NICHE HYPERGRAPHS OF PRODUCTS OF DIGRAPHS

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Abstract

If \( D = (V, A) \) is a digraph, its niche hypergraph \( N\mathcal{H}(D) = (V, E) \) has the edge set \( E = \{ e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N^-(D)(v) \lor e = N^+(D)(v) \} \). Niche hypergraphs generalize the well-known niche graphs and are closely related to competition hypergraphs as well as common enemy hypergraphs.

For several products \( D_1 \circ D_2 \) of digraphs \( D_1 \) and \( D_2 \), we investigate the relations between the niche hypergraphs of the factors \( D_1, D_2 \) and the niche hypergraph of their product \( D_1 \circ D_2 \).

Keywords: niche hypergraph, product of digraphs, competition hypergraph.

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1. Introduction and Definitions

All hypergraphs \( \mathcal{H} = (V(\mathcal{H}), E(\mathcal{H})) \), graphs \( G = (V(G), E(G)) \) and digraphs \( D = (V(D), A(D)) \) considered in the following may have isolates but no multiple edges. Moreover, in digraphs loops are forbidden. With \( N_D^-(v), N_D^+(v), d_D^-(v) \) and \( d_D^+(v) \) we denote the in-neighborhood, the out-neighborhood, the in-degree
and the out-degree of $v \in V(D)$, respectively. In standard terminology we follow Bang-Jensen and Gutin [1].

In 1968, Cohen [3] introduced the competition graph $C(D) = (V, E(C(D)))$ of a digraph $D = (V, A)$ representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices $v_1$, $v_2$ are connected by an edge if and only if they compete for a common prey $w$, i.e.,

$$E(C(D)) = \{v_1, v_2 \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^-(w) \land v_2 \in N_D^+(w)\}.$$ 

Surveys of the large literature around competition graphs (and its variants) can be found in [5, 6, 11]; for (a selection of) recent results see [4, 7–10, 12–17, 21].

Meanwhile the following variants of $E$ have been investigated. The common enemy graph $CE(D)$ (cf. [11]) with the edge set

$$E(CE(D)) = \{v_1, v_2 \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^+(w) \land v_2 \in N_D^+(w)\},$$

the double competition graph or competition-common enemy graph $DC(D)$ with the edge set $E(DC(D)) = E(C(D)) \cap E(CE(D))$ (cf. [18]), and the niche graph $N(D)$ with $E(N(D)) = E(C(D)) \cup E(CE(D))$ (cf. [2]).

In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [19]. The competition hypergraph $CH(D)$ of a digraph $D = (V, A)$ has the vertex set $V$ and the edge set

$$E(CH(D)) = \{e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v)\}.$$ 

As a second hypergraph generalization, recently Park and Sano [16] defined the double competition hypergraph $DCH(D)$ of a digraph $D = (V, A)$, which has the vertex set $V$ and the edge set

$$E(DCH(D)) = \{e \subseteq V \mid |e| \geq 2 \land \exists v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^+(v_2)\}.$$ 

Our paper [5] was a third step in this direction; there we considered the niche hypergraph $NH(D)$ of a digraph $D = (V, A)$, again with the vertex set $V$ and the edge set

$$E(NH(D)) = \{e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \lor e = N_D^+(v)\}.$$ 

Note that $NH(D) = NH(D)$ holds for any digraph $D$, if $\overline{D}$ denotes the digraph obtained from $D$ by reversing all arcs.

In [5] we present results on several properties of niche hypergraphs and the so-called niche number $\hat{n}$ of hypergraphs. In most of the investigations in [5] the generating digraph $D$ of $NH(D)$ is assumed to be acyclic.
Niche Hypergraphs of Products of Digraphs

For technical reasons, we define another hypergraph generalization. The common enemy hypergraph \( CEH(D) \) of a digraph \( D = (V, A) \) has the vertex set \( V \) and the edge set

\[
E(CEH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v) \}.
\]

In the hypergraphs \( CH(D) \), \( CEH(D) \) and \( NH(D) \) no loops are allowed. Therefore, by definition the in-neighborhoods and out-neighborhoods of cardinality 1 in the digraph \( D \) play no role in the corresponding hypergraphs. This loss of information proved to be disadvantageous in the investigation of competition hypergraphs of products of digraphs (cf. [20]).

So, considering niche hypergraphs of products of digraphs, it seems to be consequent to allow loops in niche hypergraphs, too. Therefore, we define the \( l \)-competition hypergraph \( CH^l(D) \), the \( l \)-common enemy hypergraph \( CEH^l(D) \) and the \( l \)-niche hypergraph \( NH^l(D) \) (with loops) having the edge sets

\[
E(CH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \},
E(CEH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \} \quad \text{and}
E(NH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \lor e = N_D^+(v) \neq \emptyset \}
\]

\[
= E(CH^l(D)) \cup E(CEH^l(D)).
\]

For the sake of brevity, in the following we often use the term \((l)\)-competition hypergraph (sometimes in connection with the notation \( CH^l(D) \)) for the competition hypergraph \( CH(D) \) as well as for the \( l \)-competition hypergraph \( CH^l(D) \), analogously for \((l)\)-common enemy and \((l)\)-niche hypergraphs with the notations \( CEH^l(D) \) and \( NH^l(D) \), respectively.

For five products \( D_1 \circ D_2 \) (Cartesian product \( D_1 \times D_2 \), Cartesian sum \( D_1 + D_2 \), normal product \( D_1 \ast D_2 \), lexicographic product \( D_1 \cdot D_2 \) and disjunction \( D_1 \lor D_2 \)) of digraphs \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) we investigate the construction of the \((l)\)-niche hypergraph \( NH^{(l)}(D_1 \circ D_2) = \left(V, E^{(l)}_v\right)\) from \( NH^{(l)}(D_1) = \left(V_1, E^{(l)}_1\right)\), \( NH^{(l)}(D_2) = \left(V_2, E^{(l)}_2\right)\) and vice versa.

The products considered here have always the vertex set \( V := V_1 \times V_2 \); using the notation \( \tilde{A} := \{((a, b), (a', b')) \mid a, a' \in V_1, b, b' \in V_2\} \) their arc sets are defined as follows:

\[
A(D_1 \times D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \land (b, b') \in A_2\},
A(D_1 + D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid ((a, a') \in A_1 \land b = b') \lor (a = a' \land (b, b') \in A_2)\},
A(D_1 \ast D_2) := A(D_1 \times D_2) \cup A(D_1 + D_2),
A(D_1 \cdot D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (a = a' \land (b, b') \in A_2)\},
A(D_1 \lor D_2) := \{((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (b, b') \in A_2\}.
\]
It follows immediately that \( A(D_1 + D_2) \subseteq A(D_1 \ast D_2) \subseteq A(D_1 \lor D_2) \) and \( A(D_1 \times D_2) \subseteq A(D_1 \ast D_2) \). Except the lexicographic product all these products are commutative in the sense that \( D_1 \circ D_2 \simeq D_2 \circ D_1 \), where \( \circ \in \{ \times, +, \ast, \lor \} \).

Usually we number the vertices of \( D_1 \) and \( D_2 \) such that \( V_1 = \{1, 2, \ldots, r\} \), \( V_2 = \{1, 2, \ldots, s\} \) and arrange the vertices of \( V = V_1 \times V_2 \) according to the places of an \( (r, s) \)-matrix.

In analogy with the rows and the columns of the described \( (r, s) \)-matrix we call the set \( Z_i = \{(i, j) \mid j \in V_2\} \) \( (i \in V_1) \) and the set \( S_j = \{(i, j) \mid i \in V_1\} \) \( (j \in V_2) \) the \( i \)-th row and the \( j \)-th column of \( D_1 \circ D_2 \), respectively.

Then, for each \( \circ \in \{+, \ast, \cdot, \lor\} \), the subdigraph \( \langle S_j \rangle_{D_1 \circ D_2} \) of \( D_1 \circ D_2 \) induced by the vertices of a column \( S_j \) is isomorphic to \( D_1 \), and, analogously, the subdigraph \( \langle Z_i \rangle_{D_1 \circ D_2} \) of \( D_1 \circ D_2 \) induced by the vertices of a row \( Z_i \) is isomorphic to \( D_2 \). Moreover, if an arc \( a \in A(D_1 \circ D_2) \) consists only of vertices of one row \( Z_i \) \( (i \in V_1) \), we refer to \( a \) as a horizontal arc. Analogously, an arc \( a \) containing only vertices of one column \( S_j \) \( (j \in V_2) \) is called a vertical arc.

Considering \( (l) \)-niche hypergraphs, the question arises, whether or not \( NH^{(l)}(D_1 \circ D_2) \) can be obtained from \( NH^{(l)}(D_1) \) and \( NH^{(l)}(D_2) \) and vice versa.

As an instance for competition hypergraphs \( CH^{(l)} \), we cite two results from [20].

**Theorem 1** [20]. The \( l \)-competition hypergraph \( CH^l(D_1 \times D_2) = (V, \mathcal{E}^l_1) \) of the Cartesian product can be obtained from the \( l \)-competition hypergraphs \( CH^l(D_1) = (V_1, \mathcal{E}^l_1) \) and \( CH^l(D_2) = (V_2, \mathcal{E}^l_2) \) of \( D_1 \) and \( D_2 : \mathcal{E}^l_1 = \{ e_1 \times e_2 \mid e_1 \in \mathcal{E}^l_1 \land e_2 \in \mathcal{E}^l_2 \} \).

**Theorem 2** [20]. The \( l \)-competition hypergraph \( CH^l(D_1 \lor D_2) = (V, \mathcal{E}^l_1) \) of the disjunction can be obtained from the \( l \)-competition hypergraphs \( CH^l(D_1) = (V_1, \mathcal{E}^l_1) \) and \( CH^l(D_2) = (V_2, \mathcal{E}^l_2) \) of \( D_1 \) and \( D_2 \), if for each of the following conditions is known whether it is true or not:

1. \( \exists v_2 \in V_2 : N^-_2(v_2) = \emptyset \) and \( \exists v_1 \in V_1 : N^-_1(v_1) = \emptyset \).

In general, \( CH^l(D_1 \lor D_2) \) cannot be obtained from \( CH^l(D_1) \) and \( CH^l(D_2) \) without the extra information on points (a) and (b).

Note that in some cases under certain conditions \( D_1 \circ D_2 \) and even \( D_1 \) and \( D_2 \) can be reconstructed from \( CH^l(D_1 \circ D_2) \). For niche hypergraphs such strong results are not expectable.

The main reason why the reconstruction of \( D_1 \) and \( D_2 \) from \( NH^{(l)}(D_1 \circ D_2) \) is much more difficult is the following. In general, for any hyperedge \( e \in \mathcal{E}(NH^{(l)}(D)) \) it is not possible to see whether \( e \) is a set of predecessors \( e = N^-_D(v) \) or a set of successors \( e = N^+_D(v) \) of a certain vertex \( v \in V(D) \).

It is interesting that, in general, for the same reason also the construction of \( NH(D_1 \circ D_2) \) from \( NH^l(D_1) \) and \( NH^l(D_2) \) is impossible.
2. Construction of \( NH^{(l)}(D_1 \circ D_2) \) from \( NH^{(l)}(D_1) \) and \( NH^{(l)}(D_2) \)

The digraphs \( D = (V, A) \) and \( D' = (V, A') \) are \((l)\)-niche equivalent if and only if \( D \) and \( D' \) have the same \((l)\)-niche hypergraph, i.e., \( NH^{(l)}(D) = NH^{(l)}(D') \).

**Theorem 3.** Let \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) be digraphs. In general, for \( \circ \in \{\times, +, \cdot, \vee\} \), the niche hypergraph \( NH(D_1 \circ D_2) = (V, E') \) of \( D_1 \circ D_2 \) cannot be obtained from the \( l \)-niche hypergraphs \( NH^{l}(D_1) = (V_1, E'_1) \) and \( NH^{l}(D_2) = (V_2, E'_2) \) of \( D_1 \) and \( D_2 \).

**Proof.** It suffices to present digraphs \( D_1 = (V_1, A_1) \), \( D'_1 = (V_1, A'_1) \), \( D_2 = (V_2, A_2) \) with \( A_1 \neq A_2 \), such that \( D_1 \) and \( D'_1 \) are \((l)\)-niche equivalent, but the niche hypergraphs \( NH^{l}(D_1) \) and \( NH^{l}(D'_1) \) of \( D_1 \circ D_2 \) and \( D'_1 \circ D_2 \) are distinct, i.e., \( NH^{(l)}(D_1 \circ D_2) \neq NH^{(l)}(D'_1 \circ D_2) \).

So let us consider the following digraphs and their niche hypergraphs:

\[ D_1 = (V_1, A_1) = (V_1, (1, 2), (1, 2), (2, 4), (4, 5), (2, 4)) \]
\[ D'_1 = (V_1, A'_1) = (V_1, (1, 2), (3, 2), (1, 2), (4, 5)) \]
\[ D_2 = (V_2, A_2) = (V_2, (1, 2, 3)) \]
\[ D'_2 = (V_2, A'_2) = (V_2, (1, 2, 3)) \]

Clearly, \( D_1 \) and \( D'_1 \) are \((l)\)-niche equivalent, they have the \((l)\)-niche hypergraph \( NH^{l}(D_1) = NH^{l}(D'_1) = (V_1, E'_1) \), where \( E'_1 = \{(1, 3), \{2\}, \{4\}, \{5\}\} \).

In detail, looking at \( D_1 \) we have

\[ E'_1 = E(NH^{l}(D_1)) = \{(1, 3) = N_{D_1}(2), \{2\} = N_{D_1}(4) = N_{D_1}^+(1) = N_{D_1}^+(3), \{4\} = N_{D_1}(5) = N_{D_1}^+(2), \{5\} = N_{D_1}^+(4)\} \]

regarding \( D'_1 \) we get

\[ E'_1 = E(NH^{l}(D'_1)) = \{(1, 3) = N_{D'_1}(2), \{2\} = N_{D'_1}(4) = N_{D'_1}^+(1) = N_{D'_1}^+(3), \{4\} = N_{D'_1}(5), \{5\} = N_{D'_1}^+(4)\} \]

Note that \( D_1 \) and \( D'_1 \) — despite having one and the same \((l)\)-niche hypergraph — are significantly different in the sense that \( D'_1 \neq D_1, D_1 \neq D'_1 \), and, moreover, \( D_1 \) is connected but \( D'_1 \) consists of two components. Of course, using \( D_1 \) and \( D'_1 \) instead of \( D_1 \) and \( D'_1 \) could be an alternative approach for proving Theorem 3.

For the sake of completeness, we give the \((l)\)-niche hypergraph \( NH^{l}(D_2) = (V_2, E'_2) \), with \( E'_2 = \{\{1, 2\} = N_{D_2}(3), \{3\} = N_{D_2}(4) = N_{D_2}^+(3)\} \).

Now we compare the \((l)\)-niche hypergraphs of the products \( D_1 \circ D_2 \) and \( D'_1 \circ D_2 \).

- **Cartesian product** \( D_1^{(l)} \times D_2 \).

Since the Cartesian product has not so many arcs and, consequently, its niche hypergraph \( NH \left( D_1^{(l)} \times D_2 \right) \) includes only few hyperedges, we present the whole edge sets \( E \left( NH \left( D_1^{(l)} \times D_2 \right) \right) \) here (in case of the other four products the edge sets of \( NH \left( D_1^{(l)} \circ D_2 \right) \) will be considerably larger, hence in these cases we will give up on writing down these sets completely).
\[ E(NH(D_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D_1 \times D_2}^-(2, 3), \]
\[ \{(2, 1), (2, 2)\} = N_{D_1 \times D_2}^-((2, 3)), \]
\[ \{(4, 1), (4, 2)\} = N_{D_1 \times D_2}^-((5, 3)) \]

and
\[ E(NH(D'_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D'_1 \times D_2}^-((2, 3)), \]
\[ \{(4, 1), (4, 2)\} = N_{D'_1 \times D_2}^-((5, 3)) \} \].

- **Cartesian sum** \( D_1^{(r)} + D_2 \), **normal product** \( D_1^{(r)} \ast D_2 \) and **lexicographic product** \( D_1^{(r)} \cdot D_2 \).

Since \( D_1 \) is connected, the Cartesian sum \( D_1 + D_2 \), the normal product \( D_1 \ast D_2 \) as well as the lexicographic product \( D_1 \cdot D_2 \) are connected, too. Considering the (disconnected) digraph \( D'_1 \), obviously \( D'_1 + D_2, D'_1 \ast D_2 \) and \( D'_1 \cdot D_2 \) are disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected. In detail, each of the products \( D'_1 \circ D_2, D'_1 \circ D_2 \) is disconnected.

Therefore, in the niche hypergraph \( NH(D'_1 \circ D_2) \) hyperedges containing vertices of both components cannot exist:

\[ \forall e \in E(NH(D'_1 \circ D_2)) : e \cap (Z_1 \cup Z_2 \cup Z_3) = \emptyset \ \text{and} \ \ e \cap (Z_4 \cup Z_5) = \emptyset. \]

Consequently, to show \( NH(D_1 \circ D_2) \neq NH(D'_1 \circ D_2) \), it suffices to find a hyperedge \( e \in E(NH(D_1 \circ D_2)) \) such that both \( e \cap (Z_1 \cup Z_2 \cup Z_3) \) and \( e \cap (Z_4 \cup Z_5) \) are nonempty.

For each of the three products \( D_1 \circ D_2 \) we will obtain such a hyperedge by considering the set of the predecessors of the vertex \((4, 3) \in V(D_1 \circ D_2)\), i.e., \( e = N_{D_1 \circ D_2}^-((4, 3)) \).

Clearly, \( e \) results from \( N_{D_1}^-((4, 3)) = \{2\} \) and \( N_{D_2}^-((4, 3)) = \{1, 2\} \).

For the Cartesian sum \( D_1 + D_2 \), we have
\[ e = \{(2, 3), (4, 1), (4, 2)\} = N_{D_1 + D_2}^-((4, 3)). \]

In case of the normal product \( D_1 \ast D_2 \), we obtain
\[ e = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\} = N_{D_1 \ast D_2}^-((4, 3)). \]

It is easy to see that in the lexicographic product \( D_1 \cdot D_2 \) the vertex \((4, 3) \) has the same predecessors as in the normal product, hence
\[ e = N_{D_1 \cdot D_2}^-((4, 3)) = N_{D_1 \ast D_2}^-((4, 3)) = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\}. \]

- **Disjunction** \( D_1^{(r)} \lor D_2 \).

Now both \( D_1 \lor D_2 \) and \( D'_1 \lor D_2 \) are connected. Nevertheless, as in the previous cases, we consider the predecessors of the vertex \((4, 3) \) and get the hyperedge
\[ e = N_{D_1 \lor D_2}^-((4, 3)) \]
\[ = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\} \]
\[ = S_1 \cup S_2 \cup \{(2, 3)\} = S_1 \cup S_2 \cup Z_2 \in E(NH(D_1 \lor D_2)). \]
Niche Hypergraphs of Products of Digraphs

Note that \( S_1 \cup S_2 \) in the result from \( N_{D_2}^{-}(3) = \{1, 2\} \) and \( Z_2 \) from \( N_{D_1}^{-}(4) = \{2\} \).

We search for this hyperedge \( e \) in \( N\mathcal{H}(D_1' \lor D_2) \).

Assume \( e = N_{D_1' \lor D_2}^{+}((i, j)) \) or \( e = N_{D_1' \lor D_2}^{-}((i, j)) \). Since \( D_1' \) and \( D_2 \) are loopless digraphs, we obtain \( (i, j) \notin e \) and \( (i, j) \in \{ (1, 3), (3, 3), (4, 3), (5, 3) \} \), i.e., \( j = 3 \).

Let \( e = N_{D_1' \lor D_2}^{+}((i, 3)) \). Because of \( N_{D_2}^{+}(3) = \emptyset \) and \( S_1 \subseteq e \), all vertices of \( S_1 \) have to be successors of \((i, 3)\) in \( D_1' \lor D_2 \) and \( \{1, 2, \ldots, 5\} = N_{D_1'}^{+}(i) \), where \( i \in \{1, 2, \ldots, 5\} \). This contradicts the fact that \( D_1' \) is loopless.

Consequently, \( e = N_{D_1' \lor D_2}^{-}((i, 3)) \). Then, \( S_1 \cup S_2 \subseteq e \) holds trivially. Owing to \((2, 3) \in e \) we get \((2, 3) \in N_{D_1' \lor D_2}^{-}((i, 3)) \), i.e., \( 2 \in N_{D_1'}^{+}(i) \) with \( i \in \{1, 2, \ldots, 5\} \). This contradicts \( N_{D_1'}^{+}(2) = \emptyset \).

Hence, \( e \notin \mathcal{E}(N\mathcal{H}(D_1' \lor D_2)) \), thus \( D_1 \lor D_2 \) and \( D_1' \lor D_2 \) are not niche equivalent. Therefore, the niche hypergraph of the disjunction \( D_1 \lor D_2 \) cannot be constructed from the niche hypergraphs of \( D_1 \) and \( D_2 \) in general.

Using Theorems 1 and 2, for the Cartesian product and the disjunction some positive construction results can be derived. For this end we have to make use of \( \mathcal{E}(N\mathcal{H}^{(l)}(D)) = \mathcal{E}(C\mathcal{H}^{(l)}(D)) \cup \mathcal{E}(C\mathcal{E}H^{(l)}(D)) \) and \( C\mathcal{E}H^{(l)}(D) = C\mathcal{H}^{(l)}(\overline{D}) \).

Remark 4. The \( l \)-niche hypergraph \( N\mathcal{H}^{(l)}(D_1 \times D_2) \) of the Cartesian product can be obtained from the \( l \)-competition hypergraphs \( C\mathcal{H}^{(l)}(D_1), C\mathcal{H}^{(l)}(D_2) \) and the \( l \)-common enemy hypergraphs \( C\mathcal{E}H^{(l)}(D_1), C\mathcal{E}H^{(l)}(D_2) \):

\[
\mathcal{E}(N\mathcal{H}^{(l)}(D_1 \times D_2)) = \mathcal{E}(C\mathcal{H}^{(l)}(D_1 \times D_2)) \cup \mathcal{E}(C\mathcal{E}H^{(l)}(D_1 \times D_2)) \]

\[= \{ e_1 \times e_2 | e_1 \in \mathcal{E}(C\mathcal{H}^{(l)}(D_1)) \land e_2 \in \mathcal{E}(C\mathcal{H}^{(l)}(D_2)) \} \]

\[\cup \{ e_1 \times e_2 | e_1 \in \mathcal{E}(C\mathcal{E}H^{(l)}(D_1)) \land e_2 \in \mathcal{E}(C\mathcal{E}H^{(l)}(D_2)) \} \].

Remark 5. The \( l \)-niche hypergraph \( N\mathcal{H}^{(l)}(D_1 \lor D_2) \) of the disjunction can be obtained from the \( l \)-competition hypergraphs \( C\mathcal{H}^{(l)}(D_1), C\mathcal{H}^{(l)}(D_2) \) and the \( l \)-common enemy hypergraphs \( C\mathcal{E}H^{(l)}(D_1), C\mathcal{E}H^{(l)}(D_2) \) provided that each of the following conditions is known to be true or false:

(a) \( \exists v_2 \in V_2 : N_{D_2}^{-}(v_2) = \emptyset \) and \( \exists v_1 \in V_1 : N_{D_1}^{-}(v_1) = \emptyset \) and

(c) \( \exists v_2 \in V_2 : N_{D_2}^{+}(v_2) = \emptyset \) and \( \exists v_1 \in V_1 : N_{D_1}^{+}(v_1) = \emptyset \).

In general, \( N\mathcal{H}^{(l)}(D_1 \lor D_2) \) cannot be obtained from \( C\mathcal{H}^{(l)}(D_1), C\mathcal{H}^{(l)}(D_2), C\mathcal{E}H^{(l)}(D_1) \) and \( C\mathcal{E}H^{(l)}(D_2) \) without the extra information on points (a)–(d).

3. Reconstruction of \( N\mathcal{H}^{(l)}(D_1) \) and \( N\mathcal{H}^{(l)}(D_2) \) from \( N\mathcal{H}^{(l)}(D_1 \circ D_2) \)

In the following, for a set \( e = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \subseteq V_1 \times V_2 \) we define \( \pi_1(e) := \)
\{i_1, \ldots, i_k\} \text{ and } \pi_2(e) := \{j_1, \ldots, j_k\}, \text{ respectively, } i.e., \pi_i \text{ denotes the projection of vertices of } NH^{(i)}(D_1 \circ D_2) \text{ onto their } i-th \text{ components, for } i \in \{1, 2\}.

**Theorem 6** (Cartesian product \(D_1 \times D_2\)).

(a) If \(E(NH(D_1 \times D_2)) \neq \emptyset\), then \(NH(D_1)\) and \(NH(D_2)\) can be obtained from \(NH(D_1 \times D_2)\).

(b) If \(E(NH^l(D_1 \times D_2)) \neq \emptyset\), then \(NH^l(D_1)\) and \(NH^l(D_2)\) can be obtained from \(NH^l(D_1 \times D_2)\).

**Proof.** Note that \(E(NH(D_1 \times D_2)) \neq \emptyset\) implies \(A_1 \neq \emptyset \neq A_2\) and \(\max(|A_1|, |A_2|) \geq 2\). Moreover, \(E(NH^l(D_1 \times D_2)) \neq \emptyset\) is equivalent to \(A_1 \neq \emptyset \neq A_2\) and, consequently, to \(E(NH^l(D_1)) \neq \emptyset \neq E(NH^l(D_2))\).

(b) Let \(e \in E(NH^l(D_1 \times D_2))\). This is equivalent to \(e \in E(CNH^l(D_1 \times D_2))\) or \(e \in E(CNH^l(D_1 \times D_2))\), i.e., \(e = N_{D_1 \times D_2}^{-}(i, j)) \text{ or } e = N_{D_1 \times D_2}^{+}(i, j))\), with a certain \((i, j) \in V_1 \times V_2\).

This holds if and only if there is a vertex \((i, j) \in V_1 \times V_2\) such that

\[\pi_1(e) = N_{D_1}^{-}(i) \text{ and } \pi_2(e) = N_{D_2}^{-}(j) \text{ or } \pi_1(e) = N_{D_1}^{+}(i) \text{ and } \pi_2(e) = N_{D_2}^{+}(j),\]

which implies \(\pi_1(e) \in E(NH^l(D_1))\) and \(\pi_2(e) \in E(NH^l(D_2))\).

Clearly, this way we can get all hyperedges \(e_1 \in E(NH^l(D_1))\) and \(e_2 \in E(NH^l(D_2))\).

(a) An analog argumentation holds if we consider the niche hypergraphs \(NH\) instead of the \(l\)-niche hypergraphs \(NH^l\), since hyperedges \(e \in E(NH^l(D_1 \times D_2))\) of cardinality 1 can be omitted if we are interested only in hyperedges \(e_i \in E(NH(D_i))\) (which have cardinality greater than 1), for \(i = 1, 2\).

**Theorem 7** (Cartesian sum \(D_1 + D_2\)).

(a) \(NH(D_1)\) and \(NH(D_2)\) can be obtained from \(NH(D_1 + D_2)\).

(b) \(NH^l(D_1)\) and \(NH^l(D_2)\) can be obtained from \(NH^l(D_1 + D_2)\), provided that one of the following conditions is true:

1. \(E(NH^l(D_1 + D_2)) = \emptyset\);
2. \(\forall e \in E(NH^l(D_1 + D_2)): |\pi_1(e)| \geq 1\) and \(\exists e \in E(NH^l(D_1 + D_2)): |\pi_2(e)| \geq 2\);
3. \(\forall e \in E(NH^l(D_1 + D_2)): |\pi_2(e)| \geq 1\) and \(\exists e \in E(NH^l(D_1 + D_2)): |\pi_1(e)| \geq 2\);
4. \(\exists (i, j) \in V_1 \times V_2 \forall e \in E(NH^l(D_1 + D_2)): (i, j) \notin e\).

**Proof.** (a) Let \(e \in E(NH(D_1 + D_2))\) and \((i, j) \in V_1 \times V_2\) with \(e = N_{D_1 + D_2}^{-}(i, j)) \text{ or } e = N_{D_1 + D_2}^{+}(i, j))\). Then \(e = \{(i, j), \ldots, (i, j), (i_1, j), \ldots, (i_2, j)\}\), where \(i, i_1, \ldots, i_l\) and \(j, j_1, \ldots, j_k\) are pairwise distinct vertices in \(V_1\) and \(V_2\), respectively.

...
To construct $E(NH(D_1))$, we need only those hyperedges $e \in E(NH(D_1 + D_2))$ which contain $l \geq 2$ vertices with one and the same second component:

$$E(NH(D_1)) = \left\{ \pi_1(e) \mid I \in E(NH(D_1 + D_2)) \wedge e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\} \wedge l \geq 2 \wedge I = \left\{ \{i\}, k \geq 1 \right\} \{\emptyset, k = 0\} \right\}.$$  

Analogously, we obtain $E(NH(D_2))$:

$$E(NH(D_2)) = \left\{ \pi_2(e) \mid J \in E(NH(D_1 + D_2)) \wedge e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\} \wedge k \geq 2 \wedge J = \left\{ \{j\}, l \geq 1 \right\} \{\emptyset, l = 0\} \right\}.$$  

(b) The proof of (1)–(3) is similar to the proof of (1)–(3) of Proposition 2 in [20].

Case (1): $E(NH^l(D_1 + D_2)) = \emptyset$. Obviously, $A(D_1 + D_2) = \emptyset = A(D_1) = A(D_2) = E(NH^l(D_1)) = E(NH^l(D_2))$.

Case (2): $\forall e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| = 1$ and $\exists e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| \geq 2$.

Let $e \in E(NH^l(D_1 + D_2))$ with $|\pi_2(e)| \geq 2$, i.e., $e = \{(i, j_1), \ldots, (i, j_k)\} = N_{D_1+D_2}^+(\{i, j\})$ or $e = \{(i, j_1), \ldots, (i, j_k)\} = N_{D_1+D_2}^-(\{i, j\})$ with $k \geq 2$ and suitable $i \in V_1$, $j \in V_2$ and $j_1, \ldots, j_k \in V_2$.

We discuss only the situation $e = N_{D_1+D_2}^+(\{i, j\})$, since $e = N_{D_1+D_2}^-(\{i, j\})$ can be proved analogously.

Clearly, $N_{D_2}^-(\{j_1, \ldots, j_k\}) = \pi_2(e)$. The assumption that there are $i' \in V_1$, $l \geq 1$ and $i'_1, \ldots, i'_l \in V_1$ with $N_{D_1}^-(i') = \{i'_1, \ldots, i'_l\} \neq \emptyset$ would lead to $e' = N_{D_1+D_2}^-(\{i', j\}) = \{(i'_1, j), \ldots, (i'_l, j), (i', j_1), \ldots, (i', j_k)\}$ with $|\pi_1(e')| \geq 2$, a contradiction.

Therefore, $E(NH^l(D_1)) = \emptyset$ and $E(NH^l(D_2)) = \left\{ \pi_2(e) \mid e \in E(NH^l(D_1 + D_2)) \wedge |\pi_1(e)| \geq 2 \right\}$.

Case (3): $\forall e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| = 1$ and $\exists e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| \geq 2$.

This can be treated in the same way as Case (2).

Case (4): $\exists (i, j) \in V_1 \times V_2 \forall e \in E(NH^l(D_1 + D_2)) : (i, j) \notin e$. Since for every $e \in E(NH^l(D_1 + D_2))$ we have $(i, j) \notin e$, the vertex $i \in V_1$ is an isolate in
\(NH^l(D_1)\) and in \(D_1\). For the same reason, \(j \in V_2\) is an isolate in \(NH^l(D_2)\) and in \(D_2\). We discuss only the construction of \(NH^l(D_2)\), the rest follows analogously.

Since \(i\) is an isolate, in \(D_1 + D_2\) there is no arc between the \(i\)-th row \(Z_i\) and any other row. Therefore, all arcs with an initial or a terminal vertex in \(Z_i\) result from arcs in \(D_2\) and we have

\[
\forall a \in A(D_1 + D_2): V(a) \cap Z_i \neq \emptyset \Rightarrow V(a) \subseteq Z_i.
\]

Hence, denoting by \(\langle Z_i \rangle_{D_1 + D_2}\) and by \(\langle Z_i \rangle_{NH^l(D_1 + D_2)}\) the subdigraph of \(D_1 + D_2\) and the subhypergraph of \(NH^l(D_1 + D_2)\) generated by the vertices of \(Z_i\), respectively, we obtain

- \(\langle Z_i \rangle_{D_1 + D_2} \simeq D_2\);
- \(\langle Z_i \rangle_{NH^l(D_1 + D_2)} \simeq NH^l(D_2)\) and
- \(E(NH^l(D_2)) = \{\pi_2(e) | e \in E(NH^l(D_1 + D_2)) \land e \subseteq Z_i\}\).

Note that, being interested in \(l\)-niche hypergraphs, loops \(e = \{(i, j)\} \in E(NH^l(D_1 + D_2))\) could lead to the problem that \(\{(i, j)\}\) can be a loop in \(NH^l(D_1 + D_2)\) either because of \(\langle Z_i \rangle_{NH^l(D_1 + D_2)} \simeq NH^l(D_2)\) and \(j\) is an isolate in \(D_2\) or because of \(i\) is an isolate in \(D_1\) and \(\{j\} \in E(NH^l(D_2))\) — and without further information it cannot be decided which of these cases occurs.

In comparison with Proposition 2(4) of our paper [20] we see that for the reconstruction of the \(l\)-competition graphs \(CH^l(D_1)\) and \(CH^l(D_2)\) from \(CH^l(D_1 + D_2)\) there is another sufficient condition, namely:

\[
\exists e \in E(CH^l(D_1 + D_2)) : |\pi_1(e)| \geq 3 \land |\pi_2(e)| \geq 3.
\]

**Remark 8.** In general, for niche hypergraphs an analogous condition to Proposition 2(4) in [20], i.e.,

\[
(a) \quad \exists e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| \geq 3 \land |\pi_2(e)| \geq 3
\]

is unsuited to ensure that \(NH^l(D_1)\) and \(NH^l(D_2)\) can be reconstructed from \(NH^l(D_1 + D_2)\).

**Proof.** Without loss of generality, let \(e = \{(i, j_1), \ldots, (i, j_l), (i_1, j), \ldots, (i_l, j)\}\) be a hyperedge in \(NH^l(D_1 + D_2)\) with \(k \geq 2\) and \(l \geq 2\).

There are two possibilities for the hyperedge \(e\), namely \(e = \{N^{-}_{D_1 + D_2}((i, j))\}\),

\[
\pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = \begin{cases} 
N^-_{D_1}(i) \\
N^+_{D_1}(i) 
\end{cases}, \quad \text{and}
\]

\[
\pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = \begin{cases} 
N^-_{D_2}(j) \\
N^+_{D_2}(j) 
\end{cases}.
\]
Then we have \( e \in E \left( CH^l(D_1 + D_2) \right) \), which is equivalent to \( e = N_{D_1 + D_2}^{-}(i, j) \), or otherwise \( e \in E \left( CEH^l(D_1 + D_2) \right) \), i.e., \( e = N_{D_1 + D_2}^{+}(i, j) \). In the first case it follows \( \pi_1(e) \setminus \{i\} = N_{D_1}^{-}(i) \) and \( \pi_2(e) \setminus \{j\} = N_{D_2}^{-}(j) \), in the second case \( \pi_1(e) \setminus \{i\} = N_{D_1}^{+}(i) \) and \( \pi_2(e) \setminus \{j\} = N_{D_2}^{+}(j) \) is valid.

In both cases we obtain \( \pi_1(e) \setminus \{i\} \in E \left( NH^{l}(D_1) \right) \) and \( \pi_2(e) \setminus \{j\} \in E \left( NH^{l}(D_2) \right) \) and both sets \( \pi_1(e) \setminus \{i\} \) and \( \pi_2(e) \setminus \{j\} \) are hyperedges in the corresponding competition hypergraph \( CH^l(D_\tau) \) (\( \tau \in \{1, 2\} \)) or both are hyperedges in the common enemy hypergraph \( CEH^l(D_\tau) \) (\( \tau \in \{1, 2\} \)).

Our argumentation is the following.

- The above implies that, in this sense, "competition hyperedges" \( e \in E \left( CH^l(D_1 + D_2) \right) \) include only information on "competition hyperedges" in \( E \left( CH^l(D_1) \right) \subseteq E \left( NH^{l}(D_1) \right) \) and \( E \left( CH^l(D_2) \right) \subseteq E \left( NH^{l}(D_2) \right) \), respectively. The same applies to "common enemy hyperedges" \( e \in E \left( CEH^{l}(D_1 + D_2) \right) \) and "common enemy hyperedges" in \( E \left( CEH^{l}(D_1) \right) \subseteq E \left( NH^{l}(D_1) \right) \) and \( E \left( CEH^{l}(D_2) \right) \subseteq E \left( NH^{l}(D_2) \right) \).

- Below, we will describe the reconstruction of the hyperedges of \( CH^l(D_1) \) and \( CH^l(D_2) \) from \( CH^l(D_1 + D_2) \) according to Case 4 of the proof of Proposition 2 in [20]. We will see that in this reconstruction procedure the conditions \( |\pi_1(e)| \geq 3 \) and \( |\pi_2(e)| \geq 3 \) (for a certain hyperedge \( e \in E \left( CH^l(D_1 + D_2) \right) \)) are essential. Obviously, an analog reconstruction procedure can be used to obtain \( CEH^l(D_1) \) and \( CEH^l(D_2) \) from \( CEH^l(D_1 + D_2) \), if there is a hyperedge \( e \in E \left( CEH^{l}(D_1 + D_2) \right) \) with \( |\pi_1(e)| \geq 3 \) and \( |\pi_2(e)| \geq 3 \). Clearly, the described reconstruction will fail if there is no such hyperedge \( e \) with the required properties.

- Now let \( D_1 \) and \( D_2 \) be digraphs fulfilling (a). Note that, in general, for an arbitrarily chosen hyperedge \( e \) in \( NH^{l}(D_1 + D_2) \) it cannot be found out whether \( e \) is a "competition hyperedge", i.e., \( e \in E \left( CH^l(D_1 + D_2) \right) \), or a "common enemy hyperedge", i.e., \( e \in E \left( CEH^{l}(D_1 + D_2) \right) \).

- We additionally assume that in \( NH^{l}(D_1 + D_2) \) all hyperedges fulfilling (a) are edges of the competition hypergraph \( CH^l(D_1 + D_2) \) but not edges of the common enemy hypergraph \( CEH^l(D_1 + D_2) \). Then, clearly, the reconstruction method from Proposition 2 in [20] has to fail for hyperedges in \( E \left( CEH^{l}(D_2) \right) \setminus E \left( CH^l(D_2) \right) \).

It remains to describe the reconstruction method from Case 4 of the proof of Proposition 2 in [20].

Under the assumptions given above, let \( e \in E \left( NH^{l}(D_1 + D_2) \right) \) be a hyperedge with (a), i.e., \( e \in E \left( CH^l(D_1 + D_2) \right) \). Because of \( |\pi_1(e)| \geq 3 \) and \( |\pi_2(e)| \geq 3 \), there are vertices \( i \in V_1 \) and \( j \in V_2 \) with \( k := |\{(i, j') \mid j' \in V_2 \} \cap e| \geq 2 \) and \( l := |\{(i', j) \mid i' \in V_1 \} \cap e| \geq 2 \).
Then \( e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\} = \mathcal{N}_{D_1+D_2}((i, j)) \) and therefore \( \mathcal{N}_{D_1}^{-1}(i) = \{i_1, \ldots, i_l\} = \pi_1(e) \setminus \{i\} \) and \( \mathcal{N}_{D_2}(-1)(j) = \{j_1, \ldots, j_k\} = \pi_2(e) \setminus \{j\} \).

For each \( x \in V_1 \) let \( e^x := \{(x, j_1), \ldots, (x, j_k), (x_1, j), \ldots, (x_l, j)\} \in \mathcal{E}(\mathcal{C}\mathcal{H}^l(D_1 + D_2)) \) with \( l_x \geq 0 \). Obviously, \( e^x = \mathcal{N}_{D_1+D_2}((x, j)) \) and \( \mathcal{N}_{D_1}(x) = \{x_1, \ldots, x_l\} = \pi_1(e^x) \setminus \{x\} \). This way we obtain \( D_1 = (V_1, A_1) \) as well as \( \mathcal{E}(\mathcal{C}\mathcal{H}^l(D_1)) = \left\{ \mathcal{N}_{D_1}^{-1}(x) \mid x \in V_1 \land \mathcal{N}_{D_1}(x) \neq \emptyset \right\} \).

Analogously, for each \( y \in V_2 \) let \( e^y := \{(i_1, y), \ldots, (i, y), (i, y_1), \ldots, (i, y_{k_y})\} \in \mathcal{E}(\mathcal{C}\mathcal{H}^l(D_1 + D_2)) \) with \( k_y \geq 0 \). Then \( e^y = \mathcal{N}_{D_1+D_2}((i, y)) \) and \( \mathcal{N}_{D_2}(y) = \{y_1, \ldots, y_{k_y}\} = \pi_2(e^y) \setminus \{y\} \).

\textbf{Theorem 9} (Normal product \( D_1 \ast D_2 \)).

(a) \( \mathcal{N}\mathcal{H}(D_1) \) and \( \mathcal{N}\mathcal{H}(D_2) \) can be obtained from \( \mathcal{N}\mathcal{H}(D_1 \ast D_2) \).

(b) If there is a hyperedge \( e \in \mathcal{E}(\mathcal{N}\mathcal{H}(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \), then \( \mathcal{N}\mathcal{H}^l(D_1) \) and \( \mathcal{N}\mathcal{H}^l(D_2) \) can be obtained from \( \mathcal{N}\mathcal{H}(D_1 \ast D_2) \).

\textbf{Proof.} (b) The existence of a hyperedge \( e \in \mathcal{E}(\mathcal{N}\mathcal{H}(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \) is equivalent to \( A_1 \neq \emptyset \neq A_2 \). Let

\[ e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i, j), (i, j_1), (i_1, j_1), \ldots, (i_1, j), \ldots, (i, j_1), (i_1, j_1), (i, j_1), \ldots, (i, j), \ldots, (i, j_1), (i_1, j_1), \ldots, (i_1, j)\} \]

\[ \in \mathcal{E}(\mathcal{N}\mathcal{H}(D_1 \ast D_2)) = \mathcal{E}(\mathcal{C}\mathcal{H}(D_1 \ast D_2)) \cup \mathcal{E}(\mathcal{C}\mathcal{E}\mathcal{H}(D_1 \ast D_2)), \]

with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \).

We will follow the idea of the proof of Case 2 of Corollary 2 in our paper [20], where a similar result for competition hypergraphs was given.

But by contrast to Corollary 2 in [20], in the case of niche hypergraphs it is impossible to reconstruct the digraphs \( D_1 \) and \( D_2 \) themselves in general. The reason is the same as mentioned before for the Cartesian sum (see the proof of Remark 8). Although for a hyperedge \( e \in \mathcal{E}(\mathcal{N}\mathcal{H}(D_1 \ast D_2)) \) we can find out the vertex \((i, j)\) with \( e = \mathcal{N}_{D_1+D_2}^+((i, j)) \) or \( e = \mathcal{N}_{D_1+D_2}^-((i, j)) \), in general it will be impossible to determine whether \( e \) is the set of predecessors (\( e \) is a ”competition hyperedge”) or the set of successors (\( e \) is a ”common enemy hyperedge”) of the vertex \((i, j)\) in \( D_1 \ast D_2 \).

Note that, in spite of the distinction of cases below, it is unnecessary to know for the actual hyperedge \( e \in \mathcal{E}(\mathcal{N}\mathcal{H}(D_1 \ast D_2)) \) under investigation whether or not it is a ”competition hyperedge“ (\( e \in \mathcal{E}(\mathcal{C}\mathcal{H}(D_1 \ast D_2)) \)) or it is an ”common enemy hyperedge“ (\( e \in \mathcal{E}(\mathcal{C}\mathcal{E}\mathcal{H}(D_1 \ast D_2)) \)). This will become clear by the remarks to Case (2) below.

\textit{Case} (1): \( e \in \mathcal{E}(\mathcal{C}\mathcal{H}(D_1 \ast D_2)) \). With some modifications of the proof of Case 2 of Corollary 2 in [20] we get the following.
(a) Because of \( l = |\pi_1(e)| - 1 \geq 1 \) and \( k = |\pi_2(e)| - 1 \geq 1 \), the vertices \( i \in V_1 \) and \( j \in V_2 \) with \( N_{D_1 \ast D_2}(i, j) = e \) can be identified as the only vertices which occur exactly \( k \) and \( l \) times in \( \pi_1(e) \) and \( \pi_2(e) \), respectively. Moreover, \( \pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = N_{D_1}(i) \) and \( \pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = N_{D_2}(j) \).

(b) Obviously, for every vertex of \( x \), there are at least 3 vertices: \( (x, j_1), (x', j), (x', j_1) \), where \( x' \in N_{D_1}(x) \). Therefore \( N_{D_1 \ast D_2}((x, j)) \in E(\mathcal{CH}(D_1 \ast D_2)) \subseteq E(\mathcal{NH}(D_1 \ast D_2)) \). Analogously, for each \( y \in V_2 \) we get \( N_{D_1 \ast D_2}((i, y)) \in E(\mathcal{CH}(D_1 \ast D_2)) \subseteq E(\mathcal{NH}(D_1 \ast D_2)) \).

(c) Note that if \( x \in V_1 \) with \( N_{D_1}(x) = \emptyset \), then \( N_{D_1 \ast D_2}((x, j)) = \{(x, j_1), \ldots, (x, j_k)\} \); i.e., \( N_{D_1 \ast D_2}((x, j)) \in E(\mathcal{CH}(D_1 \ast D_2)) \subseteq E(\mathcal{NH}(D_1 \ast D_2)) \) if and only if \( k \geq 2 \). Analogously, for every \( y \in V_2 \) we get \( N_{D_1 \ast D_2}((i, y)) \in E(\mathcal{CH}(D_1 \ast D_2)) \subseteq E(\mathcal{NH}(D_1 \ast D_2)) \) if and only if \( l \geq 2 \).

Because of (b), for all vertices of \( D_1 \) and \( D_2 \), respectively, with positive indegree we get their sets of predecessors applying the procedure described in (a) to all hyperedges \( e \in E(\mathcal{CH}(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \). (In general, for a vertex \( v_1 \in V_1 \) and \( v_2 \in V_2 \), respectively, with positive indegree, procedure (a) will produce its set of predecessors more than once.) Trivially, each vertex for which (a) does not provide a set of predecessors has indegree 0 (cf. (c)).

Thus we obtain the edge set \( E(\mathcal{CH}(D_1)) \) and \( E(\mathcal{CH}(D_2)) \) of the \( l \)-competition hypergraph \( \mathcal{CH}(D_1) \) and \( \mathcal{CH}(D_2) \), respectively.

Note that we did not need hyperedges \( e \in E(\mathcal{CH}(D_1 \ast D_2)) \setminus E(\mathcal{CH}(D_1 \ast D_2)) \), i.e., hyperedges of cardinality 1.

**Case (2):** \( e \in E(\mathcal{CEH}(D_1 \ast D_2)) \). Note that \( \mathcal{CH}(D) = \mathcal{CEH}(\overline{D}) \), for any digraph \( D \). Applying the following substitutions to the proof of Case (1), word-for-word we obtain the verification of Case (2):

\[
\begin{align*}
\mathcal{CH} & \mapsto \mathcal{CEH}, \\
N^− & \mapsto N^+, \\
\text{indegree} & \mapsto \text{outdegree} \quad \text{and} \\
\text{predecessor} & \mapsto \text{successor}.
\end{align*}
\]

(a) Because of (b) it suffices to consider the case when \( A_1 = \emptyset \) or \( A_2 = \emptyset \) holds. Replacing "+" by "*" in (1)–(3) of Theorem 7, we see that the occurrence of (1), (2) or (3) is equivalent to \( A_1 = \emptyset \) or \( A_2 = \emptyset \) and we can use an analog argumentation as in the corresponding part of the proof of Theorem 7. So using (2) we obtain \( E(\mathcal{NH}(D_2)) = \{\pi_2(e) \mid e \in E(\mathcal{NH}(D_1 \ast D_2))\} \) and \( E(\mathcal{NH}(D_2)) = \{\pi_2(e) \mid e \in E(\mathcal{NH}(D_1 \ast D_2)) \land |\pi_2(e)| \geq 2\} \), respectively. ■
Note that \( A_1 = \emptyset \) or \( A_2 = \emptyset \) implies \( D_1 \ast D_2 = D_1 + D_2 \). Therefore, the last part of the above proof in connection with Theorem 7 lead to the following consequence.

**Corollary 10.** \( NH^l(D_1) \) and \( NH^l(D_2) \) can be obtained from \( NH^l(D_1 \ast D_2) \), provided that one of the following conditions is true:

1. \( \exists e \in \mathcal{E}(NH^l(D_1 \ast D_2)) \) such that \( |\pi_1(e)| = 1 \) and \( |\pi_2(e)| \geq 2 \);
2. \( \exists e \in \mathcal{E}(NH^l(D_1 \ast D_2)) \) such that \( |\pi_2(e)| = 1 \) and \( |\pi_1(e)| \geq 2 \).

**Theorem 11** (Lexicographic product : \( NH^l(D_1 \ast D_2) \)).

(a) \( NH(D_1) \) and \( NH(D_2) \) can be obtained from \( NH(D_1 \ast D_2) \).

(b) If \( |V_2| \geq 2 \), then \( NH^l(D_1) \) can be obtained from \( NH(D_1 \ast D_2) \).

(c) \( NH^l(D_1) \) and \( NH^l(D_2) \) can be obtained from \( NH(D_1 \ast D_2) \).

**Proof.** First we will show (c), i.e., \( NH^l(D_1) \) and \( NH^l(D_2) \) can be reconstructed from \( NH^l(D_1 \ast D_2) \). Then we obtain (b) and (a) as follows:

Since for \( |V_2| \geq 2 \) every loop \( e_1 = \{i\} \) in \( NH^l(D_1) \) leads to a non-loop \( e \) in \( NH^l(D_1 \ast D_2) \) (containing at least all vertices of the row \( Z_i \)), we will see that we need no loops of \( NH^l(D_1 \ast D_2) \) in order to obtain \( NH^l(D_1) \), this includes (b).

Analogously, it is obvious that non-loops \( e_i \) of \( NH^l(D_1) \) and \( NH^l(D_2) \), respectively, result in non-loops in \( NH^l(D_1 \ast D_2) \). In our considerations it will become clear that for the reconstruction of \( NH^l(D_1) \) and \( NH^l(D_2) \) we do not need the loops in \( NH^l(D_1 \ast D_2) \), so we get (a).

In order to prove (c), we consider a hyperedge \( e \in \mathcal{E}(NH^l(D_1 \ast D_2)) \). Then there is a vertex \((i, j) \in V_1 \times V_2 \) such that \( e = N_{D_1 \ast D_2}((i, j)) \) or \( e = N_{D_1 \ast D_2}^+(i, j) \). In order to simplify our depictions, we write down the considerations only for the case \( e = N_{D_1 \ast D_2}((i, j)) \in \mathcal{E}(CH^l(D_1 \ast D_2)) \); the hyperedges \( e = N_{D_1 \ast D_2}^+(i, j) \in \mathcal{E}(CEH^l(D_1 \ast D_2)) \) can be treated analogously.

In \( NH^l(D_1 \ast D_2) \) there are two possibilities for the hyperedge \( e \).

**Case 1.** \( \exists l \geq 1 \exists i_1, \ldots, i_l \in V_1 : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \). Without loss of generality let \( i_1, \ldots, i_l \) be pairwise distinct.

Hence, \( e \) is the union of the complete rows \( Z_{i_1}, \ldots, Z_{i_l} \) of \( D_1 \ast D_2 \) and from the definition of \( D_1 \ast D_2 \) it follows \( i \notin \{i_1, \ldots, i_l\} \), \( N_{D_1}^{-}(i) = \{i_1, \ldots, i_l\} \) and \( N_{D_2}^{-}(j) = \emptyset \).

**Case 2.** \( \exists l \geq 0 \exists i_1, \ldots, i_l, i' \in V_1 \exists Z' \subseteq Z_{i'} : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \cup Z' \wedge Z' \neq \emptyset \). We get \( i = i' \in V_1 \setminus \{i_1, \ldots, i_l\} \) as well as \( N_{D_1}^{-}(i') = \{i_1, \ldots, i_l\} \).
\[ \mathcal{E}(N \mathcal{H}_1^l(D_1)) \text{ and } N^-_{D_2}(j) = \pi_2(e \cap Z') = \pi_2(Z') \in \mathcal{E}(N \mathcal{H}_1^l(D_2)) \] with a certain \( j \in V_2 \). In general, if \(|Z'| < |V_2| - 1\) holds, the vertex \( j \) cannot be determined.

Again, for any hyperedge \( e \in \mathcal{E}(N \mathcal{H}_1^l(D_1 \cdot D_2)) \) it cannot be found out whether \( e \) is a competition hyperedge (i.e., \( e \in \mathcal{E}(C \mathcal{H}_1^l(D_1 \cdot D_2)) \)) or \( e \) is a common enemy hyperedge (i.e., \( e \in \mathcal{E}(C \mathcal{E} \mathcal{H}_1^l(D_1 \cdot D_2)) \)) in general. But for the reconstruction of \( N \mathcal{H}_1^l(D_1) \) and \( N \mathcal{H}_1^l(D_2) \) this plays no role, since the considerations of Case 1 and Case 2 are valid for competition hyperedges (i.e., sets of predecessors) as well as, analogously, for common enemy hyperedges (i.e., sets of successors).

Moreover, we remark that Cases 1 and 2 (together with their analogs for the common enemy hyperedges) provide all hyperedges of the \((l)\)-niche hypergraphs \( N \mathcal{H}_1^l(D_1) \) and \( N \mathcal{H}_1^l(D_2) \).

Now we discuss the disjunction \( D_1 \vee D_2 \). The case \(|V_1| = 1 \) or \(|V_2| = 1\) implies \( D_1 \vee D_2 = D_1 \cdot D_2 \). Therefore, because of Theorem 11 it suffices to investigate the case \(|V_1|, |V_2| \geq 2\).

**Theorem 12** (Disjunction \( D_1 \vee D_2 \)). If \(|V_1|, |V_2| \geq 2\), then \( N \mathcal{H}_1^l(D_1) \) and \( N \mathcal{H}_1^l(D_2) \) can be obtained from \( N \mathcal{H}(D_1 \vee D_2) \).

**Proof.** Since both \( V_1 \) and \( V_2 \) contain at least two vertices, in \( N \mathcal{H}_1^l(D_1 \vee D_2) \) there are no loops and \( N \mathcal{H}_1^l(D_1 \vee D_2) = N \mathcal{H}(D_1 \vee D_2) \).

Moreover, for every hyperedge \( e \in \mathcal{E}(N \mathcal{H}(D_1 \vee D_2)) \) it holds
\[ \exists l \geq 0 \exists i_1, \ldots, i_l \in V_1 \exists k \geq 0 \exists j_1, \ldots, j_k \in V_2 : e = Z_{i_1} \cup \ldots \cup Z_{i_l} \cup S_{j_1} \cup \ldots \cup S_{j_k} \]
and, clearly, \( \min(l, k) > 0 \).

By analogy with the proof of Theorem 11 let \( (i,j) \in V_1 \times V_2 \) be a vertex such that \( e = N^-_{D_1 \vee D_2}(i,j) \) or \( e = N^+_{D_1 \vee D_2}(i,j) \). Now we follow the idea of the proof of Proposition 2 in [20], subsection 3.5, and use the abbreviations \( \mathcal{E}_1^l := \mathcal{E}(N \mathcal{H}_1^l(D_1)) \), \( \mathcal{E}_2^l := \mathcal{E}(N \mathcal{H}_1^l(D_2)) \) and \( \mathcal{E}_v := \mathcal{E}(N \mathcal{H}(D_1 \vee D_2)) \).

In case of \( \mathcal{E}_v = \emptyset \) both \( \mathcal{E}_1^l \) and \( \mathcal{E}_2^l \) are empty, too.

So let \( \mathcal{E}_v \neq \emptyset \). Additionally, for an arbitrary hyperedge \( e \in \mathcal{E}_v \) we define \( \pi_1(e) := \{ i \mid (i,j) \in e \} \) (for \( j \in \pi_2(e) \)) and \( \pi_2(e) := \{ j \mid (i,j) \in e \} \) (for \( i \in \pi_1(e) \)).

In \( \mathcal{N}(D_1 \vee D_2) \) we have three types of hyperedges:
\[ A := \{ e \in \mathcal{E}_v \mid \pi_1(e) \subset V_1 \}, \]
\[ B := \{ e \in \mathcal{E}_v \mid \pi_2(e) \subset V_2 \} \text{ and } \]
\[ C := \{ e \in \mathcal{E}_v \mid \pi_1(e) = V_1 \land \pi_2(e) = V_2 \}. \]

We obtain
\[ A = C = \emptyset \text{ if and only if } A_1 = \emptyset, \mathcal{E}_1^l = \emptyset \text{ and } \mathcal{E}_2^l = \{ \pi_2(e) \mid e \in \mathcal{E}_v \}; \]
\[ B = C = \emptyset \text{ if and only if } A_2 = \emptyset, \mathcal{E}_2^l = \emptyset \text{ and } \mathcal{E}_1^l = \{ \pi_1(e) \mid e \in \mathcal{E}_v \}; \]
\[ C \neq \emptyset \text{ if and only if } A_1 \neq \emptyset \neq A_2. \]

It remains to investigate the case \( C \neq \emptyset \). Here we see that, to determine \( \mathcal{E}_1^l \) and \( \mathcal{E}_2^l \), it suffices to make use of the hyperedges in \( C \):
\[ E_1^i = \{ (i \in V_1 \mid \pi_2^i(e) = V_2) \mid e \in C \} \]

\[ and \]

\[ E_2^j = \{ (j \in V_2 \mid \pi_1^j(e) = V_1) \mid e \in C \}. \]

(Note that in case \( A \neq \emptyset \) we have \( E_1^i = \{ \pi_1^i(e) \mid e \in A \} \) and, analogously, if \( B \neq \emptyset \) it follows \( E_2^j = \{ \pi_2^j(e) \mid e \in B \}. \)

\[ \Box \]

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