4-regular graphs and let \( G_2 \) have a loop \( vv \). Construct \( G \) by subdividing an edge \( uw \) in \( G_1 \), identifying the new vertex with \( v \), and removing the loop \( vv \), so

\[
\begin{align*}
V(G) &= V(G_1) \cup V(G_2) ; \\
E(G) &= (E(G_1) \setminus \{uw\}) \cup (E(G_2) \setminus \{vv\}) \cup \{uv, wv\}.
\end{align*}
\]

(See Figure 5.) Then \( G \) is (3,1)-colorable.
**Proof.** Fix $(3, 1)$-colorings $c_i$ of $G_i$ for $i \in \{1, 2\}$. Note that $v$ in $G_2$ is incident to only one loop and that the two non-loop edges incident to $v$ have different colors under $c_2$. Without loss of generality, assume that $c_1(uw)$ is equal to the color of one of the non-loop edges incident to $v$. Therefore the colorings $c_1$ and $c_2$ extend to a $(3, 1)$-coloring of $G$ by coloring the edges $uv$ and $uw$ with color $c_1(uw)$.

**Corollary 10.** Suppose exactly one of the connected $4$-regular graphs $G_1$ and $G_2$ is $(3, 1)$-colorable. Then for any $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, $(G_1, e_1) + (G_2, e_2)$ is $(3, 1)$-colorable.

**Proof.** Without loss of generality, we assume that $G_1$ is $(3, 1)$-colorable and $G_2$ is not. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. By Theorem 3 and Lemma 8, the graph $G' = (G_2, e_2) + O$ is $(3, 1)$-colorable. Applying Lemma 9 to $G_1$ and $G'$, we see that $(G_1, e_1) + (G_2, e_2)$ is $(3, 1)$-colorable.

**Lemma 11.** Let $G$ be a $4$-regular graph that is not $(3, 1)$-colorable. If $G$ has a non-double, non-loop edge, then $G$ is not $2$-connected.

**Proof.** Let $uv$ be a non-double, non-loop edge, and suppose for contradiction that $G$ is $2$-connected. By Lemma 7, since $G$ is not $(3, 1)$-colorable, $G' = G - \{u, v\}$ is disconnected. Since $G$ is $2$-connected, neither $u$ nor $v$ is a cut-vertex. Therefore, every component of $G'$ must contain at least one vertex from $N_G(u)$ and at least one vertex from $N_G(v)$. Since the sum of the degrees of the vertices must be even in each component, the $4$-regularity of $G$ implies that each component of $G'$ must have been connected to $\{u, v\}$ by an even number of edges. Let $N_G(u) \setminus \{v\} = \{u_1, u_2, u_3\}$ and $N_G(v) \setminus \{u\} = \{v_1, v_2, v_3\}$. Without loss of generality, $G'$ is the disjoint union of a component $G_1$ containing $u_1$ and $v_1$ and a subgraph $G_2$ of one or two components containing $u_2, u_3, v_2, v_3$.

Let $G_1' = (G_1 + u_1v_1, u_1v_1) + O$ and $G_2' = ((G - G_1) + uv, uv) + O$. (See Figure 6.) That is, $V(G_1') = V(G_1) \cup \{w_1\};$ $E(G_1') = E(G_1) \cup \{u_1w_1, v_1w_1, w_1w_1\};$ $V(G_2') = V(G_2) \cup \{u, v, w_2\};$ $E(G_2') = E(G_2) \cup \{uw_2, uu_3, uv, vv_2, vv_3, uw_2, vw_2, w_2w_2\}.$
By the assumption of 2-connectedness, the vertex $u_1$ is not a cut-vertex of $G$. If $u_1 = v_1$, then the vertex also has a loop (so as not to be a cut vertex) and then $G'_1$ is trivially $(3,1)$-colorable. Otherwise, $u_1 \neq v_1$ and $G'_1 - \{u_1, w_1\}$ is connected. Thus by Lemma 7, $G'_1$ is $(3,1)$-colorable. Likewise, $G'_2 - \{u, w_2\}$ is connected, so $G'_2$ is $(3,1)$-colorable. Select $(3,1)$-coloring $c_i$ of $G'_i$ for $i \in \{1, 2\}$. Note that because of the loops, $c_1(u_1 w_1) \neq c_1(v_1 w_1)$ and $c_2(u w_2) \neq c_2(v w_2)$. We can assume that $c_1(u_1 w_1) = c_2(u w_2)$ and $c_1(v_1 w_1) = c_2(v w_2)$. Therefore, the colorings $c_1$ and $c_2$ easily extend to a $(3,1)$-coloring $c$ of $G$, which is a contradiction.

Lemma 12. Let $G$ be a connected 4-regular graph that is not 2-connected. Then $G = (G_1, e_1) + (G_2, e_2)$ for some 4-regular graphs $G_1$, $G_2$ and edges $e_1 \in E(G_1)$, $e_2 \in E(G_2)$.

Proof. Indeed, let $w \in V(G)$ be a cut-vertex. Now the lemma is implied by the following observation. Since the number of vertices with odd degrees in a graph is always even, $G - w$ consists of exactly two components and each of these components receives exactly two of the edges incident to $w$.

Proof of Theorem 4. Consider 4-regular graphs $G_1$ and $G_2$ and edges $e_1$ in $G_1$, $e_2$ in $G_2$. Any $(3,1)$-coloring of $(G_1, e_1) + (G_2, e_2)$ yields a $(3,1)$-coloring of $G_1$ or $G_2$, since the edges obtained by subdividing $e_1$ or $e_2$ are of the same color. Therefore every graph that is obtained from odd double cycles via edge adhesion is not $(3,1)$-colorable due to Lemma 6.

Now let $G$ be a connected 4-regular graph that is not $(3,1)$-colorable. We use induction on $|V(G)|$ to prove that $G$ is constructed from odd double cycles via edge adhesion. If $|V(G)| = 1$, then $G$ is a double cycle of one vertex and the theorem trivially holds. Assume that $|V(G)| \geq 2$. We may also assume that
Let \( G \) be a regular graph.

If each non-double edge is a loop, then one can easily check that \( G \) is not 2-connected. If \( G \) has a non-double non-loop edge, Lemma 11 implies that it is not 2-connected. By Lemma 12, \( G = (G_1, e_1) + (G_2, e_2) \) for some 4-regular graphs \( G_1, G_2 \) and edges \( e_1 \in E(G_1), e_2 \in E(G_2) \). Corollary 10 implies that either both \( G_1 \) and \( G_2 \) are \( (3, 1) \)-colorable or neither of them is \( (3, 1) \)-colorable. In the latter case, by the inductive hypothesis, we are done.

Assume that both \( G_1 \) and \( G_2 \) are \( (3, 1) \)-colorable. Let \( G'_1 = (G_1, e_1) + O \) and observe that \( G \) is obtained from \( G'_1 \) and \( G_2 \) as in the statement of Lemma 9. Since \( G_2 \) is \( (3, 1) \)-colorable, but \( G \) is not, Lemma 9 implies that \( G'_1 \) is not \( (3, 1) \)-colorable. Therefore, by the inductive hypothesis, \( G'_1 \) is obtained from odd double cycles via edge adhesion. Since \( G'_1 \) contains a loop and at least two vertices, it is not a double cycle. Thus, \( G'_1 = (G'_{11}, e'_{11}) + (G'_{12}, e'_{12}) \), where neither \( G'_{11} \) nor \( G'_{12} \) is \( (3, 1) \)-colorable. Note that, without loss of generality, \( G'_{11} \) does not contain the subdivided edge \( e_1 \), and so \( G = (G'_{11}, e'_{11}) + (H, f) \) for some graph \( H \) and edge \( f \) in \( H \). Since both \( G \) and \( G'_{11} \) are not \( (3, 1) \)-colorable, neither is \( H \) by Corollary 10. We have shown that \( G \) is obtained from two graphs that are not \( (3, 1) \)-colorable via edge adhesion, and so the inductive step is complete.

3. \( r \)-Regular Graphs for \( r \geq 5 \)

Question 2 for \( r = 5 \) remains open at this time. However, in this section we demonstrate that there are \( r \)-regular graphs with no \( (1, r-1) \)-coloring for each \( r \geq 6 \). More generally, for each odd \( t \) and each even \( r \), as well as for each odd \( t \) and each odd \( r \geq (t+2)(t+1) \), we construct an \( r \)-regular graph with no \( (r-t, t) \)-coloring. Note that for even \( t \), every \( r \)-regular graph has an \( (r-t, t) \)-coloring and for odd \( t \geq \frac{r}{2} \) and even \( r \) every \( r \)-regular graph has a \( (r-t, t) \)-coloring due to Theorem 2.

**Theorem 13.** Let \( r \) and \( t \) be positive integers with \( t \leq \frac{r}{2} \) odd. If \( r \) is even or \( r \geq (t+2)(t+1) \), then there exists a connected \( r \)-regular graph that is not \( (r-t, t) \)-colorable.

Observe that this is the same upper bound on odd \( r \) as in Theorem 2(b) (due to [3]) for the existence of \( r \)-regular graphs without \( \{r-t, t\} \)-factors.

**Proof.** First, if \( r \) is even, then the \( r \)-regular graph with one vertex and \( \frac{r}{2} \) loops has no \( (r-t, t) \)-coloring, since \( t \) is odd.

Now suppose that \( r \geq (t+2)(t+1) \geq 6 \) is odd. Let \( G \) be a graph on vertices \( v, u, u_1, \ldots, u_{t+1} \) with \( t+2 \) edges between \( v \) and \( u \) and \( \frac{r-t-2}{2} \) loops incident to \( u_i, 1 \leq i \leq t+1 \), and \( r-(t+2)(t+1) \geq 0 \) edges between \( v \) and \( u \) and \( \frac{(t+2)(t+1)}{2} \).
loops incident to \( u \). Observe that \( G \) is \( r \)-regular. Suppose that \( G \) admits an \((r-t,t)\)-coloring. Then there is an \( i \) such that all \( t+2 \) edges between \( v \) and \( u_i \) are of the same color. However, this is a contradiction, because there is no coloring of the loops incident to this \( u_i \) such that there are exactly \( t \) edges of another color incident to \( u_i \), as \( t \) is odd.

Now we will exhibit \( r \)-regular graphs of even order that have \((r-1,1)\)-colorings but not \( \{r-1,1\} \)-factors. The constructions are similar to constructions in [9].

**Theorem 14.** For every even \( r \geq 6 \) there exists a connected \((r-1,1)\)-colorable \( r \)-regular graph of even order without an \( \{r-1,1\} \)-factor.

**Proof.** Note that \( K_{r+1} \) has an odd number of vertices and thus does not have an \( \{r-1,1\} \)-factor, as \( r-1 \) is odd. However, there is an \((r-1,1)\)-coloring with 3 colors. Indeed color a copy of \( K_r \) red, \( r-1 \) of the remaining edges blue, and the last edge green.

If \( \frac{r}{2} \) is odd, then let \( G_1, \ldots, G_{\frac{r}{2}} \) be vertex-disjoint copies of \( K_{r+1} - e \). Form a graph \( G \) from the union of the \( G_i \) by connecting all vertices of degree \( r \) in the \( G_i \) to a new vertex \( u \). Then \( G \) has an even number of vertices and is \( r \)-regular. Moreover there is an \((r-1,1)\)-coloring with 3 colors. Indeed, start by coloring \( r-1 \) of the edges incident to \( u \) green, and the other blue. For each of the \( \frac{r}{2} - 1 \) copies of \( K_{r+1} - e \) with two incoming green edges, color red a copy of \( K_r \) that contains exactly one of the neighbors of \( u \), and color the other \( r-1 \) edges (incident to the other neighbor of \( u \)) blue. In the final copy of \( K_{r+1} - e \), do the same, making sure that the \( K_r \) contains the neighbor of \( u \) with the incoming blue end. Now that we have shown \( G \) to be \((r-1,1)\)-colorable, assume that \( G \) has an \( \{r-1,1\} \)-factor, i.e., an \((r-1,1)\)-coloring in two colors. Then there is an \( i, 1 \leq i \leq \frac{r}{2} \), such that both edges between \( G_i \) and \( u \) are of the same color. This yields an \((r-1,1)\)-coloring of \( K_{r+1} \) in two colors, a contradiction.

If \( \frac{r}{2} \) is even, then let \( t = 3(\frac{r}{2} - 1) \). Let \( G_1, \ldots, G_t \) be vertex-disjoint copies of \( K_{r+1} - e \). Form a graph \( G \) from the union of the \( G_i \) and a disjoint copy of \( K_3 \) with vertex set \( \{u_0, u_1, u_2\} \) by connecting both vertices of degree \( r-1 \) in \( G_i \) to \( u_j \) if \( j(\frac{r}{2} - 1) < i \leq (j+1)(\frac{r}{2} - 1) \). Then \( G \) has an even number of vertices and is \( r \)-regular. One can show that \( G \) has an \((r-1,1)\)-coloring but no \( \{r-1,1\} \)-factor with arguments similar to those given above.

4. Concluding Remarks

Here we state a number of open problems related to our work. Recall from the Introduction that Tashkinov [12] showed that every 4-regular graph with no
multiple edges and at most one loop contains a 3-regular subgraph. It is not known whether the restriction on the number of loops is necessary.

**Question 4.** Does every 4-regular graph with no multiple edges have a 3-regular subgraph?

Let us note that Question 4 is open even for the class of 4-regular graphs with no multiple edges and at most two loops. (Note that we regard two loops at a single vertex as a pair of multiple edges.)

Most of our unanswered questions concern 5-regular graphs. The first case of Conjecture 1 that Theorem 2 does not address is when $r = 5$ and $t = 1$.

**Conjecture 15.** Every 5-regular graph has a $\{4, 1\}$-factor.

Weakening this, we have the following unresolved case of Question 2.

**Question 5.** Does every 5-regular graph have a $(4, 1)$-coloring?

Another variation of this question concerns colorings with a bounded number of colors. Bernshteyn [4] showed that if $G$ is a 4-regular graph that has a $(3, 1)$-coloring, then $G$ has a $(3, 1)$-coloring that uses at most three colors.

**Question 6.** Is there a positive integer $K$ such that every 5-regular graph has a $(4, 1)$-coloring using at most $K$ colors?

Question 6 lies “between” Conjecture 15 and Question 5 in the following sense. An affirmative answer to Question 6 clearly gives an affirmative answer to Question 5. On the other hand, as observed in the Introduction, Conjecture 15 implies an affirmative answer to Question 6 with $K = 2$.

Our final question concerns ordered $(r - 1, 1)$-colorings.

**Question 7.** For $r \geq 5$, if $G$ is an $r$-regular graph with an $(r - 1)$-regular subgraph, does $G$ admit an ordered $(r - 1, 1)$-coloring?

As observed in the Introduction, the converse to this statement always holds (see Figure 1). Also, Theorem 3 implies that the corresponding statement is true for $r = 4$.

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References


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