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CRITICAL AND FLOW-CRITICAL SNARKS COINCIDE

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Abstract

Over the past twenty years, critical and bicritical snarks have been appearing in the literature in various forms and in different contexts. Two main variants of criticality of snarks have been studied: criticality with respect to the non-existence of a 3-edge-colouring and criticality with respect to the non-existence of a nowhere-zero 4-flow. In this paper we show that these two kinds of criticality coincide, thereby completing previous partial results of de Freitas *et al.* [Electron. Notes Discrete Math. 50 (2015) 199–204] and Fiol *et al.* [Electron. J. Combin. 25 (2017) #P4.54].

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1. Introduction

A snark is a connected cubic graph whose edges cannot be properly coloured with three colours; equivalently, it is a connected cubic graph that admits no nowhere-zero 4-flow. This definition follows Cameron *et al.* [4], Nedela and Škoviera [14], Šámal [15], Steffen [16], and others, rather than the traditional more restrictive definition that excludes small cycle-separating edge-cuts and short circuits in order to avoid trivial cases. As suggested by several authors, the idea of non-triviality of snarks is rather subtle and seems to be best captured by various reductions and decompositions of snarks [4, 5, 14, 16]. The concept of a snark reduction is, in turn, closely related to that of criticality of a snark, which naturally takes one of two forms: criticality with respect to the non-existence of a 3-edge-colouring [3, 5, 14] and criticality with respect to the non-existence of a nowhere-zero 4-flow [6, 8, 9, 10]. The purpose of the present paper is to show that these two types of criticality coincide. Although the discussed relationship is not complicated, it has been generally overlooked and, so far, the two types of criticality have been considered separately, se for example Šámal [15, Section 3.1]. Even the most recent survey paper of Fiol *et al.* brings only a partial result in this direction [12, Theorem 4.5].

The concept of a critical snark first appeared in 1996 in the work of Nedela and Škoviera [14] within the context of snark reductions. According to their definition, a snark is critical if the removal of any two adjacent vertices produces a 3-edge-colourable graph, and is bicritical if the removal of any two distinct vertices produces a 3-edge-colourable graph.

Explicit occurrence of flow-critical snarks in the literature is of much later date. It first appears in a 2008 paper of da Silva and Lucchesi [8] investigating graphs critical with respect to the existence of a nowhere-zero k-flow for an arbitrary integer $k \geq 2$. They defined a graph to be k-edge-critical if it does not admit a nowhere-zero k-flow but the graph obtained by the contraction of any edge does. They further defined a graph to be k-vertex-critical if it does not admit a nowhere-zero k-flow but the graph obtained by the identification of any two distinct vertices does. If we take into account the fact that contracting an edge has the same effect on the existence of a nowhere-zero k-flow as identifying its end-vertices, the later two definitions of da Silva and Lucchesi [8], with $k = 4$, can be viewed as natural counterparts of critical and bicritical snarks of Nedela and Škoviera. Nevertheless, our main result shows that for snarks flow-criticality does not bring anything substantially new.

Theorem 1. *A snark is* 4*-edge-critical if and only if it is critical. A snark is* 4*-vertex-critical if and only if it is bicritical.*

There have been a number of papers following either of the two approaches to the criticality; see for example [5, 13, 17, 18] and [6, 9, 10], respectively. In several other works, critical snarks have emerged in forms different from those explained above, yet in all the cases the definitions turn out to be equivalent to one of those given above. For example, DeVos *et al.* [7] and more recently Šámal [15] define a snark to be critical if the subgraph obtained by the removal of an arbitrary edge admits a cycle-continuous mapping onto the graph consisting of two vertices joined by three parallel edges. The latter condition easily translates to the one requiring the existence of a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on each edge-deleted subgraph, which in turn implies that critical snarks in the sense of DeVos *et al.* and Šámal [7, 15] coincide with 4-edge-critical snarks of da Silva and Lucchesi [8]. The same family of snarks, under the name of 4-flow-critical snarks, occurs in a recent survey of edge-uncolourability measures by Fiol *et al.* [12, Section 4.1] without any reference to previous work.

To sum up, during the past twenty years critical snarks were rediscovered several times in one form or another, and within different contexts. Although partial results concerning the relationship between the different definitions exist, see Freitas *et al.* [10, Theorem 3.1] and Fiol *et al.* [12, Theorem 4.5], it has not been fully realised that all of them actually coincide with either critical or bicritical snarks of Nedela and Škoviera [14]. In the present paper we therefore establish this fact in full generality and explain the relations between various versions of criticality in detail. Instead of just proving that critical snarks coincide with 4-edge-critical snarks and bicritical snarks coincide with 4-vertex-critical ones we investigate the corresponding reduction operations locally on the pairs of vertices and show that different operations have the same effect.

2. Definitions and Preliminaries

In this section we fix the terminology for the rest of this paper. We assume the reader to have the basic knowledge of edge-colourings and nowhere-zero flows on graphs. For more details we recommend the reader to consult Diestel [11].

Our graphs may have parallel edges and loops. Occasionally we also allow dangling edges. All concepts needed for graphs with dangling edges are straightforward modifications of those known for ordinary graphs. In particular, the definitions of a 3-edge-colouring and a nowhere-zero flow extend to graphs with dangling edges without any difficulty. For more details see, for example, [5, Section 2].

As mentioned above, we define a *snark* to be a connected cubic graph that does not admit a proper edge-colouring with three colours; equivalently, a snark is a connected cubic graph that does not admit a nowhere-zero 4-flow. The smallest snark is the *dumbbell graph*, which has two vertices joined by an edge and a loop attached to each vertex. The smallest bridgeless snark is the Petersen graph.

We now introduce the operations related to critical and flow-critical snarks. Given a graph G and an edge e of G, we let $G-e$ denote the subgraph of G obtained by the removal of e, and $G\sim e$ the cubic graph which arises from $G-e$ by suppressing the resulting two vertices of degree two. By G/e we denote the graph obtained from G by the contraction of e.

Let u and v be two distinct vertices of G. By $G-\{u,v\}$ we denote a graph created from G by removing u and v but retaining the dangling edges. Note that this deviation from the standard meaning of the vertex removal has no effect on the existence of a 3-edge-colouring but is important for the existence of nowhere-zero flows. By $G/\{u,v\}$ we denote the graph obtained from G by identifying u and v. If u and v are connected by an edge e, then G/e arises from $G/\{u,v\}$ by removing the loop resulting from e.

We proceed to the central concepts of this paper. Following Nedela and Škoviera [14] we define a snark G to be *critical* if $G-\{u,v\}$ is 3-edge-colourable for every pair of adjacent vertices u and v, and to be *bicritical* if $G-\{u,v\}$ is

3-edge-colourable for every pair of distinct vertices u and v. Thus every bicritical snark is critical, but not necessarily vice versa.

Let G be an arbitrary graph, not necessarily cubic, and let $k \geq 2$ be an integer. Following da Silva and Lucchesi [8] (see also Carneiro *et al.* [6]) we say that G is *k-edge-critical* if G does not admit a nowhere-zero k-flow but for each edge e the graph G/e does. We further say that G is *k-vertex-critical* if it does not admit a nowhere-zero k-flow but for any two distinct vertices u and v the graph $G/\{u, v\}$ does. We now apply these definitions to snarks with $k = 4$. Taking into account the fact that the presence of a loop at a vertex has no effect on the existence of a nowhere-zero k-flow, we can define a snark to be *4-edge-critical* if $G/\{u, v\}$ has a nowhere-zero 4-flow for any two adjacent vertices, and *4-vertex-critical* if $G/\{u, v\}$ has a nowhere-zero 4-flow for any two distinct vertices. Although these snarks can be encountered in the literature under different names, we have decided to adopt the terminology used in [6, 8] and also in the snark section of the database "House of Graphs" [1].

Before proceeding further we need to mention an important connection of critical snarks to irreducible snarks, which were introduced in [14] and thoroughly studied in [5]. A snark G is said to be *k-irreducible* for a given integer $k \geq 1$ if removing fewer than k edges from G does not produce a component with chromatic index 4 which could be completed to a cubic graph H of order smaller than G. The resulting graph H is a snark and is called a *k-reduction* of G. A snark is called *irreducible* if it is k-irreducible for every $k \geq 1$. The next result was proved in [14].

Theorem 2. *The following statements are true for an arbitrary snark G.*

(i) *If $1 \leq k \leq 4$, then G is k-irreducible if and only if G is either cyclically k-connected or the dumbbell graph.*
(ii) *If $k \in \{5, 6\}$, then G is k-irreducible if and only if it is critical.*
(iii) *If $k \geq 7$, then G is k-irreducible if and only if it is bicritical.*

As a direct consequence of Theorem 2 we obtain the fact that a snark is irreducible if and only if it is bicritical. Furthermore, in a bicritical snark the removal of every nontrivial edge-cut (one that is different from three edges incident with a vertex) produces only colourable components. The following corollary proved in [14] is also interesting.

Corollary 3. *Every critical snark, other than the dumbbell graph, is cyclically 4-edge-connected and has girth at least 5.*

3. Critical and Flow-Critical Snarks

We start by exploring the effect of various operations occurring in the definitions of critical and flow-critical snarks.

Theorem 4. *Let G be a snark and let u and v be two distinct vertices of G. The following statements* (i)–(iii) *are equivalent. If, in addition, u and v are adjacent and joined by an edge e, then all the statements* (i)–(vi) *are equivalent.*

(i) $G - \{u, v\}$ *is* 3*-edge-colourable.*

(ii) $G - \{u, v\}$ *admits a nowhere-zero* 4*-flow.*

(iii) $G/\{u, v\}$ *admits a nowhere-zero* 4*-flow.*

(iv) $G - e$ *admits a nowhere-zero* 4*-flow.*

(v) G/e *admits a nowhere-zero* 4*-flow.*

(vi) $G \sim e$ *is* 3*-edge-colourable.*

Proof. (i)⇒(ii): Assume that the graph $G - \{u, v\}$ admits a 3-edge-colouring. If the colours for this colouring are taken to be the non-zero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the colouring is at the same time a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on $G - \{u, v\}$. By Kirchhoff's law, the sum of colours on the dangling edges of $G - \{u, v\}$ must be 0. It follows that if we attach the dangling edges to a new vertex w, the resulting graph G^+ will have no dangling edges and will carry a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Tutte's equivalence theorem (see [11, Theorem 6.3.3 and Corollary 6.3.2]) now implies that G^+ admits a nowhere-zero 4-flow. After removing w from G^+ and retaining the dangling edges we obtain a nowhere-zero 4-flow on $G - \{u, v\}$.

(ii)⇒(iii): Assume that $G - \{u, v\}$ admits a nowhere-zero 4-flow. Without loss of generality we may assume that the underlying orientation has all the dangling edges of $G - \{u, v\}$ directed outward. Kirchhoff's law now implies that the sum of values on the dangling edges is 0. Since $G/\{u, v\}$ arises from $G - \{u, v\}$ by attaching the dangling edges to a new vertex, we infer that the induced valuation is a nowhere-zero 4-flow on $G/\{u, v\}$.

(iii)⇒(i): Assume that $G/\{u, v\}$ admits a nowhere-zero 4-flow. By Tutte's equivalence theorem it also admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. If we regard the flow values in $\mathbb{Z}_2 \times \mathbb{Z}_2$ as colours and remove the vertex $u = v$ from $G/\{u, v\}$, we immediately obtain a 3-edge-colouring of $G - \{u, v\}$.

For the rest of the proof we assume that the vertices u and v are joined by an edge e.

(i)⇒(vi): Assume that $G - \{u, v\}$ admits a 3-edge colouring. Again, we regard the colours as non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which makes the colouring a nowhere-zero flow. By Kirchhoff's law, the sum $x + y$ of the colours on the dangling edges formerly incident with u must be the same as the sum $x' + y'$ of the colours formerly incident with v. We claim that $x + y = 0$. If we had $x + y \neq 0$, then we could assign $x + y$ to e and thus extend the colouring of $G - \{u, v\}$ to a proper 3-edge-colouring of the entire G. Since G is a snark, this is impossible. Therefore $x + y = 0$, and hence both $x = y$ and $x' = y'$, which means that the

3-edge-colouring of $G - \{u, v\}$ in question induces a 3-edge-colouring of $G \sim e$.

(vi)⇒(iv): Assume that $G \sim e$ is 3-edge-colourable. Then $G \sim e$ has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow and hence a nowhere-zero 4-flow. Since $G - e$ is a subdivision of $G \sim e$, it follows that $G - e$ has a nowhere-zero 4-flow as well.

(iv)⇒(v): If $G-e$ has a nowhere-zero 4-flow, then so does the graph obtained from $G - e$ by identifying u and v, which is exactly G/e.

(v)⇒(i): If G/e has a nowhere-zero 4-flow, then by Tutte's equivalence theorem it also has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Obviously, the same is true for the graph obtained from G/e by removing the vertex corresponding to e and by retaining the dangling edges. The latter graph is isomorphic to $G - \{u, v\}$ and carries a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, which at the same time is a 3-edge-colouring of $G - \{u, v\}$. ■

Remarks.

1. Statements (i) and (ii) of Theorem 4 are equivalent for every cubic graph, not just for snarks. Statement (iv) implies statements (iii) and (v) for every graph, be it cubic or not, but the reverse implications (iii)⇒(iv) and (v)⇒(iv) are true only when G is a cubic graph which is not 3-edge-colourable. The only place in the proof of Theorem 4 where the assumption that G is a snark is used is the implication (i)⇒(vi).

2. The equivalence of statements (i) and (vi) (under the assumption of uncolourability) was first observed in [14, Proposition 4.2].

The next two theorems are immediate consequences of Theorems 4 and 2.

Theorem 5. *The following statements are equivalent for an arbitrary snark G.*

(i) *G is critical.*
(ii) *G is 4-edge-critical.*
(iii) *G is 5-irreducible.*
(iv) *G is 6-irreducible.*
(v) *$G \sim e$ is 3-edge-colourable for each edge e of G.*

The equivalence (i)⇔(ii) in Theorem 5 explains that the property of being a 4-edge-critical snark is the same as being critical. This fact has been recently observed by de Freitas *et al.* [10, Theorem 3.1] and independently by Fiol *et al.* [12, Theorem 4.5]. Surprisingly, an analogous statement for 4-vertex-critical and bicritical snarks has so far escaped attention. We formulate this fact in the next theorem, thereby completing the relationship between critical and flow-critical snarks.

Theorem 6. *The following statements are equivalent for an arbitrary snark G.*

(i) *G is bicritical.*

(ii) *G is 4-vertex-critical.*
(iii) *G is 7-irreducible.*
(iv) *G is irreducible.*

It is worth mentioning that there exist *strictly critical* snarks — snarks that are critical but not bicritical. The first known examples were found independently by Chladný in his Master Thesis (see [5]) and by Steffen [17]. Somewhat later, Grünewald and Steffen [13] presented a construction of cyclically 5-edge-connected strictly critical snarks. Strictly critical snarks whose cyclic connectivity equals 4 were completely characterised by Chladný and Škoviera [5, Section 6], providing a deeper insight into what makes snarks strictly critical. They also showed that there exist strictly critical snarks of order n for every even integer $n \geq 32$. On the other hand, an exhaustive computer search performed by Brinkmann and Steffen [3] revealed that there are no strictly critical snarks of any order smaller than 32.

Strictly critical snarks have resurfaced within the flow-critical context in a recent work of Carneiro *et al.* [6]. They devised an exponential-time algorithm that verifies whether a snark is 4-edge-critical or 4-vertex-critical, and applied the algorithm to the body of all cyclically 4-edge-connected snarks of order at most 36 with girth at least 5 generated by Brinkmann *et al.* [2]. The use of this algorithm allowed them to compile complete lists of critical, bicritical, and strictly critical snarks of every order not exceeding 36. The lists are available in the snark section of the database "House of Graphs" [1]. It transpires that among all snarks of order at most 36 there are exactly 55172 critical snarks, but only 846 of them are strictly critical, just slightly over 1.5 percent. We have verified that all of them have cyclic connectivity 4. (The number 837 of strictly critical snarks of order not exceeding 36 mentioned in [6, Section 3] is incorrect.)

Theorems 5 and 6 suggest that the algorithm of Carneiro *et al.* [6] to check flow-criticality of a given snark G can be simplified if we consider criticality instead. Indeed, the algorithm for flow-criticality fixes an orientation of G and for a chosen pair (u, v) of vertices it attempts to construct a weight function with values in $\mathbb{Z}_4$ under which the Kirchhoff law fails only at u and v. This requires, in particular, checking several possibilities for balanced weight assignments at each vertex w different from u and v. Our Theorems 5 and 6 imply that if we verify criticality instead, then no orientation is required, and for each vertex there is only one possibility for a balanced assignment, up to a permutation of colours. This approach might prove useful in testing irreducibility of large individual snarks or large sets of snarks.

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