NICHE HYPERGRAPHS OF PRODUCTS OF DIGRAPHS

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Abstract

If $D = (V, A)$ is a digraph, its niche hypergraph $N\mathcal{H}(D) = (V, \mathcal{E})$ has the edge set $\mathcal{E} = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N^+_{D}(v) \lor e = N^-_{D}(v) \}$. Niche hypergraphs generalize the well-known niche graphs and are closely related to competition hypergraphs as well as common enemy hypergraphs. For several products $D_1 \circ D_2$ of digraphs $D_1$ and $D_2$, we investigate the relations between the niche hypergraphs of the factors $D_1$, $D_2$ and the niche hypergraph of their product $D_1 \circ D_2$.

Keywords: niche hypergraph, product of digraphs, competition hypergraph.

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1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered in the following may have isolates but no multiple edges. Moreover, in digraphs loops are forbidden. With $N^-_{D}(v)$, $N^+_{D}(v)$, $d^+_D(v)$ and $d^-_D(v)$ we denote the in-neighborhood, the out-neighborhood, the in-degree
and the out-degree of \( v \in V(D) \), respectively. In standard terminology we follow Bang-Jensen and Gutin [1].

In 1968, Cohen [3] introduced the competition graph \( C(D) = (V, E(C(D))) \) of a digraph \( D = (V, A) \) representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices \( v_1, v_2 \) are connected by an edge if and only if they compete for a common prey \( w \), i.e.,

\[
E(C(D)) = \{ v_1, v_2 \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^+(w) \land v_2 \in N_D^+(w) \}.
\]

Surveys of the large literature around competition graphs (and its variants) can be found in [5, 6, 11]; for (a selection of) recent results see [4, 7–10, 12–17, 21].

Meanwhile the following variants of \( C(D) \) have been investigated. The common enemy graph \( CE(D) \) (cf. [11]) with the edge set

\[
E(CE(D)) = \{ v_1, v_2 \mid v_1 \neq v_2 \land \exists w \in V : v_1 \in N_D^+(w) \land v_2 \in N_D^+(w) \}.
\]

the double competition graph or competition-common enemy graph \( DC(D) \) with the edge set \( E(DC(D)) = E(C(D)) \land E(CE(D)) \) (cf. [18]), and the niche graph \( N(D) \) with \( E(N(D)) = E(C(D)) \lor E(CE(D)) \) (cf. [2]).

In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [19]. The competition hypergraph \( CH(D) \) of a digraph \( D = (V, A) \) has the vertex set \( V \) and the edge set

\[
E(CH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \}.
\]

As a second hypergraph generalization, recently Park and Sano [16] defined the double competition hypergraph \( DCH(D) \) of a digraph \( D = (V, A) \), which has the vertex set \( V \) and the edge set

\[
E(DCH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^+(v_2) \}.
\]

Our paper [5] was a third step in this direction; there we considered the niche hypergraph \( NH(D) \) of a digraph \( D = (V, A) \), again with the vertex set \( V \) and the edge set

\[
E(NH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^-(v) \lor e = N_D^+(v) \}.
\]

Note that \( NH(D) = NH(\overrightarrow{D}) \) holds for any digraph \( D \), if \( \overrightarrow{D} \) denotes the digraph obtained from \( D \) by reversing all arcs.

In [5] we present results on several properties of niche hypergraphs and the so-called niche number \( \hat{n} \) of hypergraphs. In most of the investigations in [5] the generating digraph \( D \) of \( NH(D) \) is assumed to be acyclic.
For technical reasons, we define another hypergraph generalization. The common enemy hypergraph \( CEH(D) \) of a digraph \( D = (V, A) \) has the vertex set \( V \) and the edge set

\[
\mathcal{E}(CEH(D)) = \{ e \subseteq V \mid |e| \geq 2 \land \exists v \in V : e = N_D^+(v) \}.
\]

In the hypergraphs \( CH(D) \), \( CEH(D) \) and \( NH(D) \) no loops are allowed. Therefore, by definition the in-neighborhoods and out-neighborhoods of cardinality 1 in the digraph \( D \) play no role in the corresponding hypergraphs. This loss of information proved to be disadvantageous in the investigation of competition hypergraphs of products of digraphs (cf. [20]). So, considering niche hypergraphs of products of digraphs, it seems to be consequent to allow loops in niche hypergraphs, too. Therefore, we define the \( l \)-competition hypergraph \( CH^l(D) \), the \( l \)-common enemy hypergraph \( CEH^l(D) \) and the \( l \)-niche hypergraph \( NH^l(D) \) (with loops) having the edge sets

\[
\mathcal{E}(CH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \},
\]

\[
\mathcal{E}(CEH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \} \quad \text{and}
\]

\[
\mathcal{E}(NH^l(D)) = \{ e \subseteq V \mid \exists v \in V : e = N_D^+(v) \neq \emptyset \lor e = N_D^-(v) \neq \emptyset \} = \mathcal{E}(CH^l(D)) \cup \mathcal{E}(CEH^l(D)).
\]

For the sake of brevity, in the following we often use the term \( (l) \)-competition hypergraph (sometimes in connection with the notation \( CH^l(D) \)) for the competition hypergraph \( CH(D) \) as well as for the \( l \)-competition hypergraph \( CH^l(D) \), analogously for \( (l) \)-common enemy and \( (l) \)-niche hypergraphs with the notations \( CEH^l(D) \) and \( NH^l(D) \), respectively.

For five products \( D_1 \circ D_2 \) (Cartesian product \( D_1 \times D_2 \), Cartesian sum \( D_1 + D_2 \), normal product \( D_1 \circ_2 D_2 \), lexicographic product \( D_1 \cdot D_2 \) and disjunction \( D_1 \lor D_2 \)) of digraphs \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) we investigate the construction of the \( (l) \)-niche hypergraph \( NH^l(D_1 \circ D_2) = (V, \mathcal{E}_l) \) from \( NH^l(D_1) = (V_1, \mathcal{E}_1^l) \), \( NH^l(D_2) = (V_2, \mathcal{E}_2^l) \) and vice versa.

The products considered here have always the vertex set \( V := V_1 \times V_2 \); using the notation \( \tilde{A} := \{ ((a, b), (a', b')) \mid a, a' \in V_1 \land b, b' \in V_2 \} \) their arc sets are defined as follows:

\[
A(D_1 \times D_2) := \{ ((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \land (b, b') \in A_2 \},
\]

\[
A(D_1 + D_2) := \{ ((a, b), (a', b')) \in \tilde{A} \mid ((a, a') \in A_1 \land b = b') \lor (a = a' \land (b, b') \in A_2) \},
\]

\[
A(D_1 \circ_2 D_2) := A(D_1 \times D_2) \cup A(D_1 + D_2),
\]

\[
A(D_1 \cdot D_2) := \{ ((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \land (a = a' \land (b, b') \in A_2) \},
\]

\[
A(D_1 \lor D_2) := \{ ((a, b), (a', b')) \in \tilde{A} \mid (a, a') \in A_1 \lor (b, b') \in A_2 \}.
\]
It follows immediately that \( A(D_1 + D_2) \subseteq A(D_1 \ast D_2) \subseteq A(D_1 \vee D_2) \) and \( A(D_1 \times D_2) \subseteq A(D_1 \ast D_2) \). Except the lexicographic product all these products are commutative in the sense that \( D_1 \circ D_2 \simeq D_2 \circ D_1 \), where \( \circ \in \{ \times, +, \ast, \vee \} \).

Usually we number the vertices of \( D_1 \) and \( D_2 \) such that \( V_1 = \{1, 2, \ldots, r\}, V_2 = \{1, 2, \ldots, s\} \) and arrange the vertices of \( V = V_1 \times V_2 \) according to the places of an \((r, s)\)-matrix.

In analogy with the rows and the columns of the described \((r, s)\)-matrix we call the set \( Z_i = \{(i, j) \mid j \in V_2 \} \) (\( i \in V_1 \)) and the set \( S_j = \{(i, j) \mid i \in V_1 \} \) (\( j \in V_2 \)) the \( i \)-th row and the \( j \)-th column of \( D_1 \circ D_2 \), respectively.

Then, for each \( \circ \in \{+, \ast, \cdot, \vee\} \), the subdigraph \( \langle S_j \rangle_{D_1 \circ D_2} \) of \( D_1 \circ D_2 \) induced by the vertices of a column \( S_j \) is isomorphic to \( D_1 \), and, analogously, the subdigraph \( \langle Z_i \rangle_{D_1 \circ D_2} \) of \( D_1 \circ D_2 \) induced by the vertices of a row \( Z_i \) is isomorphic to \( D_2 \). Moreover, if an arc \( a \in A(D_1 \circ D_2) \) consists only of vertices of one row \( Z_i \) (\( i \in V_1 \)), we refer to \( a \) as a horizontal arc. Analogously, an arc \( a \) containing only vertices of one column \( S_j \) (\( j \in V_2 \)) is called a vertical arc.

Considering \((l)\)-niche hypergraphs, the question arises, whether or not \( NH(l)(D_1 \circ D_2) \) can be obtained from \( NH(l)(D_1) \) and \( NH(l)(D_2) \) and vice versa.

As an instance for competition hypergraphs \( CH(l) \), we cite two results from [20].

**Theorem 1** [20]. The \( l \)-competition hypergraph \( CH(l)(D_1 \times D_2) = (V, E^l_1) \) of the Cartesian product can be obtained from the \( l \)-competition hypergraphs \( CH(l)(D_1) = (V_1, E^l_1) \) and \( CH(l)(D_2) = (V_2, E^l_2) \) of \( D_1 \) and \( D_2 \): \( E^l_1 = \{ e_1 \times e_2 \mid e_1 \in E^l_1 \wedge e_2 \in E^l_2 \} \).

**Theorem 2** [20]. The \( l \)-competition hypergraph \( CH(l)(D_1 \vee D_2) = (V, E^l_1) \) of the disjunction can be obtained from the \( l \)-competition hypergraphs \( CH(l)(D_1) = (V_1, E^l_1) \) and \( CH(l)(D_2) = (V_2, E^l_2) \) of \( D_1 \) and \( D_2 \), if for each of the following conditions is known whether it is true or not:

(a) \( \exists v_2 \in V_2 : N^-_{\overline{l}}(v_2) = \emptyset \) and (b) \( \exists v_1 \in V_1 : N^-_{\overline{l}}(v_1) = \emptyset \).

In general, \( CH(l)(D_1 \vee D_2) \) cannot be obtained from \( CH(l)(D_1) \) and \( CH(l)(D_2) \) without the extra information on points (a) and (b).

Note that in some cases under certain conditions \( D_1 \circ D_2 \) and even \( D_1 \) and \( D_2 \) can be reconstructed from \( CH(l)(D_1 \circ D_2) \). For niche hypergraphs such strong results are not expectable.

The main reason why the reconstruction of \( D_1 \) and \( D_2 \) from \( NH(l)(D_1 \circ D_2) \) is much more difficult is the following. In general, for any hyperedge \( e \in E(NH(l)(D)) \) it is not possible to see whether \( e \) is a set of predecessors \( e = N^-_{\overline{l}}(v) \) or a set of successors \( e = N^+_{\overline{l}}(v) \) of a certain vertex \( v \in V(D) \).

It is interesting that, in general, for the same reason also the construction of \( NH(D_1 \circ D_2) \) from \( NH(l)(D_1) \) and \( NH(l)(D_2) \) is impossible.
2. Construction of $\mathcal{N}H^{(l)}(D_1 \circ D_2)$ from $\mathcal{N}H^{(l)}(D_1)$ and $\mathcal{N}H^{(l)}(D_2)$

The digraphs $D = (V, A)$ and $D' = (V, A')$ are (l)-niche equivalent if and only if $D$ and $D'$ have the same (l)-niche hypergraph, i.e., $\mathcal{N}H^{(l)}(D) = \mathcal{N}H^{(l)}(D')$.

**Theorem 3.** Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be digraphs. In general, for $\circ \in \{\times, +, \cdot, \vee\}$, the niche hypergraph $\mathcal{N}H(D_1 \circ D_2) = (V, E)$ of $D_1 \circ D_2$ cannot be obtained from the l-niche hypergraphs $\mathcal{N}H^l(D_1) = (V_1, E_1^l)$ and $\mathcal{N}H^l(D_2) = (V_2, E_2^l)$ of $D_1$ and $D_2$.

**Proof.** It suffices to present digraphs $D_1 = (V_1, A_1)$, $D'_1 = (V_1, A'_1)$, $D_2 = (V_2, A_2)$ such that $D_1$ and $D'_1$ are l-niche equivalent, but the niche hypergraphs of $D_1 \circ D_2$ and $D'_1 \circ D_2$ are distinct, i.e., $\mathcal{N}H(D_1 \circ D_2) \neq \mathcal{N}H(D'_1 \circ D_2)$.

So let us consider the following digraphs and their niche hypergraphs:

- $D_1 = (V_1, A_1)$ with $V_1 = \{1, 2, 3, 4, 5\}$ and $A_1 = \{(1, 2), (3, 2), (4, 5), (2, 4)\}$,
- $D'_1 = (V_1, A'_1)$ with $A'_1 = \{(1, 2), (3, 2), (4, 5)\}$ and
- $D_2 = (V_2, A_2)$ with $V_2 = \{1, 2, 3\}$ and $A_2 = \{(1, 3), (2, 3)\}$.

Obviously, $D_1$ and $D'_1$ are l-niche equivalent, they have the l-niche hypergraph $\mathcal{N}H^l(D_1) = \mathcal{N}H^l(D'_1) = (V_1, E_1^l)$, where $E_1^l = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\}$.

In detail, looking at $D_1$ we have

$$E_1^l = E(\mathcal{N}H^l(D_1)) = \{\{1, 3\} = N^-_{D_1}(2), \{2\} = N^+_{D_1}(4) = N^+_{D_1}(1) = N^+_{D_1}(3), \{4\} = N^-_{D_1}(5) = N^+_{D_1}(2), \{5\} = N^+_{D_1}(4)\};$$

regarding $D'_1$ we get

$$E'_1^l = E(\mathcal{N}H^l(D'_1)) = \{\{1, 3\} = N^-_{D'_1}(2), \{2\} = N^+_{D'_1}(1) = N^+_{D'_1}(3), \{4\} = N^-_{D'_1}(5), \{5\} = N^+_{D'_1}(4)\}.$$

Note that $D_1$ and $D'_1$ — despite having one and the same l-niche hypergraph — are significantly different in the sense that $D'_1 \neq \overline{D_1}$, $l \neq D'_1$, and, moreover, $D_1$ is connected but $D'_1$ consists of two components. Of course, using $D_1$ and $\overline{D_1}$ instead of $D_1$ and $D'_1$ could be an alternative approach for proving Theorem 3.

For the sake of completeness, we give the l-niche hypergraph $\mathcal{N}H^l(D_2) = (V_2, E_2^l)$, with $E_2^l = \{\{1, 2\} = N^-_{D_2}(3), \{3\} = N^+_{D_2}(1) = N^+_{D_2}(2)\}$.

Now we compare the niche hypergraphs of the products $D_1 \circ D_2$ and $D'_1 \circ D_2$.

- **Cartesian product** $D_1^{(l)} \times D_2$.

Since the Cartesian product has not so many arcs and, consequently, its niche hypergraph $\mathcal{N}H\left(D_1^{(l)} \times D_2\right)$ includes only few hyperedges, we present the whole edge sets $E\left(\mathcal{N}H\left(D_1^{(l)} \times D_2\right)\right)$ here (in case of the other four products the edge sets of $\mathcal{N}H\left(D_1^{(l)} \circ D_2\right)$ will be considerably larger, hence in these cases we will give up on writing down these sets completely).
\[ E(NH(D_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D_1 \times D_2}^-(2, 3), \]
\[ \{(2, 1), (2, 2)\} = N_{D_1 \times D_2}^-(4, 3), \]
\[ \{(4, 1), (4, 2)\} = N_{D_1 \times D_2}^-(5, 3) \}\]
and
\[ E(NH(D'_1 \times D_2)) = \{(1, 1), (1, 2), (3, 1), (3, 2)\} = N_{D'_1 \times D_2}^-(2, 3), \]
\[ \{(4, 1), (4, 2)\} = N_{D'_1 \times D_2}^-(5, 3) \}\]

- **Cartesian sum** \( D^{\text{(	ext{c})}}_1 + D_2 \), **normal product** \( D^{\text{(	ext{n})}}_1 \times D_2 \) **and lexicographic product** \( D^{\text{(	ext{l})}}_1 \cdot D_2 \).

Since \( D_1 \) is connected, the Cartesian sum \( D_1 + D_2 \), the normal product \( D_1 \times D_2 \) as well as the lexicographic product \( D_1 \cdot D_2 \) are connected, too. Considering the (disconnected) digraph \( D'_1 \), obviously \( D'_1 + D_2 \), \( D'_1 \times D_2 \) and \( D'_1 \cdot D_2 \) are disconnected. In detail, each of the products \( D'_1 \circ D_2 (\circ \in \{+, \cdot, \circ\}) \) consists of the two components \( (Z_1 \cup Z_2 \cup Z_3)_{D'_1 \circ D_2} \) and \( (Z_4 \cup Z_5)_{D'_1 \circ D_2} \).

Therefore, in the niche hypergraph \( NH(D'_1 \circ D_2) \) hyperedges containing vertices of both components cannot exist:

\[ \forall e \in E(NH(D'_1 \circ D_2)) : e \cap (Z_1 \cup Z_2 \cup Z_3) = \emptyset \lor e \cap (Z_4 \cup Z_5) = \emptyset. \]

Consequently, to show \( NH(D_1 \circ D_2) \neq NH(D'_1 \circ D_2) \), it suffices to find a hyperedge \( e \in E(NH(D_1 \circ D_2)) \) such that both \( e \cap (Z_1 \cup Z_2 \cup Z_3) \) and \( e \cap (Z_4 \cup Z_5) \) are nonempty.

For each of the three products \( D_1 \circ D_2 \) we will obtain such a hyperedge by considering the set of the predecessors of the vertex \( (4, 3) \in V(D_1 \circ D_2) \), i.e., \( e = N_{D_1 \circ D_2}^-(4, 3) \). Clearly, \( e \) results from \( N_{D_1}^-(4) = \{2\} \) and \( N_{D_2}^-(3) = \{1, 2\} \).

For the Cartesian sum \( D_1 + D_2 \), we have

\[ e = \{(2, 3), (4, 1), (4, 2)\} = N_{D_1 + D_2}^-(4, 3). \]

In case of the normal product \( D_1 \times D_2 \), we obtain

\[ e = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\} = N_{D_1 \times D_2}^-(4, 3). \]

It is easy to see that in the lexicographic product \( D_1 \cdot D_2 \) the vertex \( (4, 3) \) has the same predecessors as in the normal product, hence

\[ e = N_{D_1 \cdot D_2}^-(4, 3) = N_{D_1 \circ D_2}^-(4, 3) = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2)\}. \]

- **Disjunction** \( D^{\text{(	ext{d})}}_1 \lor D_2 \).

Now both \( D_1 \lor D_2 \) and \( D'_1 \lor D_2 \) are connected. Nevertheless, as in the previous cases, we consider the predecessors of the vertex \( (4, 3) \) and get the hyperedge

\[ e = N_{D_1 \lor D_2}^-(4, 3) \]
\[ = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\} \]
\[ = S_1 \cup S_2 \cup \{(2, 3)\} = S_1 \cup S_2 \cup Z_2 \in E(NH(D_1 \lor D_2)). \]
Note that $S_1 \cup S_2$ in $e$ result from $N_{D_2}^-(3) = \{1, 2\}$ and $Z_2$ from $N_{D_1}^-(4) = \{2\}$.

We search for this hyperedge $e$ in $NH(D_1' \lor D_2)$.

Assume $e = N_{D_1' \lor D_2}^+(i, j)$ or $e = N_{D_1' \lor D_2}^-(i, j)$. Since $D_1'$ and $D_2$ are loopless digraphs, we obtain $(i, j) \notin e$ and $(i, j) \in \{(1, 3), (3, 3), (4, 3), (5, 3)\}$, i.e., $j = 3$.

Let $e = N_{D_1' \lor D_2}^+(i, 3)$. Because of $N_{D_2}^+(3) = \emptyset$ and $S_1 \subseteq e$, all vertices of $S_1$ have to be successors of $(i, 3)$ in $D_1' \lor D_2$ and $\{1, 2, \ldots, 5\} = N_{D_1}^+(i)$, where $i \in \{1, 2, \ldots, 5\}$. This contradicts the fact that $D_1'$ is loopless.

Consequently, $e = N_{D_1' \lor D_2}^+(i, 3))$. Then, $S_1 \cup S_2 \subseteq e$ holds trivially. Owing to $(2, 3) \in e$ we get $(2, 3) \in N_{D_1' \lor D_2}^-(i, 3))$, i.e., $2 \in N_{D_1}^+(i)$ with $i \in \{1, 2, \ldots, 5\}$. This contradicts $N_{D_1}^+(2) = \emptyset$.

Hence, $e \notin \mathcal{E}(NH(D_1' \lor D_2))$, thus $D_1 \lor D_2$ and $D_1' \lor D_2$ are not niche equivalent. Therefore, the niche hypergraph of the disjunction $D_1 \lor D_2$ cannot be constructed from the niche hypergraphs of $D_1$ and $D_2$ in general.

Using Theorems 1 and 2, for the Cartesian product and the disjunction some positive construction results can be derived. For this end we have to make use of

$$
\mathcal{E}(NH^{(l)}(D)) = \mathcal{E}(CH^{(l)}(D)) \cup \mathcal{E}(CEH^{(l)}(D))
$$

and

$$
CEH^{(l)}(D) = CH^{(l)}(\overline{D}).
$$

**Remark 4.** The $l$-niche hypergraph $NH^{(l)}(D_1 \times D_2)$ of the Cartesian product can be obtained from the $l$-competition hypergraphs $CH^{(l)}(D_1), CH^{(l)}(D_2)$ and the $l$-common enemy hypergraphs $CEH^{(l)}(D_1), CEH^{(l)}(D_2)$:

$$
\mathcal{E}(NH^{(l)}(D_1 \times D_2)) = \mathcal{E}(CH^{(l)}(D_1 \times D_2)) \cup \mathcal{E}(CEH^{(l)}(D_1 \times D_2)) = \{e_1 \times e_2 \mid e_1 \in \mathcal{E}(CH^{(l)}(D_1)) \land e_2 \in \mathcal{E}(CH^{(l)}(D_2))\} \\
\cup \{e_1 \times e_2 \mid e_1 \in \mathcal{E}(CEH^{(l)}(D_1)) \land e_2 \in \mathcal{E}(CEH^{(l)}(D_2))\}.
$$

**Remark 5.** The $l$-niche hypergraph $NH^{(l)}(D_1 \lor D_2)$ of the disjunction can be obtained from the $l$-competition hypergraphs $CH^{(l)}(D_1), CH^{(l)}(D_2)$ and the $l$-common enemy hypergraphs $CEH^{(l)}(D_1), CEH^{(l)}(D_2)$ provided that each of the following conditions is known to be true or false:

(a) $\exists v_2 \in V_2 : N_{D_2}^-(v_2) = \emptyset$ \quad (b) $\exists v_1 \in V_1 : N_{D_1}^-(v_1) = \emptyset$ \quad (c) $\exists v_2 \in V_2 : N_{D_2}^+(v_2) = \emptyset$ \quad (d) $\exists v_1 \in V_1 : N_{D_1}^+(v_1) = \emptyset$.

In general, $NH^{(l)}(D_1 \lor D_2)$ cannot be obtained from $CH^{(l)}(D_1), CH^{(l)}(D_2), CEH^{(l)}(D_1)$ and $CEH^{(l)}(D_2)$ without the extra information on points (a)–(d).

3. **Reconstruction of $NH^{(l)}(D_1)$ and $NH^{(l)}(D_2)$ from $NH^{(l)}(D_1 \circ D_2)$**

In the following, for a set $e = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \subseteq V_1 \times V_2$ we define $\pi_1(e) :=$
$\{i_1, \ldots, i_k\}$ and $\pi_2(e) := \{j_1, \ldots, j_k\}$, respectively, i.e., $\pi_i$ denotes the projection of vertices of $NH^{(i)}(D_1 \circ D_2)$ onto their $i$-th components, for $i \in \{1, 2\}$.

**Theorem 6** (Cartesian product $D_1 \times D_2$).

(a) If $\mathcal{E}(NH(D_1 \times D_2)) \neq \emptyset$, then $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 \times D_2)$.

(b) If $\mathcal{E}(NH^l(D_1 \times D_2)) \neq \emptyset$, then $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 \times D_2)$.

**Proof.** Note that $\mathcal{E}(NH(D_1 \times D_2)) \neq \emptyset$ implies $A_1 \neq \emptyset \neq A_2$ and $\max(|A_1|, |A_2|) \geq 2$. Moreover, $\mathcal{E}(NH^l(D_1 \times D_2)) \neq \emptyset$ is equivalent to $A_1 \neq \emptyset \neq A_2$ and, consequently, $\mathcal{E}(NH^l(D_1)) \neq \emptyset \neq \mathcal{E}(NH^l(D_2))$.

(b) Let $e \in \mathcal{E}(NH^l(D_1 \times D_2))$. This is equivalent to $e \in \mathcal{E}(CH^l(D_1 \times D_2))$ or $e \in \mathcal{E}(CEH^l(D_1 \times D_2))$, i.e., $e = N_{D_1 \times D_2}^{-}(i, j)$ or $e = N_{D_1 \times D_2}^{+}(i, j)$, with a certain $(i, j) \in V_1 \times V_2$.

This holds if and only if there is a vertex $(i, j) \in V_1 \times V_2$ such that $\pi_1(e) = N_{D_1}^{-}(i)$ or $\pi_2(e) = N_{D_2}^{-}(j)$. Note that $\pi_1(e) = N_{D_1}^{+}(i)$ and $\pi_2(e) = N_{D_2}^{+}(j)$, which implies $\pi_1(e) \in \mathcal{E}(NH^l(D_1))$ and $\pi_2(e) \in \mathcal{E}(NH^l(D_2))$.

Clearly, this way we can get all hyperedges $e_1 \in \mathcal{E}(NH^l(D_1))$ and $e_2 \in \mathcal{E}(NH^l(D_2))$.

(a) An analog argumentation holds if we consider the niche hypergraphs $NH$ instead of the l-niche hypergraphs $NH^l$, since hyperedges $e \in \mathcal{E}(NH^l(D_1 \times D_2))$ of cardinality 1 can be omitted if we are interested only in hyperedges $e_i \in \mathcal{E}(NH(D_i))$ (which have cardinality greater than 1), for $i = 1, 2$. $lacksquare$

**Theorem 7** (Cartesian sum $D_1 + D_2$).

(a) $NH(D_1)$ and $NH(D_2)$ can be obtained from $NH(D_1 + D_2)$.

(b) $NH^l(D_1)$ and $NH^l(D_2)$ can be obtained from $NH^l(D_1 + D_2)$, provided that one of the following conditions is true:

1. $\mathcal{E}(NH^l(D_1 + D_2)) = \emptyset$;
2. $\forall e \in \mathcal{E}(NH^l(D_1 + D_2)) : |\pi_1(e)| = 1$ and $\exists e \in \mathcal{E}(NH^l(D_1 + D_2)) : |\pi_2(e)| \geq 2$;
3. $\forall e \in \mathcal{E}(NH^l(D_1 + D_2)) : |\pi_2(e)| = 1$ and $\exists e \in \mathcal{E}(NH^l(D_1 + D_2)) : |\pi_1(e)| \geq 2$;
4. $\exists (i, j) \in V_1 \times V_2 \forall e \in \mathcal{E}(NH^l(D_1 + D_2)) : (i, j) \notin e$.

**Proof.** (a) Let $e \in \mathcal{E}(NH(D_1 + D_2))$ and $(i, j) \in V_1 \times V_2$ with $e = N_{D_1 + D_2}^{-}(i, j)$ or $e = N_{D_1 + D_2}^{+}(i, j)$. Then $e = \{(i, j), \ldots, (i, j), (i, j), \ldots, (i, j)\}$, where $i, i_1, \ldots, i_l$ and $j, j_1, \ldots, j_k$ are pairwise distinct vertices in $V_1$ and $V_2$, respectively.
To construct $E(NH(D_1))$, we need only those hyperedges $e \in E(NH(D_1 + D_2))$ which contain $l \geq 2$ vertices with one and the same second component:

$$E(NH(D_1)) = \left\{ \pi_1(e) \mid I \in E(NH(D_1 + D_2)) \land e = \{(i, j_1), \ldots, (i, j_k),(i_1, j), \ldots, (i_l, j)\} \land l \geq 2 \land \left\{ \begin{array}{l} k \geq 1 \quad \text{or} \quad k = 0 \end{array} \right\} \right\}.$$

Analogously, we obtain $E(NH(D_2))$:

$$E(NH(D_2)) = \left\{ \pi_2(e) \mid J \in E(NH(D_1 + D_2)) \land e = \{(i, j_1), \ldots, (i, j_k),(i_1, j), \ldots, (i_l, j)\} \land k \geq 2 \land \left\{ \begin{array}{l} l \geq 1 \quad \text{or} \quad l = 0 \end{array} \right\} \right\}.$$

(b) The proof of (1)–(3) is similar to the proof of (1)–(3) of Proposition 2 in [20].

Case (1): $E(NH^l(D_1 + D_2)) = \emptyset$. Obviously, $A(D_1 + D_2) = \emptyset = A(D_1) = A(D_2) = E(NH^l(D_1)) = E(NH^l(D_2))$.

Case (2): $\forall e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| = 1$ and $\exists e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| \geq 2$.

Let $e \in E(NH^l(D_1 + D_2))$ with $|\pi_2(e)| \geq 2$, i.e., $e = \{(i, j_1), \ldots, (i, j_k)\} = N_{D_1+D_2}^-(i,j)$ or $e = \{(i_1, j), \ldots, (i_k, j)\} = N_{D_1+D_2}^+(i,j)$ with $k \geq 2$ and suitable $i \in V_1$, $j \in V_2$ and $i_1, \ldots, i_k \in V_2$.

We discuss only the situation $e = N_{D_1+D_2}^-(i,j)$, since $e = N_{D_1+D_2}^+(i,j)$ can be proved analogously.

Clearly, $N_{D_2}^-(j) = \{j_1, \ldots, j_k\} = \pi_2(e)$. The assumption that there are $i' \in V_1$, $l \geq 1$ and $i_1', \ldots, i'_l \in V_1$ with $N_{D_1}^-(i') = \{i_1', \ldots, i'_l\} \neq \emptyset$ would lead to $e' = N_{D_1+D_2}^-(i',j)$.

Therefore, $E(NH^l(D_1)) = \emptyset$ and $E(NH^l(D_2)) = \left\{ \pi_2(e) \mid e \in E(NH^l(D_1 + D_2)) \right\}.$

Case (3): $\forall e \in E(NH^l(D_1 + D_2)) : |\pi_2(e)| = 1$ and $\exists e \in E(NH^l(D_1 + D_2)) : |\pi_1(e)| \geq 2$.

This can be treated in the same way as Case (2).

Case (4): $\exists (i, j) \in V_1 \times V_2 \forall e \in E(NH^l(D_1 + D_2)) : (i, j) \notin e$. Since for every $e \in E(NH^l(D_1 + D_2))$ we have $(i, j) \notin e$, the vertex $i \in V_1$ is an isolate in
$NH^l(D_1)$ and in $D_1$. For the same reason, $j \in V_2$ is an isolate in $NH^l(D_2)$ and in $D_2$. We discuss only the construction of $NH^l(D_2)$, the rest follows analogously.

Since $i$ is an isolate, in $D_1 + D_2$ there is no arc between the $i$-th row $Z_i$ and any other row. Therefore, all arcs with an initial or a terminal vertex in $Z_i$ result from arcs in $D_2$ and we have

$$\forall a \in A(D_1 + D_2) : V(a) \cap Z_i \neq \emptyset \Rightarrow V(a) \subseteq Z_i.$$

Hence, denoting by $(Z_i)_{D_1 + D_2}$ and by $(Z_i)_{NH^l(D_1 + D_2)}$ the subdigraph of $D_1 + D_2$ and the subhypergraph of $NH^l(D_1 + D_2)$ generated by the vertices of $Z_i$, respectively, we obtain

- $(Z_i)_{D_1 + D_2} \simeq D_2$,
- $(Z_i)_{NH^l(D_1 + D_2)} \simeq NH^l(D_2)$ and
- $\mathcal{E}(NH^l(D_2)) = \{ \pi_2(e) \mid e \in \mathcal{E}(NH^l(D_1 + D_2)) \land e \subseteq Z_i \}.$

Note that, being interested in $l$-niche hypergraphs, loops $e = \{(i, j)\} \in \mathcal{E}(NH^l(D_1 + D_2))$ could lead to the problem that $\{(i, j)\}$ can be a loop in $NH^l(D_1 + D_2)$ either because of $\{i\} \in \mathcal{E}(NH^l(D_1))$ and $j$ is an isolate in $D_2$ or because of $i$ is an isolate in $D_1$ and $\{j\} \in \mathcal{E}(NH^l(D_2))$ — and without further information it cannot be decided which of these cases occurs.

In comparison with Proposition 2(4) of our paper [20] we see that for the reconstruction of the $l$-competition graphs $CH^l(D_1)$ and $CH^l(D_2)$ from $CH^l(D_1 + D_2)$ there is another sufficient condition, namely:

$$\exists e \in \mathcal{E}(CH^l(D_1 + D_2)) : \lvert \pi_1(e) \rvert \geq 3 \land \lvert \pi_2(e) \rvert \geq 3.$$

**Remark 8.** In general, for niche hypergraphs an analogous condition to Proposition 2(4) in [20], i.e.,

$$(a) \quad \exists e \in \mathcal{E}(NH^l(D_1 + D_2)) : \lvert \pi_1(e) \rvert \geq 3 \land \lvert \pi_2(e) \rvert \geq 3$$

is unsuited to ensure that $NH^l(D_1)$ and $NH^l(D_2)$ can be reconstructed from $NH^l(D_1 + D_2)$.

**Proof.** Without loss of generality, let $e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\}$ be a hyperedge in $NH^l(D_1 + D_2)$ with $k \geq 2$ and $l \geq 2$.

There are two possibilities for the hyperedge $e$, namely $e = \{N_{D_1+D_2}^-(i, j), N_{D_1+D_2}^+(i, j)\}$, i.e.,

$$\pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = \begin{cases} N_{D_1}^-(i) \\ N_{D_1}^+(i) \end{cases}, \text{ and}$$

$$\pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = \begin{cases} N_{D_2}^-(j) \\ N_{D_2}^+(j) \end{cases}.$$
Then we have \( e \in \mathcal{E}(CH^1(D_1 + D_2)) \), which is equivalent to \( e = N_{D_1 + D_2}^{-}(i, j) \), or otherwise \( e \in \mathcal{E}(C\mathcal{H}^1(D_1 + D_2)) \), i.e., \( e = N_{D_1 + D_2}^{+}(i, j) \). In the first case it follows \( \pi_1(e) \setminus \{i\} = N_{D_1}^{-}(i) \) and \( \pi_2(e) \setminus \{j\} = N_{D_2}^{-}(j) \), in the second case \( \pi_1(e) \setminus \{i\} = N_{D_1}^{+}(i) \) and \( \pi_2(e) \setminus \{j\} = N_{D_2}^{+}(j) \) is valid.

In both cases we obtain \( \pi_1(e) \setminus \{i\} \in \mathcal{E}(NH^1(D_1)) \) and \( \pi_2(e) \setminus \{j\} \in \mathcal{E}(NH^1(D_2)) \) and both sets \( \pi_1(e) \setminus \{i\} \) and \( \pi_2(e) \setminus \{j\} \) are hyperedges in the corresponding competition hypergraph \( CH^1(D_\tau) \) \((\tau \in \{1, 2\})\) or both are hyperedges in the common enemy hypergraph \( C\mathcal{H}^1(D_\tau) \) \((\tau \in \{1, 2\})\).

Our argumentation is the following.

- The above implies that, in this sense, "

\( \text{competition hyperedges} \) e \in \mathcal{E}(CH^1(D_1 + D_2)) \subseteq \mathcal{E}(NH^1(D_1 + D_2)) \) include only information on "competition hyperedges" in \( \mathcal{E}(CH^1(D_1)) \subseteq \mathcal{E}(NH^1(D_1)) \) and \( \mathcal{E}(CH^1(D_2)) \subseteq \mathcal{E}(NH^1(D_2)) \), respectively. The same applies to "common enemy hyperedges" e \in \mathcal{E}(C\mathcal{H}^1(D_1 + D_2)) \subseteq \mathcal{E}(NH^1(D_1 + D_2)) \) and "common enemy hyperedges" in \( \mathcal{E}(C\mathcal{H}^1(D_1)) \subseteq \mathcal{E}(NH^1(D_1)) \) and \( \mathcal{E}(C\mathcal{H}^1(D_2)) \subseteq \mathcal{E}(NH^1(D_2)) \).

- Below, we will describe the reconstruction of the hyperedges of \( CH^1(D_1) \) \) and \( C\mathcal{H}^1(D_2) \) from \( CH^1(D_1 + D_2) \) according to Case 4 of the proof of Proposition 2 in [20]. We will see that in this reconstruction procedure the conditions \(|\pi_1(e)| \geq 3 \) and \(|\pi_2(e)| \geq 3 \) (for a certain hyperedge e \in \mathcal{E}(CH^1(D_1 + D_2))) are essential. Obviously, an analog reconstruction procedure can be used to obtain \( C\mathcal{H}^1(D_1) \) \) and \( C\mathcal{H}^1(D_2) \) from \( C\mathcal{H}^1(D_1 + D_2) \), if there is a hyperedge e \in \mathcal{E}(C\mathcal{H}^1(D_1 + D_2)) \) with \(|\pi_1(e)| \geq 3 \) and \(|\pi_2(e)| \geq 3 \). Clearly, the described reconstruction will fail if there is no such hyperedge e with the required properties.

- Now let \( D_1 \) \) and \( D_2 \) be digraphs fulfilling \((\alpha)\). Note that, in general, for an arbitrarily chosen hyperedge e in \( NH^1(D_1 + D_2) \) it cannot be found out whether e is a "competition hyperedge", i.e., e \in \mathcal{E}(CH^1(D_1 + D_2)) \), or a "common enemy hyperedge", i.e., e \in \mathcal{E}(C\mathcal{H}^1(D_1 + D_2)) \).

- We additionally assume that in \( NH^1(D_1 + D_2) \) all hyperedges fulfilling \((\alpha)\) are edges of the competition hypergraph \( CH^1(D_1 + D_2) \) but not edges of the common enemy hypergraph \( C\mathcal{H}^1(D_1 + D_2) \). Then, clearly, the reconstruction method from Proposition 2 in [20] has to fail for hyperedges in \( \mathcal{E}(C\mathcal{H}^1(D_2)) \setminus \mathcal{E}(CH^1(D_2)) \subseteq \mathcal{E}(NH^1(D_2)) \).

It remains to describe the reconstruction method from Case 4 of the proof of Proposition 2 in [20].

Under the assumptions given above, let e \in \mathcal{E}(NH^1(D_1 + D_2)) be a hyperedge with \((\alpha)\), i.e., e \in \mathcal{E}(CH^1(D_1 + D_2)). Because of \(|\pi_1(e)| \geq 3 \) and \(|\pi_2(e)| \geq 3 \), there are vertices \( i \in V_1 \) \) and \( j \in V_2 \) with \( k := |\{(i, j') | j' \in V_2\} \cap e| \geq 2 \) and \( l := |\{(i', j) | i' \in V_1\} \cap e| \geq 2 \).
Then \( e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\} = N_{D_1+D_2}^-((i, j)) \) and therefore \( N_{D_1}^-(i) = \{i_1, \ldots, i_l\} = \pi_1(e) \setminus \{i\} \) and \( N_{D_2}^-(j) = \{j_1, \ldots, j_k\} = \pi_2(e) \setminus \{j\} \).

For each \( x \in V_1 \) let \( e^x := \{(x, j_1), \ldots, (x, j_k), (x_1, j), \ldots, (x_{l_x}, j)\} \in \mathcal{E}(CH^l(D_1 + D_2)) \) with \( l_x \geq 0 \). Obviously, \( e^x = N_{D_1+D_2}^-((x, j)) \) and \( N_{D_1}^-(x) = \{x_1, \ldots, x_{l_x}\} = \pi_1(e^x) \setminus \{x\} \). This way we obtain \( D_1 = (V_1, A_1) \) as well as \( \mathcal{E}(CH^l(D_1)) = \{N_{D_1}^-(x) \mid x \in V_1 \cap N_{D_1}^-(x) \neq \emptyset\} \).

Analogously, for each \( y \in V_2 \) let \( e^y := \{(i_1, y), \ldots, (i, y), (i, y_1), \ldots, (i, y_{k_y})\} \in \mathcal{E}(CH^l(D_1 + D_2)) \) with \( k_y \geq 0 \). Then \( e^y = N_{D_1+D_2}^-((i, y)) \) and \( N_{D_2}^-(y) = \{y_1, \ldots, y_{k_y}\} = \pi_2(e^y) \setminus \{y\} \).

**Theorem 9** (Normal product \( D_1 \ast D_2 \)).

(a) \( NH(D_1) \) and \( NH(D_2) \) can be obtained from \( NH(D_1 \ast D_2) \).

(b) If there is a hyperedge \( e \in \mathcal{E}(NH(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \), then \( NH^l(D_1) \) and \( NH^l(D_2) \) can be obtained from \( NH(D_1 \ast D_2) \).

**Proof.** (b) The existence of a hyperedge \( e \in \mathcal{E}(NH(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \) is equivalent to \( A_1 \neq \emptyset \neq A_2 \). Let

\[
e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), (i_1, j_1), (i_1, j_2), \ldots, (i_1, j_k), (i_2, j), \ldots, (i_l, j_k)\}
\]

\( \in \mathcal{E}(NH(D_1 \ast D_2)) = \mathcal{E}(CH(D_1 \ast D_2)) \cup \mathcal{E}(CEH(D_1 \ast D_2)) \),

with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \).

We will follow the idea of the proof of Case 2 of Corollary 2 in our paper [20], where a similar result for competition hypergraphs was given.

But by contrast to Corollary 2 in [20], in the case of niche hypergraphs it is impossible to reconstruct the digraphs \( D_1 \) and \( D_2 \) themselves in general. The reason is the same as mentioned before for the Cartesian sum (see the proof of Remark 8). Although for a hyperedge \( e \in \mathcal{E}(NH(D_1 \ast D_2)) \) we can find out the vertex \((i, j)\) with \( e = N_{D_1 \ast D_2}^+(i, j) \) or \( e = N_{D_1 \ast D_2}^+(i, j) \), in general it will be impossible to determine whether \( e \) is the set of predecessors (\( e \) is a "competition hyperedge") or the set of successors (\( e \) is a "common enemy hyperedge") of the vertex \((i, j)\) in \( D_1 \ast D_2 \).

Note that, in spite of the distinction of cases below, it is unnecessary to know for the actual hyperedge \( e \in \mathcal{E}(NH(D_1 \ast D_2)) \) under investigation whether or not it is a "competition hyperedge" (\( e \in \mathcal{E}(CH(D_1 \ast D_2)) \)) or it is an "common enemy hyperedge" (\( e \in \mathcal{E}(CEH(D_1 \ast D_2)) \)). This will become clear by the remarks to Case (2) below.

Case (1): \( e \in \mathcal{E}(CH(D_1 \ast D_2)) \). With some modifications of the proof of Case 2 of Corollary 2 in [20] we get the following.
(a) Because of \( l = |\pi_1(e)| - 1 \geq 1 \) and \( k = |\pi_2(e)| - 1 \geq 1 \), the vertices \( i \in V_1 \) and \( j \in V_2 \) with \( N_{D_1 \ast D_2}^-(\{i, j\}) = e \) can be identified as the only vertices which occur exactly \( k \) and \( l \) times in \( \pi_1(e) \) and \( \pi_2(e) \), respectively. Moreover, \( \pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = N_{D_1}^-(i) \) and \( \pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = N_{D_2}^-(j) \).

(b) Obviously, for every \( x \in V_1 \) with \( N_{D_1}^-(x) \neq \emptyset \) in \( N_{D_1 \ast D_2}^-(\{x, j\}) \) there are at least 3 vertices: \((x, j_1), (x', j), (x', j_1)\), where \( x' \in N_{D_1}^-(x) \). Therefore \( N_{D_1 \ast D_2}^-(\{x, j\}) \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 \ast D_2)) \). Analogously, for each \( y \in V_2 \) with \( N_{D_2}^-(y) \neq \emptyset \) we get \( N_{D_1 \ast D_2}^-(\{i, y\}) \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 \ast D_2)) \).

(c) Note that if \( x \in V_1 \) with \( N_{D_1}^-(x) = \emptyset \), then \( N_{D_1 \ast D_2}^-(\{x, j\}) = \{(x, j_1), \ldots, (x, j_k)\} \); i.e., \( N_{D_1 \ast D_2}^-(\{x, j\}) \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 \ast D_2)) \) if and only if \( k \geq 2 \). Analogously, for every \( y \in V_2 \) with \( N_{D_2}^-(y) = \emptyset \) it follows \( N_{D_1 \ast D_2}^-(\{i, y\}) \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \subseteq \mathcal{E}(\mathcal{NH}(D_1 \ast D_2)) \) if and only if \( l \geq 2 \).

Because of (b), for all vertices of \( D_1 \) and \( D_2 \), respectively, with positive indegree we get their sets of predecessors applying the procedure described in (a) to all hyperedges \( e \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \) with \( |\pi_1(e)| \geq 2 \) and \( |\pi_2(e)| \geq 2 \). (In general, for a vertex \( v_1 \in V_1 \) and \( v_2 \in V_2 \), respectively, with positive indegree, procedure (a) will produce its set of predecessors more than once.) Trivially, each vertex for which (a) does not provide a set of predecessors has indegree 0 (cf. (c)).

Thus we obtain the edge set \( \mathcal{E}(\mathcal{CH}(D_1)) \) and \( \mathcal{E}(\mathcal{CH}(D_2)) \) of the \( l \)-competition hypergraph \( \mathcal{CH}(D_1) \) and \( \mathcal{CH}(D_2) \), respectively.

Note that we did not need hyperedges \( e \in \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \setminus \mathcal{E}(\mathcal{CH}(D_1 \ast D_2)) \), i.e., hyperedges of cardinality 1.

Case (2): \( e \in \mathcal{E}(\mathcal{CEH}(D_1 \ast D_2)) \). Note that \( \mathcal{CH}(D) = \mathcal{CEH}(\overline{D}) \), for any digraph \( D \). Applying the following substitutions to the proof of Case (1), word-for-word we obtain the verification of Case (2):

\[
\begin{align*}
\mathcal{CH} & \quad \mapsto \quad \mathcal{CEH}, \\
N^- & \quad \mapsto \quad N^+, \\
indegree & \quad \mapsto \quad outdegree \quad \text{and} \\
predecessor & \quad \mapsto \quad successor.
\end{align*}
\]

(a) Because of (b) it suffices to consider the case when \( A_1 = \emptyset \) or \( A_2 = \emptyset \) holds. Replacing "+" by "*" in (1)–(3) of Theorem 7, we see that the occurrence of (1), (2) or (3) is equivalent to \( A_1 = \emptyset \) or \( A_2 = \emptyset \) and we can use an analog argumentation as in the corresponding part of the proof of Theorem 7. So using (2) we obtain \( \mathcal{E}(\mathcal{NH}(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(\mathcal{NH}(D_1 \ast D_2))\} \) and \( \mathcal{E}(\mathcal{NH}(D_2)) = \{\pi_2(e) \mid e \in \mathcal{E}(\mathcal{NH}(D_1 \ast D_2)) \land |\pi_2(e)| \geq 2\} \), respectively. ■
Note that $A_1 = \emptyset$ or $A_2 = \emptyset$ implies $D_1 \ast D_2 = D_1 + D_2$. Therefore, the last part of the above proof in connection with Theorem 7 lead to the following consequence.

**Corollary 10.** \(NH^l(D_1)\) and \(NH^l(D_2)\) can be obtained from \(NH^l(D_1 \ast D_2)\), provided that one of the following conditions is true:

1. \(\forall e \in E(NH^l(D_1 \ast D_2)) : |\pi_1(e)| = 1\) and \(\exists e \in E(NH^l(D_1 \ast D_2)) : |\pi_2(e)| \geq 2\);
2. \(\forall e \in E(NH^l(D_1 \ast D_2)) : |\pi_2(e)| = 1\) and \(\exists e \in E(NH^l(D_1 \ast D_2)) : |\pi_1(e)| \geq 2\).

**Theorem 11** (Lexicographic product \(D_1 \cdot D_2\)).

(a) \(NH(D_1)\) and \(NH(D_2)\) can be obtained from \(NH(D_1 \cdot D_2)\).

(b) If \(|V_2| \geq 2\), then \(NH^l(D_1)\) can be obtained from \(NH^l(D_1 \cdot D_2)\).

(c) \(NH^l(D_1)\) and \(NH^l(D_2)\) can be obtained from \(NH^l(D_1 \cdot D_2)\).

**Proof.** First we will show (c), i.e., \(NH^l(D_1)\) and \(NH^l(D_2)\) can be reconstructed from \(NH^l(D_1 \cdot D_2)\). Then we obtain (b) and (a) as follows:

Since for \(|V_2| \geq 2\) every loop \(e_1 = \{i\}\) in \(NH^l(D_1)\) leads to a non-loop \(e\) in \(NH^l(D_1 \cdot D_2)\) (containing at least all vertices of the row \(Z_i\)), we will see that we need no loops of \(NH^l(D_1 \cdot D_2)\) in order to obtain \(NH^l(D_1)\), this includes (b).

Analogously, it is obvious that non-loops \(e_i\) of \(NH^l(D_1)\) and \(NH^l(D_2)\), respectively, result in non-loops in \(NH^l(D_1 \cdot D_2)\). In our considerations it will become clear that for the reconstruction of \(NH^l(D_1)\) and \(NH^l(D_2)\) we do not need the loops in \(NH^l(D_1 \cdot D_2)\), so we get (a).

In order to prove (c), we consider a hyperedge \(e \in E(NH^l(D_1 \cdot D_2))\). Then there is a vertex \((i, j) \in V_1 \times V_2\) such that \(e = N_{D_1 \cdot D_2}((i, j))\) or \(e = N_{D_1 \cdot D_2}^+(i, j))\).

In order to simplify our depictions, we write down the considerations only for the case \(e = N_{D_1 \cdot D_2}^-(i, j)) \in E(C^H(D_1 \cdot D_2))\); the hyperedges \(e = N_{D_1 \cdot D_2}^+(i, j)) \in E(C^H(D_1 \cdot D_2))\) can be treated analogously.

In \(NH^l(D_1 \cdot D_2)\) there are two possibilities for the hyperedge \(e\).

**Case 1.** \(\exists l \geq 1 \exists i_1, \ldots, i_l \in V_1 : e = Z_{i_1} \cup \cdots \cup Z_{i_l}\). Without loss of generality let \(i_1, \ldots, i_l\) be pairwise distinct.

Hence, \(e\) is the union of the complete rows \(Z_{i_1}, \ldots, Z_{i_l}\) of \(D_1 \cdot D_2\) and from the definition of \(D_1 \cdot D_2\) it follows \(i \notin \{i_1, \ldots, i_l\}\), \(N_{D_1}^-(i) = \{i_1, \ldots, i_l\}\) and \(N_{D_2}^-(j) = \emptyset\). Therefore, Case 1 does not provide any hyperedges of \(NH^l(D_2)\) but with \(\pi_1(e) = \{i_1, \ldots, i_l\}\) \(N_{D_1}^-(i) \in E(NH^l(D_1))\) we obtain a hyperedge of \(NH^l(D_1)\).

Note that the vertex \(i \in V_1\) is unknown if \(l < |V_1| - 1\). Moreover, Case 1 occurs if and only if there exists a vertex \(j \in V_2\) with \(N_{D_2}^-(j) = \emptyset\).

**Case 2.** \(\exists l \geq 0 \exists i_1, \ldots, i_l, i' \in V_1 : Z_i' \subset Z_i' \in Z_{i'} : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \cup Z' \land Z' \neq \emptyset\). We get \(i = i' \in V_1 \setminus \{i_1, \ldots, i_l\}\) as well as \(N_{D_1}^-(i') = \{i_1, \ldots, i_l\}\), \(\pi_1(e) \setminus \{i'\} \in E(NH^l(D_1 \cdot D_2))\).
\[E(\mathcal{NH}(D_1)) \text{ and } N^-_{D_2}(j) = \pi_2(e \cap Z') = \pi_2(Z') \in E(\mathcal{NH}(D_2)) \] with a certain \( j \in V_2 \). In general, if \(|Z'| < |V_2| - 1\) holds, the vertex \( j \) cannot be determined.

Again, for any hyperedge \( e \in E(\mathcal{NH}(D_1 \cdot D_2)) \) it cannot be found out whether \( e \) is a competition hyperedge (i.e., \( e \in \mathcal{E}(\mathcal{CH}(D_1 \cdot D_2)) \)) or \( e \) is a common enemy hyperedge (i.e., \( e \in \mathcal{E}(\mathcal{CEH}(D_1 \cdot D_2)) \)) in general. But for the reconstruction of \( \mathcal{NH}(D_1) \) and \( \mathcal{NH}(D_2) \) this plays no role, since the considerations of Case 1 and Case 2 are valid for competition hyperedges (i.e., sets of predecessors) as well as, analogously, for common enemy hyperedges (i.e., sets of successors).

Moreover, we remark that Cases 1 and 2 (together with their analogs for the common enemy hyperedges) provide all hyperedges of the \((l)\)-niche hypergraphs \( \mathcal{NH}(D_1) \) and \( \mathcal{NH}(D_2) \).

Now we discuss the disjunction \( D_1 \lor D_2 \). The case \(|V_1| = 1 \) or \(|V_2| = 1\) implies \( D_1 \lor D_2 = D_1 \cdot D_2 \). Therefore, because of Theorem 11 it suffices to investigate the case \(|V_1|, |V_2| \geq 2\).

**Theorem 12** (Disjunction \( D_1 \lor D_2 \)). If \(|V_1|, |V_2| \geq 2\), then \( \mathcal{NH}(D_1) \) and \( \mathcal{NH}(D_2) \) can be obtained from \( \mathcal{NH}(D_1 \lor D_2) \).

**Proof.** Since both \( V_1 \) and \( V_2 \) contain at least two vertices, in \( \mathcal{NH}(D_1 \lor D_2) \) there are no loops and \( \mathcal{NH}(D_1 \lor D_2) = \mathcal{NH}(D_1 \lor D_2) \).

Moreover, for every hyperedge \( e \in E(\mathcal{NH}(D_1 \lor D_2)) \) it holds
\[\exists l \geq 0 \exists i_1, \ldots, i_l \in V_1 \exists k \geq 0 \exists j_1, \ldots, j_k \in V_2 : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \cup S_{j_1} \cup \cdots \cup S_{j_k}\]
and, clearly, \( \min(l, k) > 0 \).

By analogy with the proof of Theorem 11 let \((i, j) \in V_1 \times V_2\) be a vertex such that \( e = N^-_{D_1 \lor D_2}((i, j)) \) or \( e = N^+_{D_1 \lor D_2}((i, j)) \). Now we follow the idea of the proof of Proposition 2 in [20], subsection 3.5, and use the abbreviations \( \mathcal{E}_1 := E(\mathcal{NH}(D_1)) \), \( \mathcal{E}_2 := E(\mathcal{NH}(D_2)) \) and \( \mathcal{E}_V := E(\mathcal{NH}(D_1 \lor D_2)) \).

In case of \( \mathcal{E}_V = \emptyset \) both \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are empty, too.

So let \( \mathcal{E}_V \neq \emptyset \). Additionally, for an arbitrary hyperedge \( e \in \mathcal{E}_V \) we define \( \pi_1(e) := \{ i \mid (i, j) \in e \} \) (for \( j \in \pi_2(e) \)) and \( \pi_2(e) := \{ j \mid (i, j) \in e \} \) (for \( i \in \pi_1(e) \)).

In \( \mathcal{NH}(D_1 \lor D_2) \) we have three types of hyperedges:
\[A := \{ e \in \mathcal{E}_V \mid \pi_1(e) \subset V_1 \},\]
\[B := \{ e \in \mathcal{E}_V \mid \pi_2(e) \subset V_2 \} \text{ and}\]
\[C := \{ e \in \mathcal{E}_V \mid \pi_1(e) = V_1 \land \pi_2(e) = V_2 \} \text{.}\]

We obtain
\[A = C = \emptyset \text{ if and only if } A_1 = 0, \mathcal{E}_1^l = \emptyset \text{ and } \mathcal{E}_2^l = \{ \pi_2(e) \mid e \in \mathcal{E}_V \};\]
\[B = C = \emptyset \text{ if and only if } A_2 = 0, \mathcal{E}_2^l = \emptyset \text{ and } \mathcal{E}_1^l = \{ \pi_1(e) \mid e \in \mathcal{E}_V \};\]
\[C \neq \emptyset \text{ if and only if } A_1 \neq 0 \neq A_2.\]

It remains to investigate the case \( C \neq \emptyset \). Here we see that, to determine \( \mathcal{E}_1^l \) and \( \mathcal{E}_2^l \), it suffices to make use of the hyperedges in \( C \):
\[ E_1^i = \left\{ i \in V_1 \mid \pi_2^i(e) = V_2 \right\} \mid e \in C \} \quad \text{and} \quad E_2^j = \left\{ j \in V_2 \mid \pi_1^j(e) = V_1 \right\} \mid e \in C \} .

(Note that in case \( A \neq \emptyset \) we have \( E_1^i = \{ \pi_1^i(e) \mid e \in A \} \) and, analogously, if \( B \neq \emptyset \) it follows \( E_2^j = \{ \pi_2^j(e) \mid e \in B \} \).)

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