NEIGHBOR SUM DISTINGUISHING TOTAL CHOOSABILITY OF IC-PLANAR GRAPHS

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Abstract

Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane such that every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short. A proper total-$k$-coloring of a graph $G$ is a mapping $c : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ denote the sum of the color of a vertex $v$ and the colors of all incident edges of $v$. A total-$k$-neighbor sum distinguishing-coloring of $G$ is a total-$k$-coloring of $G$ such that for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$. The least number $k$ needed for such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_c(G)$. In this paper, it is proved that if $G$ is an IC-planar graph with maximum degree $\Delta(G)$, then $\text{ch}''_c(G) \leq \max\{\Delta(G) + 3, 17\}$, where $\text{ch}''_c(G)$ is the neighbor sum distinguishing total choosability of $G$.

Keywords: neighbor sum distinguishing total choosability, maximum degree, IC-planar graph, Combinatorial Nullstellensatz.

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1. Introduction

All graphs considered are finite, simple and undirected. Let \( G \) be a graph. We use \( V(G) \), \( E(G) \), \( \Delta(G) \) and \( \delta(G) \) to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For planar graph \( G \), \( F(G) \) denotes its face set, \( d(v) \) denotes the degree of a vertex \( v \) in \( G \). The length or degree of a face \( f \), denoted by \( d(f) \), is the length of the boundary walk of \( f \) in \( G \). We call \( v \) a \( k \)-vertex, or a \( k^+ \)-vertex, or a \( k^- \)-vertex if \( d(v) = k \), or \( d(v) \geq k \), or \( d(v) \leq k \) respectively and call \( f \) a \( k \)-face, or a \( k^+ \)-face, or a \( k^- \)-face if \( d(f) = k \), or \( d(f) \geq k \), or \( d(f) \leq k \), respectively. Any undefined notation follows that of Bondy and Murty [3].

A proper total-\( k \)-coloring of a graph \( G \) is a mapping \( c : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) such that any two adjacent elements in \( V(G) \cup E(G) \) receive different colors. Let \( \Sigma_v(c) \) be the sum of the color of a vertex \( v \) and the colors of all edges incident with \( v \). If for each edge \( uv \in E(G) \), \( \Sigma_v(c) \neq \Sigma_u(c) \), then we say such total-\( k \)-coloring a neighbor sum distinguishing total-\( k \)-coloring, denoted by \( \text{tnsd-} \) \( k \)-coloring for short. The least number \( k \) needed for such a coloring of \( G \) is the neighbor sum distinguishing total chromatic number, denoted by \( \chi''_{\Sigma}(G) \). For neighbor sum distinguishing total colorings, we have the following conjecture proposed by Pilśniak and Woźniak [11].

**Conjecture 1.** For any graph \( G \), \( \chi''_{\Sigma}(G) \leq \Delta(G) + 3 \).

Loeb and Tang [10] proved that this bound was asymptotically correct by showing that \( \chi''_{\Sigma}(G) \leq \Delta(G)(1 + o(1)) \). Pilśniak and Woźniak [11] proved that Conjecture 1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. With the Combinatorial Nullstellensatz, neighbor sum distinguishing total coloring have been studied widely, see [4–6,8,9,12,19]

For a given graph \( G \), let \( L_x(x \in V \cup E) \) be a set of lists of real numbers and each of size \( k \). The neighbor sum distinguishing total choosability of \( G \) is the least number \( k \) for which for any specified collection of such lists, there exists a neighbor sum distinguish total coloring with colors from \( L_x \) for each \( x \in V \cup E \), and we denote it by \( ch''_{\Sigma}(G) \). We call such a coloring of \( G \) list neighbor sum distinguish total-k-coloring and denote it by \( \text{ltnsd-k-coloring} \). Ding et al. [4] proved that for any graph \( G \), \( ch''_{\Sigma}(G) \leq 2\Delta(G) + \text{col}(G) - 1 \), where \( \text{col}(G) \) is the coloring number of \( G \). Later Ding et al. [5] improved the bound to \( ch''_{\Sigma}(G) \leq 2\Delta(G) + \text{col}(G) - 2 \). Recently, Lu et al. [20] improved the bound to \( ch''_{\Sigma}(G) \leq \max\{\Delta(G) + \lfloor \frac{3\text{col}(G)}{2} \rfloor - 1, 3\text{col}(G) - 2 \} \). The list neighbor sum distinguish total-k-coloring of some special classes of graphs were also investigated. Graphs with bounded maximum average degree (Yao and Kong [16]); \( d \)-degenerate graphs (Yao et al. [18]); planar graphs (Qu et al. [13], Wang et al. [15]).

In this paper, we consider IC-planar graphs and prove the following result.
Theorem 2. Let $G$ be an IC-planar graph with maximum degree $\Delta(G)$. Then $\text{ch}''(G) \leq \max\{\Delta(G) + 3, 17\}$.

An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane in which every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Albertson [1] in 2008. Setting a conjecture of Albertson [1], Kráľ and Stacho [7] showed that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graph. We call $G$ a 1-planar graph if it can be drawn on a plane such that each edge is crossed by at most one other edge.

2. Preliminaries

Every IC-planar graph $G$ in this paper has been embedded on a plane such that all its crossings are independent and the number of crossings is as small as possible. In other words, we call $G$ an IC-plane graph. The associated plane graph $G^\times$ of $G$ is obtained by turning all crossings of $G$ into new 4-vertices on a plane. For convenience, a vertex in $G^\times$ is called false if it is not a vertex of $G$ and real otherwise. For a vertex $v \in V(G)$, we use $d_i(v)$ to denote the number of $i$-vertices which are adjacent to $v$. One can see that every real vertex in $G^\times$ is adjacent to at most one false vertex and incident with at most two false faces in $G^\times$.

Lemma 3 [17]. Let $G$ be a 1-plane graph and $G^\times$ be its associated plane graph. If $d_G(u) = 3$ and $v$ is a crossing vertex in $G^\times$, then either $uv \notin E(G^\times)$ or $uv$ is not incident with two 3-faces.

We define that a graph $G'$ is smaller than a graph $G$ if $|E(G')| < |E(G)|$. We call a graph minimal for a property when no smaller graph satisfies it. Let from now on $G = (V, E)$ be a minimal counterexample to Theorem 2. We set $k = \max\{\Delta(G) + 3, 17\}$. For each 5-vertex $v \in V(G)$, it is obvious that $v$ has at most five neighbors and five incident edges, so $v$ has at most 15 forbidden colors. Since $k \geq 17$, we can first erase the color of vertex $v$ and finally recolor it after arguing. In other words, we may omit the coloring for all 5-vertices of $G$ in the following discussion.

Theorem 4 (Combinatorial Nullstellensatz [2]). Let $F$ be an arbitrary field, and let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in $F[x_1, x_2, \ldots, x_n]$. Suppose the degree $\text{deg}(P)$ of $P$ equals $\sum_{i=1}^{n} k_i$, where each $k_i$ is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ in $P$ is non-zero. Then if $S_1, S_2, \ldots, S_n$
are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.

**Lemma 5** [14]. If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\leq s_1 + s_2 + \cdots + s_n$, where $s_1, s_2, \ldots, s_n$ are nonnegative integers, then

$$
\left(\frac{\partial}{\partial x_1}\right)^{s_1} \left(\frac{\partial}{\partial x_2}\right)^{s_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{s_n} P(x_1, x_2, \ldots, x_n)
$$

$$
= \sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \left(\frac{s_1}{x_1}\right) \cdots (-1)^{s_n+x_n} \left(\frac{s_n}{x_n}\right) P(x_1, x_2, \ldots, x_n).
$$

**Lemma 6** [13]. Let $L_i$ be the sets of real numbers, with $|L_i| = l_i$, where $i = 1, 2, \ldots, p$. Let $L = \left\{ \left(\sum_{i=1}^{p} x_i \mid x_i \in L_i \text{ and } \prod_{1 \leq i < j \leq p} (x_i - x_j) \neq 0 \right) \right\}$. Then $|L| \geq \sum_{i=1}^{p} (l_i + p - 1) = \sum_{i=1}^{p} l_i - p^2 + 1$.

### 3. Proof of Theorem 2

#### 3.1. Unavoidable configurations

In the following, we will often delete some edges to get a proper subgraph $G'$ of $G$, then by the minimality of $G$, there exists an ltnsd-\(k\)-coloring $c$ of $G'$. Let $W_G(v) = \sum_{e \ni v, e \in E(G)} c(e) + c(v)$. We may extend this coloring $c$ to the whole graph $G$. For any $x \in V(G) \cup E(G)$, the available colors are the remaining colors after excluding the colors of its adjacent edges and vertices in $G'$ from $L_x$.

**Claim 7.** For any vertex $v \in V(G)$, it holds that

$$
\sum_{j=1}^{t} \left[ d_j(v)(\Delta(G) + 4 - d(v) - j) \right] \leq d(v) - 1, \quad (1 \leq t \leq 5).
$$

**Claim 8.** For any vertex $v \in V(G)$, $d_{2-}(v) \leq \frac{d_{6+}(v)-1}{\Delta(G)-d(v)+1}$. Moreover, if $d(v) = \Delta(G)$, then $d_{2-}(v) \leq d_{6+}(v) - 1$.

The proof of Claim 7 and 8 are similar to that of Claim 3.1 and Claim 3.2 in [13], we omit it here. By Claim 7, we can easily get the following Corollaries.

**Corollary 9.** If $d(v) = 8$, then $d_{5-}(v) \leq 1$.

**Corollary 10.** If $d(v) = 9$, then $d_{5-}(v) \leq 2$.

**Corollary 11.** If $d(v) = 10$, then $d_{5-}(v) \leq 3$.

**Claim 12.** If $d(v) = 11$ and $d_{6+}(v) \leq 6$, then $d_{3-}(v) \leq 1$. 

**Proof.** Suppose to the contrary that $v$ is adjacent to two $3^-$-vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{11}\}$, $d(v_1) = d(v_2) = 3$ and $d(v_j) \geq 6$, $(6 \leq j \leq 11)$. Consider $G' = G - vv_1 - vv_2$, then $G'$ admits an \textit{ltsd-$k$}-coloring $c$. Now we will color the edges $vv_1, vv_2$ and recolor vertices $v_1, v_2$. Let $S_1, S_2$ be the sets of available colors for $vv_1, vv_2$, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5 > 4$. We can associate that $u, uv, v$ with the variables $x_1, x_2, x_3$, respectively. Then we

**Claim 13.** If $d(v) = 12$ and $d_{6^+}(v) \leq 6$, then $d_{3^-}(v) \leq 2$.

**Proof.** Suppose to the contrary that $v$ is adjacent to three $3^-$-vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{12}\}$, $d(v_1) = d(v_2) = d(v_3) = 3$ and $d(v_j) \geq 6$, $(7 \leq j \leq 12)$. Consider $G' = G - \{v_1, |i = 1, 2, 3}\}$, then $G'$ admits an \textit{ltsd-$k$}-coloring $c$. Now we will color the edges $vv_1, vv_2, vv_3$ and recolor vertices $v_1, v_2, v_3$. Let $S_1, S_2, S_3$ be the sets of available colors for $vv_1, vv_2, vv_3$, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, $(1 \leq i \leq 3)$. By Lemma 6, $|L| \geq |S_1| + |S_2| + |S_3| - 9 + 1 = 7 > 6$. We can choose a triple, say $(x, y, z) \in S_1 \times S_2 \times S_3$ with $x, y, z$ distinct colors, such that $x + y + z \notin \{W_G(v_j) - W_G(v) | 7 \leq j \leq 12\}$. Finally, we can recolor $v_1, v_2, v_3$ to get an \textit{ltsd-$k$}-coloring of $G$, a contradiction.

By Lemma 5, if $P(x_1, x_2, \ldots, x_n)$ is a polynomial with $\text{deg}(P) = n$, $k_1, k_2, \ldots, k_m$ are non-negative integers with $\sum_{i=1}^{m} k_i = n$ and $c \left(x_1^{k_{1}}x_2^{k_{2}} \cdots x_m^{k_{m}}\right)$ is the coefficient of $x_1^{k_{1}}x_2^{k_{2}} \cdots x_m^{k_{m}}$ in $P$, then $\frac{\partial^n P}{\partial x_1^{a_1} \cdots \partial x_m^{a_m}} = c \left(x_1^{k_{1}}x_2^{k_{2}} \cdots x_m^{k_{m}}\right) \prod_{i=1}^{m} k_i!$. In the following, we use MATLAB to calculate the coefficients of specific monomials. Moreover, we will list the codes in Appendix.

**Claim 14.** Every $5^-$-vertex is not adjacent to $7^-$-vertex in $G$.

**Proof.** Suppose to the contrary that there exists a $5^-$-vertex $u$ adjacent to a $7^-$-vertex $v$. Without loss of generality, we assume that $d(u) = 5$, $d(v) = 7$, $N(u) = \{v, u_1, \ldots, u_4\}$, $N(v) = \{u, v_1, \ldots, v_6\}$. Consider $G' = G - uv$, then $G'$ admits an \textit{ltsd-$k$}-coloring $c$. Now we will recolor the vertices $u, v$ and color the edge $uv$. Let $S_1, S_2, S_3$ be the sets of available colors for $u, uv, v$, respectively. Notice that the colors in $\{c(uu_i) | 1 \leq i \leq 4\} \cup \{c(v_i) | 1 \leq i \leq 4\}$ are forbidden for $u$, the colors in $\{c(uu_i) | 1 \leq i \leq 4\} \cup \{c(v_i) | 1 \leq i \leq 6\}$ are forbidden for $uv$, and the colors in $\{c(v_i) | 1 \leq i \leq 6\}$ are forbidden for $v$. Thus, $|S_1| = 17 - 8 = 9 > 8$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 12 = 5 > 4$. We associate that $u, uv, v$ with the variables $x_1, x_2, x_3$, respectively. Then we
consider the following polynomial.

\[ P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + \sum_{i=1}^{4} c(uu_i) - x_3 - \sum_{k=1}^{6} c(vv_k)) \]
\[
\prod_{i=1}^{4} \left( x_1 + x_2 + \sum_{l=1}^{4} c(uu_l) - W(u_i) \right) \]
\[
\prod_{j=1}^{6} \left( x_2 + x_3 + \sum_{k=1}^{6} c(vv_k) - W(v_j) \right). \]

We have \( cp \left( x_1^6 x_2^4 x_3^3 \right) = -25 \). According to Theorem 4, there exists \( x_i \in S_i \), \((1 \leq i \leq 3)\) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-\(k\)-coloring of the graph \( G \), a contradiction.

**Claim 15.** Every 6\(^{-}\)-vertex is not adjacent to 6\(^{-}\)-vertex in \( G \).

**Proof.** Suppose to the contrary that there exists a 6\(^{-}\)-vertex \( u \) adjacent to a 6\(^{-}\)-vertex \( v \). Without loss of generality, we assume that \( d(u) = 6 \), \( d(v) = 6 \), \( N(u) = \{v, u_1, \ldots, u_6\} \), \( N(v) = \{u, v_1, \ldots, v_6\} \). Consider \( G' = G - uv \), then \( G' \) admits an ltnsd-\(k\)-coloring \( c \). Now we will recolor the vertices \( u, v \) and color the edge \( uv \). Let \( S_1, S_2, S_3 \) be the sets of available colors for \( u, uv, v \), respectively. Notice that the colors in \( \{c(uu_i) | 1 \leq i \leq 5\} \cup \{c(u_i) | 1 \leq i \leq 5\} \) are forbidden for \( u \), the colors in \( \{c(uu_i) | 1 \leq i \leq 5\} \cup \{c(vv_i) | 1 \leq i \leq 5\} \) are forbidden for \( uv \), and the colors in \( \{c(vv_i) | 1 \leq i \leq 5\} \cup \{c(v_i) | 1 \leq i \leq 5\} \) are forbidden for \( v \). Thus, \( |S_1| = 17 - 10 = 7 > 6 \), \( |S_2| = 17 - 10 = 7 > 6 \), \( |S_3| = 17 - 10 = 7 > 6 \).

We associate that \( u, uv, v \) with the variables \( x_1, x_2, x_3 \), respectively, then we consider the following polynomial.

\[ P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left( x_1 + \sum_{i=1}^{5} c(uu_i) - x_3 - \sum_{k=1}^{5} c(vv_k) \right) \]
\[
\prod_{i=1}^{5} \left( x_1 + x_2 + \sum_{l=1}^{5} c(uu_l) - W(u_i) \right) \]
\[
\prod_{j=1}^{5} \left( x_2 + x_3 + \sum_{k=1}^{5} c(vv_k) - W(v_j) \right). \]

We have \( cp \left( x_1^6 x_2^4 x_3^3 \right) = -20 \). According to Theorem 4, there exists \( x_i \in S_i \), \((1 \leq i \leq 3)\) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-\(k\)-coloring of the graph \( G \), a contradiction.

**Claim 16.** Let \( d(v) = 13 \) and \( d_{6^+}(v) \leq 6 \), then \( d_{3^-}(v) \leq 5 \). Moreover, if \( d_{2^-}(v) \geq 1 \), then \( d_{3^-}(v) \leq 4 \).
Proof. Suppose to the contrary that there exists a 13-vertex v adjacent to six 3^-vertices. Without loss of generality, assume that N(v) = \{v_1, v_2, \ldots, v_{13}\}, d(v_i) = 3, (1 \leq i \leq 6) and d(v_j) \geq 6, (8 \leq j \leq 13). Consider G' = G - \{vv_i \mid i = 1, 2, \ldots, 6\}, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, (1 \leq i \leq 6). Let S_i, (1 \leq i \leq 6) be the sets of available colors for v_i (1 \leq i \leq 6), respectively. Then we consider the following polynomial.

\[ P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{l=1}^{6} x_l + \sum_{l=7}^{6} c(vv_l) + c(v) - W(v_k) \right). \]

We have cp(\prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=8}^{13} (\sum_{l=1}^{6} x_l + \sum_{l=7}^{6} c(vv_l) + c(v) - W(v_k))) = -5. According to Theorem 4, there exists x_i \in S_i, (1 \leq i \leq 6) such that \( P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0 \). We color vv_i, (1 \leq i \leq 6) correspondingly. Finally, we can recolor vertices v_i, (1 \leq i \leq 6) to get an ltnsd-k-coloring of the graph G, a contradiction.

Moreover, if d(v_i) = 2, d(v_j) = 3, (2 \leq i \leq 5) and d(v_j) \geq 6, (8 \leq j \leq 13). Consider G' = G - \{vv_i \mid i = 1, 2, \ldots, 5\}, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, (1 \leq i \leq 5). Let S_i, (1 \leq i \leq 5) be the sets of available colors for vv_i, (1 \leq i \leq 5), respectively. It is easy to obtain that |S_i| = 17 - 7 - 1 - 2 = 7 > 6, (1 \leq i \leq 6). We associate that vv_i, (1 \leq i \leq 6) with the variables x_i, (1 \leq i \leq 6), respectively. Then we consider the following polynomial.

\[ P(x_1, x_2, x_3, x_4, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{l=1}^{5} x_l + \sum_{l=6}^{5} c(vv_l) + c(v) - W(v_k) \right). \]

We have cp(\prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{k=8}^{13} (\sum_{l=1}^{5} x_l + \sum_{l=6}^{5} c(vv_l) + c(v) - W(v_k))) = 1. According to Theorem 4, there exists x_i \in S_i, (1 \leq i \leq 6) such that \( P(x_1, x_2, x_3, x_4, x_5) \neq 0 \). We color vv_i, (1 \leq i \leq 6) correspondingly. Finally, we can recolor vertices v_i, (1 \leq i \leq 6) to get an ltnsd-k-coloring of the graph G, a contradiction.

Claim 17. Let d(v) = \Delta(G) \geq 14 and d_6(v) \leq 6. If d_2-(v) \geq 1, then d_3-(v) \leq 5.

Proof. Let d(v) = d. Suppose to the contrary that there exists a d-vertex v adjacent to six 3^-vertices. Without loss of generality, assume that N(v) = \{v_1, v_2, \ldots, v_d\}, d(v_1) = 2, d(v_i) = 3, (2 \leq i \leq 6) and d(v_j) \geq 6, (d - 5 \leq j \leq d). Consider G' = G - \{vv_i \mid i = 1, 2, \ldots, 6\}, then G' admits an ltnsd-k-coloring c. Now we will color the edges vv_i and recolor vertices v_i, (1 \leq i \leq 6). Let S_i, (1 \leq i \leq 6) be the sets of available colors for vv_i (1 \leq i \leq 6), respectively.
It is easy to obtain that $|S_1| = (∆(G) + 3) − (∆(G) − 6) − 1 − 1 = 7 > 6$, $|S_i| = (∆(G) + 3) − (∆(G) − 6) − 2 = 6 > 5, (2 ≤ i ≤ 6)$. We associate that $vv_i, (1 ≤ i ≤ 6)$ with the variables $x_i, (1 ≤ i ≤ 6)$, respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 ≤ i < j ≤ 6} (x_i − x_j) \prod_{k=d=5}^{d} \left( \sum_{l=1}^{6} x_l + \sum_{l=7}^{d} c(vv_l) + c(v) − W(v_k) \right).$$

We have $cp(x_1^6x_2^5x_3^4x_4^3x_5^2x_6^1) = 1$. According to Theorem 4, there exists $x_i ∈ S_i, (1 ≤ i ≤ 6)$ such that $P(x_1, x_2, x_3, x_4, x_5, x_6) ≠ 0$. We color $vv_i, (1 ≤ i ≤ 6)$ correspondingly. Finally, we can recolor vertices $v_i, (1 ≤ i ≤ 6)$ to get an ltnsd-$k$-coloring of the graph $G$, a contradiction.

3.2. Discharging process

Let $T$ be the graph obtained by removing all $2^-$-vertices from the graph $G$ and $T^\times$ be the associated plane graph of $T$. We have $d_T(v) = d(v) − d_2^−(v)$.

**Corollary 18.** For any vertex $v$ with $d(v) ≥ 7$, it holds that $d_T(v) ≥ 7$.

**Proof.** If $7 ≤ d(v) ≤ 10$, we can easily get $d_T(v) ≥ 7$ by Claim 14 and Corollaries 9–11. When $d(v) > 10$, we just consider the situation $d_3(v) ≤ 6$. By Claim 8, $d_T(v) = d(v) − d_2^−(v) ≥ d(v) − \frac{d_3(v)−1}{2(∩(G))−d(v)+1} ≥ 11 − \frac{5}{14−11} ≥ 9$.

We apply the discharging method on associated plane graph $T^\times$ of $T$ and complete the proof by contradiction. Since $T^\times$ is a plane graph, we have

$$\sum_{v ∈ V(T^\times)} (d_{T^\times}(v) - 6) + \sum_{f ∈ F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= \sum_{v ∈ V(T)} (d_T(v) - 6) + \sum_{v ∈ V(T^\times) \setminus V(T)} (d_T(v) - 6) + \sum_{f ∈ F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= \sum_{v ∈ V(T)} (d(v) − d_2^−(v) − 6) + \sum_{v ∈ V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f ∈ F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= −12.$$

Now we define the initial charge function $ch(x)$ of $x ∈ V(T^\times) ∪ F(T^\times)$. Let $ch(v) = d_T(v) - 6 = d(v) − d_2^−(v) − 6$ if $v ∈ V(T)$, $ch(v) = d_{T^\times}(v) − 6$ if $v ∈ V(T^\times) \setminus V(T)$ and $ch(f) = 2d_{T^\times}(f) − 6$ if $f ∈ F(T^\times)$. Then we define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(T^\times) ∪ F(T^\times)$ such that $ch'(x) ≥ 0$ for all $x ∈ V(T^\times) ∪ F(T^\times)$. Notice that our discharging rules only move charge around and do not affect the sum. Thus we have $0 ≤ \sum_{x ∈ V(T^\times)∪F(T^\times)} ch'(x) = $
\[ \sum_{x \in V(T^x) \cup F(T^x)} ch(x) = -12, \] a contradiction. Since for every vertex \( v \in V(T) \), \( ch(v) = d_G(v) - d_{d-}(v) - 6 \), in the discharging process, we use \( d_G(v) \) instead of \( d_T(v) \). Similarly, for every vertex \( v \in V(T) \), when check \( ch'(v) \geq 0 \), we split the proof into cases depending on the size of \( d_G(v) \).

For \( v \in V(T^x) \) and \( f \in F(T^x) \), we define the discharging rules as follows. Note that within all the degree of a real vertex shall refer to its degree in \( G \) and the faces and their degrees correspond to the graph \( T^x \).

(R1): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 3 \), then \( u \) sends 1 to \( v \).

(R2): If the edge \( uv \) belongs to exactly one 3-face and \( d(v) = 3 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).

(R3): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 4 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).

(R4): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 5 \), then \( u \) sends \( \frac{1}{3} \) to \( v \).

(R5): Every 4-face sends 1 to each incident real 5-face in \( T^x \).

(R6): Every 5-face sends 2 to each incident real 5-face in \( T^x \).

(R7): Let \( v \) be a false vertex crossed by edge \( uw \) and \( xy \) in \( T^x \). If \( d(u) \geq 7 \), then \( u \) sends 1 to \( v \). Moreover, if \( d(w) = 3 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).

By Corollary 18 and the discharging rules, we obtain the following facts easily.

**Fact 1.** For any \( f \in F(T^x) \), \( f \) is incident with at most \( \left\lfloor \frac{d(f)}{2} \right\rfloor \) real 5-vertices in \( T^x \).

**Fact 2.** Each vertex \( v \) gives at most \( \frac{d_{\max}(v)}{2} + 1 \) away.

Let \( f \) be a face of \( T^x \). Clearly, if \( d(f) = 3 \), then \( ch'(f) = ch(f) = 2d(f) - 6 = 0 \). If \( d(f) = 4 \), then \( ch'(f) \geq ch(f) - 2 = 0 \) by Fact 1 and (R5). If \( d(f) \geq 5 \), then \( ch'(f) \geq ch(f) - \left\lfloor \frac{d(f)}{2} \right\rfloor \times 2 = 0 \) by Fact 1 and (R6).

We next check the final charge of the vertex \( v \in V(T^x) \). Obviously, \( d(v) \geq 3 \).

Recall that \( v \) has an initial weight of \( d(v) - d_{d-}(v) - 6 \).

Suppose \( d(v) = 3 \). If \( v \) is incident with three 3-faces, then \( ch'(v) = ch(v) + 3 = 0 \) by (R1). If \( v \) is incident with two 3-faces, then \( ch'(v) = ch(v) + 1 + \frac{1}{2} \times 2 + 1 = 0 \) by (R1), (R2), (R5) and (R6). If \( v \) is incident with one 3-face, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 \times 2 = 0 \) by (R2), (R5) and (R6). Otherwise, \( v \) is incident with three 4\(^+\)-faces, then \( ch'(v) \geq ch(v) + 1 \times 3 = 0 \) by (R5) and (R6).

Suppose \( d(v) = 4 \) and \( v \) is a real vertex. We have \( d_{d-}(v) = 0 \). If \( v \) is incident with four 3-faces, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 4 = 0 \) by (R3). If \( v \) is incident with three 3-faces, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 = 0 \) by (R3), (R5) and (R6). If \( v \) is incident with at most two 3-faces, then \( ch'(v) \geq ch(v) + 1 \times 2 = 0 \) by (R5) and (R6).

Suppose \( d(v) = 4 \) and \( v \) is a false vertex crossed by edge \( uw \) and \( xy \). By Claim 14 and 15, \( v \) is adjacent to at most two 6\(^{-}\)-vertices.
If $d_{v}-(v) = 2$, without loss of generality, we assume that $d(u) \leq 6$ and $d(x) \leq 6$. By Claim 15, if $4 \leq d(u) \leq 6$ and $4 \leq d(x) \leq 6$, then $uv \notin E(T)$, $v$ gives no weight away by (R3) and (R4). By the same claim, $v$ is also adjacent to two $7^+$-vertices. So $v$ receives at least $1 \times 2 = 2$ from its $7^+$-neighbors by (R7). Thus, we have $ch'(v) \geq ch(v) + 2 = 0$. If one of the vertices $x, u$ is a 3-vertex, without loss of generality, we assume that $d(u) = 2$. Then, by Claim 13, we have that $d_3^-(v) \leq 2$. If $d_2^-(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - 1$. If $d_2-(v) = 0$, then we have $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - 1$.

If $d_{v}-(v) = 1$, without loss of generality, we assume that $d(u) \leq 6$. Then $v$ is adjacent to three $7^+$-vertices. So $v$ receives at least $1 \times 3 = 3$, and $v$ gives at most $\frac{1}{2}$ away by Lemma 3 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - \frac{1}{2} > 0$.

If $v$ is adjacent to four $7^+$-vertices, $v$ receives at least $1 \times 4 = 4$ from its $7^+$-neighbors by (R7) and gives no weight away. So we have $ch'(v) \geq ch(v) + 1 \times 4 > 0$.

Suppose $d(v) = 5$. If $v$ is not incident with any $4^+$-faces, then by (R4), $ch'(v) \geq ch(v) + \frac{3}{2} \times 5 = 0$. Otherwise, if $v$ is incident with at least one $4^+$-faces, then by (R5) and (R6), $ch'(v) \geq ch(v) + 1 = 0$.

Suppose $d(v) = 6$. $v$ gives no weight away to any other vertex by the discharging rules. So $ch'(v) = ch(v) = 0$.

Suppose $d(v) = 7$. $v$ gives at most 1 to the false neighbor in $T^*$ by (R7), then $ch'(v) = ch(v) - 1 = 0$.

Suppose $d(v) = 8$. By Corollary 9, $ch'(v) = ch(v) - \max \{2, \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 9$. By Corollary 10, $ch'(v) = ch(v) - \max \{3, 1 + \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 10$. By Corollary 11, $ch'(v) = ch(v) - \max \{4, 2 + \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Next we check the final charge of the vertices with $d(v) \geq 11$. Let $w$ be a false vertex crossed by edge $uv$ and edge $xy$. According to the discharging rules, if $d(u) \leq 5$, then $v$ gives at most $d_5^-(v) + \frac{1}{2}$ away. Otherwise, $v$ gives at most $d_3^-(v) + 1$ away. Therefore, for every vertex $v$ with $d_3^+(v) \geq 7$, $ch'(v) \geq 0$. In the following discussion, we only consider the vertex with $d(v) \geq 11$ and $d_3^-(v) \leq 6$.

Suppose $d(v) = 11$. By Claim 12, we have that $d_3^-(v) \leq 1$. If $d_2^-(v) = 0$, then $d_3(v) \leq 1$. We have $ch'(v) = ch(v) - \max \{1 + d_3(v) + \left(\frac{11-1}{2} - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + \left[\frac{11-1}{2}\right] \times \frac{1}{2}\} = 5 - \max \left\{\frac{11-1}{2} + d_3(v), \frac{4}{2}\right\} > 0$ by (R1)–(R4) and (R7). If $d_2^-(v) = 1$, then $d_3(v) = 0$. We have $ch'(v) = ch(v) - \left(1 - \left[\frac{11-1}{2}\right] \times \frac{1}{2}\right) = 3 - \frac{5}{2} > 0$ by (R3) and (R7).

Suppose $d(v) = 12$. By Claim 13, we have that $d_3^-(v) \leq 2$.

If $d_2^-(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) -
max \( \{1 + d_3(v) + \left( \frac{13-1}{2} - d_3(v) \right) \times \frac{1}{2}, 3 + \left( d_3(v) - 1 \right) + \left( \frac{13-1}{2} - d_3(v) - 1 \right) \times \frac{1}{2} \} = 6 - 7 + d_3(v) \geq 0 \) by (R1)-(R4) and (R7). If \( d_3(v) = 0 \), then we have \( ch'(v) = ch(v) - 1 - \left( \frac{13-1}{2} \right) \times \frac{1}{2} > 0 \) by (R3) and (R7).

If \( d_2(v) \geq 1 \) and \( d_3(v) \geq 1 \), we have \( ch'(v) = ch(v) - \max \{1 + d_3(v) + \left( \frac{13-1}{2} - d_3(v) \right) \times \frac{1}{2}, 3 + d_3(v) - 1 + \left( \frac{13-1}{2} - d_3(v) - 1 \right) \times \frac{1}{2} \} \geq \frac{9}{3} - d_3(v) > 0 \) by (R1)-(R4) and (R7). Otherwise \( d_3(v) = 0 \), then \( d_2(v) \leq 2 \). So \( ch'(v) = ch(v) - 1 - \left( \frac{13-1}{2} \right) \times \frac{1}{2} \geq \frac{9}{3} - d_3(v) > 0 \) by (R1)-(R4) and (R7).

Suppose \( d(v) = 13 \). By Claim 16, we have that \( d_3(v) \leq 5 \). Moreover, if \( d_2(v) \geq 1 \), then \( d_3(v) \leq 4 \).

If \( d_2(v) = 0 \), then \( d_3(v) \leq 5 \). If \( d_3(v) \geq 1 \), then we have \( ch'(v) = ch(v) - \max \{1 + d_3(v) + \left( \frac{13-1}{2} - d_3(v) \right) \times \frac{1}{2}, 3 + d_3(v) - 1 + \left( \frac{13-1}{2} - d_3(v) - 1 \right) \times \frac{1}{2} \} \geq \frac{3}{3} - \frac{d_3(v)}{4} > 0 \) by (R1)-(R4) and (R7). Otherwise \( d_3(v) = 0 \), then \( d_2(v) \leq 4 \). So \( ch'(v) = ch(v) - 1 - \left( \frac{13-1}{2} \right) \times \frac{1}{2} \geq 3 - \frac{d_3(v)}{4} > 0 \) by (R1)-(R4) and (R7).

Suppose \( d(v) = \Delta(G) \geq 14 \). If \( d_2(v) = 0 \), then by Fact 2, we have \( ch'(v) = ch(v) - \frac{d_3(v)}{2} \geq 1 \) by (R1)-(R4) and (R7).

If \( d_2(v) \geq 1 \), then by Claim 17, \( d_3(v) \leq 5 \). If \( d_3(v) \geq 1 \), then we have \( ch'(v) = ch(v) - \max \{1 + d_3(v) + \left( \frac{13-1}{2} - d_3(v) \right) \times \frac{1}{2}, 3 + d_3(v) - 1 + \left( \frac{13-1}{2} - d_3(v) - 1 \right) \times \frac{1}{2} \} \geq \frac{15}{3} - \frac{d_3(v)}{4} > 0 \) by (R1)-(R4) and (R7). Otherwise \( d_3(v) = 0 \), then \( d_2(v) \leq 5 \). So \( ch'(v) = ch(v) - d_2(v) - 1 - \left( \frac{13-1}{2} \right) \times \frac{1}{2} \geq \frac{15}{3} - \frac{d_3(v)}{4} \geq 0 \) by (R1)-(R4) and (R7).

This completes the proof.

4. Remark

By the definition of IC-planar graphs, we know that every planar graphs are special IC-planar graphs. In [13], the authors proved that \( ch''_\Sigma(G) \leq \Delta(G) + 3, 16 \). So we can easily obtain the following question.

**Question 1.** Is it true that \( ch''_\Sigma(G) \leq \Delta(G) + 3 \) for IC-planar graphs with \( \Delta = 13 \)?
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Appendix A

%%% The m. file of Matlab to compute the coefficients.
% INPUT
function coefficients ()
syms x1 x2 x3 x4 x5 x6 x7 % Variables used in the following.

% Claim 3.7 %To calculate the coefficient of x1^6 x2^4 x3^4
P=(x1−x2)*(x2−x3)*(x1−x3)^2*(x1+x2)^4*(x2+x3)^6; % The polynomial
cp1=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)

% Claim 3.8
P=(x1−x2)*(x2−x3)+(x1−x3)^2*(x1+x2)^5*(x2+x3)^5;
cp2=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)

% Claim 3.9
P=(x1−x2)*(x1−x3)*(x1−x4)*(x1−x5)*
*(x2−x6)*(x3−x4)*(x3−x5)*
*(x3−x6) . . . *(x4−x5)*
*(x4−x6)*
*(x5−x6)
*(x1+x2+x3+x4+x5+x6)^6;

% Claim 3.10
P=(x1−x2)*(x1−x3)*(x1−x4)*(x1−x5)*
*(x2−x6)*(x3−x4)*(x3−x5)*(x3−x6) . . . *(x4−x5)*
*(x4−x6)*
*(x5−x6)
*(x1+x2+x3+x4+x5+x6)^6;

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