NEIGHBOR SUM DISTINGUISHING TOTAL CHOOSABILITY OF IC-PLANAR GRAPHS

WEN-YAO SONG, LIAN-YING MIAO

School of Mathematics
China University of Mining and Technology
Xuzhou 221116, P.R. China

e-mail: songwenyao@cumt.edu.cn
miaolianying@cumt.edu.cn

AND

YUAN-YUAN DUAN

School of Mathematics and Statistics
Zaozhuang University
Zaozhuang 277160, P.R. China

e-mail: duanyy0827@sina.com

Abstract

Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane such that every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short. A proper total-$k$-coloring of a graph $G$ is a mapping $c : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ denote the sum of the color of a vertex $v$ and the colors of all incident edges of $v$. A total-$k$-neighbor sum distinguishing-coloring of $G$ is a total-$k$-coloring of $G$ such that for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$. The least number $k$ needed for such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $ch_{\Sigma}^\prime(G)$. In this paper, it is proved that if $G$ is an IC-planar graph with maximum degree $\Delta(G)$, then $ch_{\Sigma}^\prime(G) \leq \max\{\Delta(G) + 3, 17\}$, where $ch_{\Sigma}^\prime(G)$ is the neighbor sum distinguishing total choosability of $G$.

Keywords: neighbor sum distinguishing total choosability, maximum degree, IC-planar graph, Combinatorial Nullstellensatz.

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1. Introduction

All graphs considered are finite, simple and undirected. Let $G$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For planar graph $G$, $F(G)$ denotes its face set, $d(v)$ denotes the degree of a vertex $v$ in $G$. The length or degree of a face $f$, denoted by $d(f)$, is the length of the boundary walk of $f$ in $G$. We call $v$ a $k$-vertex, or a $k^+$-vertex, or a $k^-$-vertex if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call $f$ a $k$-face, or a $k^+$-face, or a $k^-$-face if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A proper total-$k$-coloring of a graph $G$ is a mapping $c : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ be the sum of the color of a vertex $v$ and the colors of all edges incident with $v$. If for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$, then we say such total-$k$-coloring a neighbor sum distinguishing total-$k$-coloring, denoted by $tnsd-k$-coloring for short. The least number $k$ needed for such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi'_{tnsd}(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture proposed by Pilśniak and Woźniak [11].

**Conjecture 1.** For any graph $G$, $\chi'_{tnsd}(G) \leq \Delta(G) + 3$.

Loeb and Tang [10] proved that this bound was asymptotically correct by showing that $\chi'_{tnsd}(G) \leq \Delta(G)(1 + o(1))$. Pilśniak and Woźniak [11] proved that Conjecture 1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. With the Combinatorial Nullstellensatz, neighbor sum distinguishing total coloring have been studied widely, see [4–6,8,9,12,19].

For a given graph $G$, let $L_x (x \in V \cup E)$ be a set of lists of real numbers and each of size $k$. The neighbor sum distinguishing total choosability of $G$ is the least number $k$ for which for any specified collection of such lists, there exists a neighbor sum distinguish total coloring with colors from $L_x$ for each $x \in V \cup E$, and we denote it by $ch'_{tnsd}(G)$. We call such a coloring of $G$ list neighbor sum distinguish total-$k$-coloring and denote it by $ltnds-k$-coloring. Ding et al. [4] proved that for any graph $G$, $ch'_{tnsd}(G) \leq 2\Delta(G) + \text{col}(G) - 1$, where $\text{col}(G)$ is the coloring number of $G$. Later Ding et al. [5] improved the bound to $ch'_{tnsd}(G) \leq 2\Delta(G) + \text{col}(G) - 2$. Recently, Lu et al. [20] improved the bound to $ch'_{tnsd}(G) \leq \max\{\Delta(G) + \left\lfloor \frac{3col(G)}{2} \right\rfloor - 1, 3col(G) - 2\}$. The list neighbor sum distinguish total-$k$-coloring of some special classes of graphs were also investigated. Graphs with bounded maximum average degree (Yao and Kong [16]); $d$-degenerate graphs (Yao et al. [18]); planar graphs (Qu et al. [13], Wang et al. [15]).

In this paper, we consider IC-planar graphs and prove the following result.
**Theorem 2.** Let $G$ be an IC-planar graph with maximum degree $\Delta(G)$. Then $\text{ch}''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$.

An **IC-plane graph** is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane in which every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Albertson [1] in 2008. Setting a conjecture of Albertson [1], Král and Stacho [7] showed that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graph. We call $G$ a 1-planar graph if it can be drawn on a plane such that each edge is crossed by at most one other edge.

## 2. Preliminaries

Every IC-planar graph $G$ in this paper has been embedded on a plane such that all its crossings are independent and the number of crossings is as small as possible. In other words, we call $G$ an IC-plane graph. The **associated plane graph** $G^\times$ of $G$ is obtained by turning all crossings of $G$ into new 4-vertices on a plane. For convenience, a vertex in $G^\times$ is called **false** if it is not a vertex of $G$ and **real** otherwise. For a vertex $v \in V(G)$, we use $d_i(v)$ to denote the number of $i$-vertices which are adjacent to $v$. One can see that every real vertex in $G^\times$ is adjacent to at most one false vertex and incident with at most two false faces in $G^\times$.

**Lemma 3** [17]. Let $G$ be a 1-plane graph and $G^\times$ be its associated plane graph. If $d_G(u) = 3$ and $v$ is a crossing vertex in $G^\times$, then either $uv \notin E(G^\times)$ or $uv$ is not incident with two 3-faces.

We define that a graph $G'$ is **smaller** than a graph $G$ if $|E(G')| < |E(G)|$. We call a graph **minimal** for a property when no smaller graph satisfies it. Let from now on $G = (V, E)$ be a minimal counterexample to Theorem 2. We set $k = \max\{\Delta(G) + 3, 17\}$. For each 5$^-$-vertex $v \in V(G)$, it is obvious that $v$ has at most five neighbors and five incident edges, so $v$ has at most 15 forbidden colors. Since $k \geq 17$, we can first erase the color of vertex $v$ and finally recolor it after arguing. In other words, we may omit the coloring for all 5$^-$-vertices of $G$ in the following discussion.

**Theorem 4** (Combinatorial Nullstellensatz [2]). Let $F$ be an arbitrary field, and let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in $F[x_1, x_2, \ldots, x_n]$. Suppose the degree $\text{deg}(P)$ of $P$ equals $\sum_{i=1}^{n} k_i$, where each $k_i$ is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ in $P$ is non-zero. Then if $S_1, S_2, \ldots, S_n$
are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.

**Lemma 5** [14]. If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\leq s_1 + s_2 + \cdots + s_n$, where $s_1, s_2, \ldots, s_n$ are nonnegative integers, then

$$
\left( \frac{\partial}{\partial x_1} \right)^{s_1} \left( \frac{\partial}{\partial x_2} \right)^{s_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{s_n} P(x_1, x_2, \ldots, x_n)
$$

$$
= \sum_{i=0}^{s_1} \cdots \sum_{i=0}^{s_n} (-1)^{s_1 + s_2} \frac{s_1}{s_1} \cdots (-1)^{s_n + s_1} \frac{s_n}{s_n} P(x_1, x_2, \ldots, x_n).
$$

**Lemma 6** [13]. Let $L_i$ be the sets of real numbers, with $|L_i| = l_i$, where $i = 1, 2, \ldots, p$. Let $L = \left\{ \sum_{i=1}^{p} x_i \big| x_i \in L_i \text{ and } \prod_{1 \leq i < j \leq p} (x_i - x_j) \neq 0 \right\}$. Then $|L| \geq \sum_{i=1}^{p} (l_i - p + 1) - (p - 1) = \sum_{i=1}^{p} l_i - p^2 + 1$.

### 3. Proof of Theorem 2

#### 3.1. Unavoidable configurations

In the following, we will often delete some edges to get a proper subgraph $G'$ of $G$, then by the minimality of $G$, there exists an ltnsd-$k$-coloring $c$ of $G'$. Let $W_G(v) = \sum_{e \ni v, e \in E(G)} c(e) + c(v)$. We may extend this coloring $c$ to the whole graph $G$. For any $x \in V(G) \cup E(G)$, the available colors are the remaining colors after excluding the colors of its adjacent edges and vertices in $G'$ from $L_x$.

**Claim 7.** For any vertex $v \in V(G)$, it holds that

$$
\sum_{j=1}^{t} [d_j(v)(\Delta(G) + 4 - d(v) - j)] \leq d(v) - 1, \quad (1 \leq t \leq 5).
$$

**Claim 8.** For any vertex $v \in V(G)$, $d_{2-}(v) \leq \frac{d_{6+}(v) - 1}{\Delta(G) - d(v) + 1}$. Moreover, if $d(v) = \Delta(G)$, then $d_{2-}(v) \leq d_{6+}(v) - 1$.

The proof of Claim 7 and 8 are similar to that of Claim 3.1 and Claim 3.2 in [13], we omit it here. By Claim 7, we can easily get the following Corollaries.

**Corollary 9.** If $d(v) = 8$, then $d_{5-}(v) \leq 1$.

**Corollary 10.** If $d(v) = 9$, then $d_{5-}(v) \leq 2$.

**Corollary 11.** If $d(v) = 10$, then $d_{5-}(v) \leq 3$.

**Claim 12.** If $d(v) = 11$ and $d_{6+}(v) \leq 6$, then $d_{3-}(v) \leq 1$. 
**Proof.** Suppose to the contrary that $v$ is adjacent to two $3^-$-vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{11}\}$, $d(v_1) = d(v_2) = 3$ and $d(v_j) \geq 6$, $(6 \leq j \leq 11)$. Consider $G' = G - vv_1 - vv_2$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will color the edges $vv_1, vv_2$ and recolor vertices $v_1, v_2$. Let $S_1, S_2$ be the sets of available colors for $vv_1, vv_2$, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5 > 4$.\footnote{By Lemma 6, $|L| \geq |S_1| + |S_2| - 4 + 1 = 7 > 6$.}

We can choose a pair, say $(x, y) \in S_1 \times S_2$ with $x \neq y$, such that $x + y \notin \{W_G(v_j) - W_G(v)\} | 6 \leq j \leq 11|$. Finally, we can recolor $v_1, v_2$ to get an ltnsd-$k$-coloring of $G$, a contradiction. **Claim 13.** If $d(v) = 12$ and $d_6+(v) \leq 6$, then $d_3-(v) \leq 2$.

**Proof.** Suppose to the contrary that $v$ is adjacent to three $3^-$-vertices. Without loss of generality, we assume that $N(v) = \{v_1, v_2, \ldots, v_{12}\}$, $d(v_1) = d(v_2) = d(v_3) = 3$ and $d(v_j) \geq 6$, $(7 \leq j \leq 12)$. Consider $G' = G - \{v_i\} | i = 1, 2, 3\}$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will color the edges $vv_1, vv_2, vv_3$ and recolor vertices $v_1, v_2, v_3$. Let $S_1, S_2, S_3$ be the sets of available colors for $vv_1, vv_2, vv_3$, respectively. It is easy to obtain that $|S_i| = 17 - 12 = 5$, $(1 \leq i \leq 3)$. By Lemma 6, $|L| \geq |S_1| + |S_2| + |S_3| - 9 + 1 = 7 > 6$. We can choose a triple, say $(x, y, z) \in S_1 \times S_2 \times S_3$ with $x, y, z$ distinct colors, such that $x + y + z \notin \{W_G(v_j) - W_G(v)\} | 7 \leq j \leq 12\}$. Finally, we can recolor $v_1, v_2, v_3$ to get an ltnsd-$k$-coloring of $G$, a contradiction.

By Lemma 5, if $P(x_1, x_2, \ldots, x_n)$ is a polynomial with $\deg(P) = n$, $k_1, k_2, \ldots, k_m$ are non-negative integers with $\sum_{i=1}^{m} k_i = n$ and $cp\left(x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}\right)$ is the coefficient of $x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}$ in $P$, then $\frac{\partial^{(m)}P}{\partial x_1^{k_1}\cdots \partial x_m^{k_m}} = cp\left(x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}\right) \prod_{i=1}^{m} k_i!$. In the following, we use MATLAB to calculate the coefficients of specific monomials. Moreover, we will list the codes in Appendix.

**Claim 14.** Every $5^-$-vertex is not adjacent to $7^-$-vertex in $G$.

**Proof.** Suppose to the contrary that there exists a $5^-$-vertex $u$ adjacent to a $7^-$-vertex $v$. Without loss of generality, we assume that $d(u) = 5$, $d(v) = 7$, $N(u) = \{v, u_1, \ldots, u_4\}$, $N(v) = \{u, v_1, \ldots, v_6\}$. Consider $G' = G - uv$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will recolor the vertices $u, v$ and color the edge $uv$. Let $S_1, S_2, S_3$ be the sets of available colors for $u, w, v$, respectively. Notice that the colors in $\{c(uu_i) | 1 \leq i \leq 4\} \cup \{c(u_i) | 1 \leq i \leq 4\}$ are forbidden for $u$, the colors in $\{c(ww_i) | 1 \leq i \leq 4\} \cup \{c(vv_i) | 1 \leq i \leq 6\}$ are forbidden for $uv$, and the colors in $\{c(vv_i) | 1 \leq i \leq 6\} \cup \{c(u_i) | 1 \leq i \leq 6\}$ are forbidden for $v$. Thus, $|S_1| = 17 - 8 = 9 > 8$, $|S_2| = 17 - 10 = 7 > 6$, $|S_3| = 17 - 12 = 5 > 4$. We associate that $u, w, v$ with the variables $x_1, x_2, x_3$, respectively. Then we
consider the following polynomial.

\[
P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + \sum_{i=1}^{4} c(uu_i) - x_3 - \sum_{k=1}^{6} c(vv_k))
\]

\[
\prod_{i=1}^{4} \left( x_1 + x_2 + \sum_{l=1}^{4} c(uu_l) - W(u_i) \right)
\]

\[
\prod_{j=1}^{6} \left( x_2 + x_3 + \sum_{k=1}^{6} c(vv_k) - W(v_j) \right).
\]

We have \( cp \left( x_1^6 x_2^4 x_3^4 \right) = -25 \). According to Theorem 4, there exists \( x_i \in S_i, (1 \leq i \leq 3) \) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-k-coloring of the graph \( G \), a contradiction.

**Claim 15.** Every 6-vertex is not adjacent to 6-vertex in \( G \).

**Proof.** Suppose to the contrary that there exists a 6-vertex \( u \) adjacent to a 6-vertex \( v \). Without loss of generality, we assume that \( d(u) = 6 \), \( d(v) = 6 \), \( N(u) = \{ v, u_1, \ldots, u_5 \} \), \( N(v) = \{ u, v_1, \ldots, v_5 \} \). Consider \( G' = G - uv \), then \( G' \) admits an ltnsd-k-coloring \( c \). Now we will recolor the vertices \( u, v \) and color the edge \( uv \). Let \( S_1, S_2, S_3 \) be the sets of available colors for \( u, uv, v \), respectively. Notice that the colors in \( \{ c(uu_i) | 1 \leq i \leq 5 \} \cup \{ c(u_l) | 1 \leq i \leq 5 \} \) are forbidden for \( u \), the colors in \( \{ c(uu_i) | 1 \leq i \leq 5 \} \cup \{ c(vv_i) | 1 \leq i \leq 5 \} \) are forbidden for \( uv \), and the colors in \( \{ c(vv_i) | 1 \leq i \leq 5 \} \cup \{ c(v_l) | 1 \leq i \leq 5 \} \) are forbidden for \( v \). Thus, \( |S_1| = 17 - 10 = 7 \geq 6 \), \( |S_2| = 17 - 10 = 7 \geq 6 \), \( |S_3| = 17 - 10 = 7 \geq 6 \).

We associate that \( u, uv, v \) with the variables \( x_1, x_2, x_3 \), respectively. Then we consider the following polynomial.

\[
P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + \sum_{i=1}^{5} c(uu_i) - x_3 - \sum_{k=1}^{5} c(vv_k))
\]

\[
\prod_{i=1}^{5} \left( x_1 + x_2 + \sum_{l=1}^{5} c(uu_l) - W(u_i) \right)
\]

\[
\prod_{j=1}^{5} \left( x_2 + x_3 + \sum_{k=1}^{5} c(vv_k) - W(v_j) \right).
\]

We have \( cp \left( x_1^6 x_2^4 x_3^4 \right) = -20 \). According to Theorem 4, there exists \( x_i \in S_i, (1 \leq i \leq 3) \) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-k-coloring of the graph \( G \), a contradiction.

**Claim 16.** Let \( d(v) = 13 \) and \( d_6+(v) \leq 6 \), then \( d_3-(v) \leq 5 \). Moreover, if \( d_2-(v) \geq 1 \), then \( d_3-(v) \leq 4 \).
**Proof.** Suppose to the contrary that there exists a 13-vertex $v$ adjacent to six $3^-$-vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \ldots, v_{13}\}$, $d(v_i) = 3$, $(1 \leq i \leq 6)$ and $d(v_j) \geq 6$, $(8 \leq j \leq 13)$. Consider $G' = G - \{vv_i | i = 1, 2, \ldots, 6\}$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will color the edges $vv_i$ and recolor vertices $v_i$, $(1 \leq i \leq 6)$. Let $S_i$, $(1 \leq i \leq 6)$ be the sets of available colors for $v_i$ $(1 \leq i \leq 6)$, respectively. It is easy to obtain that $|S_i| = 17 - 7 - 1 - 2 = 7 > 6$, $(1 \leq i \leq 6)$. We associate that $vv_i$, $(1 \leq i \leq 6)$ with the variables $x_i$, $(1 \leq i \leq 6)$, respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{i=1}^{6} x_i + \sum_{l=7}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $\text{cp}(x_1^6x_2^5x_3^4x_4^3x_5^2x_6^1) = 1$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \leq i \leq 6)$ such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0$. We color $vv_i$, $(1 \leq i \leq 6)$ correspondingly. Finally, we can recolor vertices $v_i$, $(1 \leq i \leq 6)$ to get an ltnsd-$k$-coloring of the graph $G$, a contradiction.

Moreover, if $d(v_l) = 2$, $d(v_i) = 3$, $(2 \leq l \leq i \leq 5)$ and $d(v_j) \geq 6$, $(8 \leq j \leq 13)$. Consider $G' = G - \{vv_i | i = 1, 2, \ldots, 5\}$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will color the edges $vv_i$ and recolor vertices $v_i$, $(1 \leq i \leq 5)$. Let $S_i$, $(1 \leq i \leq 5)$ be the sets of available colors for $vv_i$, $(1 \leq i \leq 5)$, respectively. It is easy to obtain that $|S_i| = 17 - 8 - 1 - 1 = 7 > 6$, $|S_i| = 17 - 8 - 1 - 2 = 6 > 5$, $(2 \leq i \leq 5)$. We associate that $vv_i$, $(1 \leq i \leq 5)$ with the variables $x_i$, $(1 \leq i \leq 5)$, respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{i=1}^{5} x_i + \sum_{l=6}^{13} c(vv_l) + c(v) - W(v_k) \right).$$

We have $\text{cp}(x_1^6x_2^5x_3^4x_4^3x_5^2x_6^1) = -5$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \leq i \leq 5)$ such that $P(x_1, x_2, x_3, x_4, x_5) \neq 0$. We color $vv_i$, $(1 \leq i \leq 5)$ correspondingly. Finally, we can recolor vertices $v_i$, $(1 \leq i \leq 5)$ to get an ltnsd-$k$-coloring of the graph $G$, a contradiction.

**Claim 17.** Let $d(v) = \Delta(G) \geq 14$ and $d_6^+(v) \leq 6$. If $d_2^+(v) \geq 1$, then $d_3^-(v) \leq 5$.

**Proof.** Let $d(v) = d$. Suppose to the contrary that there exists a $d$-vertex $v$ adjacent to six $3^-$-vertices. Without loss of generality, assume that $N(v) = \{v_1, v_2, \ldots, v_d\}$, $d(v_1) = 2$, $d(v_i) = 3$, $(2 \leq i \leq 6)$ and $d(v_j) \geq 6$, $(d - 5 \leq j \leq d)$. Consider $G' = G - \{vv_l | l = 1, 2, \ldots, 6\}$, then $G'$ admits an ltnsd-$k$-coloring $c$. Now we will color the edges $vv_i$ and recolor vertices $v_i$, $(1 \leq i \leq 6)$. Let $S_i$, $(1 \leq i \leq 6)$ be the sets of available colors for $vv_i$ $(1 \leq i \leq 6)$, respectively.
It is easy to obtain that $|S_1| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 = 7 > 6$, $|S_i| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 - 2 = 6 > 5, (2 \leq i \leq 6)$. We associate that $vv_i, (1 \leq i \leq 6)$ with the variables $x_i, (1 \leq i \leq 6)$, respectively. Then we consider the following polynomial.

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=d-5}^{d} \left( \sum_{t=1}^{6} x_t + \sum_{l=7}^{d} c(vv_l) + c(v) - W(v_k) \right).$$

We have $cp(x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1) = 1$. According to Theorem 4, there exists $x_i \in S_i, (1 \leq i \leq 6)$ such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0$. We color $vv_i, (1 \leq i \leq 6)$ correspondingly. Finally, we can recolor vertices $v_i, (1 \leq i \leq 6)$ to get an 1tnsd-$k$-coloring of the graph $G$, a contradiction.

3.2. Discharging process

Let $T$ be the graph obtained by removing all $2^-$-vertices from the graph $G$ and $T^\times$ be the associated plane graph of $T$. We have $d_T(v) = d(v) - d_2^-(v)$.

Corollary 18. For any vertex $v$ with $d(v) \geq 7$, it holds that $d_T(v) \geq 7$.

Proof. If $7 \leq d(v) \leq 10$, we can easily get $d_T(v) \geq 7$ by Claim 14 and Corollaries 9–11. When $d(v) > 10$, we just consider the situation $d_0^+(v) \leq 6$. By Claim 8, $d_T(v) = d(v) - d_2^-(v) \geq d(v) - \frac{d_0^+(v) - 1}{\Delta(G) - d(v) + 1} \geq 11 - \frac{5}{14 - 11} \geq 9$.

We apply the discharging method on associated plane graph $T^\times$ of $T$ and complete the proof by contradiction. Since $T^\times$ is a plane graph, we have

$$\sum_{v \in V(T^\times)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= \sum_{v \in V(T)} (d_T(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= \sum_{v \in V(T)} (d(v) - d_2^-(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)$$

$$= -12.$$

Now we define the initial charge function $ch(x)$ of $x \in V(T^\times) \cup F(T^\times)$. Let $ch(v) = d_T(v) - 6 = d(v) - d_2^-(v) - 6$ if $v \in V(T)$, $ch(v) = d_{T^\times}(v) - 6$ if $v \in V(T^\times) \setminus V(T)$ and $ch(f) = 2d_{T^\times}(f) - 6$ if $f \in F(T^\times)$. Then we define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(T^\times) \cup F(T^\times)$ such that $ch'(x) \geq 0$ for all $x \in V(T^\times) \cup F(T^\times)$. Notice that our discharging rules only move charge around and do not affect the sum. Thus we have $0 \leq \sum_{x \in V(T^\times) \cup F(T^\times)} ch'(x) =$
\[ \sum_{x \in V(T^x) \cup F(T^x)} ch(x) = -12 \text{, a contradiction. Since for every vertex } v \in V(T), \]
\[ ch(v) = d_G(v) - d_{T^-}(v) - 6 \text{, in the discharging process, we use } d_G(v) \text{ instead of } d_T(v). \]
Similarly, for every vertex } v \in V(T) \text{, when check } ch'(v) \geq 0 \text{, we split the proof into cases depending on the size of } d_G(v).

For } v \in V(T^x) \text{ and } f \in F(T^x) \text{, we define the discharging rules as follows. Note that within all the degree of a real vertex shall refer to its degree in } G \text{ and the faces and their degrees correspond to the graph } T^x. \]

(R1): If the edge } uv \text{ belongs to two 3-faces and } d(v) = 3 \text{, then } u \text{ sends } 1 \text{ to } v.

(R2): If the edge } uv \text{ belongs to exactly one 3-face and } d(v) = 3 \text{, then } u \text{ sends } \frac{1}{2} \text{ to } v.

(R3): If the edge } uv \text{ belongs to two 3-faces and } d(v) = 4 \text{, then } u \text{ sends } \frac{1}{2} \text{ to } v.

(R4): If the edge } uv \text{ belongs to two 3-faces and } d(v) = 5 \text{, then } u \text{ sends } \frac{1}{5} \text{ to } v.

(R5): Every 4-face sends 1 to each incident real 5^-vertex in } T^x.

(R6): Every 5^-face sends 2 to each incident real 5^-vertex in } T^x.

(R7): Let } v \text{ be a false vertex crossed by edge } uw \text{ and } xy \text{ in } T^x. \text{ If } d(u) \geq 7 \text{, then } u \text{ sends } 1 \text{ to } v. \text{ Moreover, if } d(w) = 3 \text{, then } u \text{ sends } \frac{1}{2} \text{ to } v.

By Corollary 18 and the discharging rules, we obtain the following facts easily.

**Fact 1.** For any } f \in F(T^x) \text{, } f \text{ is incident with at most } \left\lfloor \frac{d(f)}{2} \right\rfloor \text{ real 5^-vertices in } T^x.

**Fact 2.** Each vertex } v \text{ gives at most } \frac{d_{2^-}(v)}{2} + 1 \text{ away.

Let } f \text{ be a face of } T^x. \text{ Clearly, if } d(f) = 3 \text{, then } ch'(f) = ch(f) = 2d(f) - 6 = 0. \text{ If } d(f) = 4 \text{, then } ch'(f) \geq ch(f) - 2 = 0 \text{ by Fact 1 and (R5). If } d(f) \geq 5 \text{, then } ch'(f) \geq ch(f) - \left\lfloor \frac{d(f)}{2} \right\rfloor \times 2 = 0 \text{ by Fact 1 and (R6).}

We next check the final charge of the vertex } v \in V(T^x). \text{ Obviously, } d(v) \geq 3 \text{. Recall that } v \text{ has an initial weight of } d(v) - d_{2^-}(v) - 6.

Suppose } d(v) = 3. \text{ If } v \text{ is incident with three 3-faces, then } ch'(v) \geq ch(v) + 3 = 0 \text{ by (R1). If } v \text{ is incident with two 3-faces, then } ch'(v) \geq ch(v) + 1 + \frac{1}{2} \times 2 + 1 = 0 \text{ by (R1), (R2), (R5) and (R6). If } v \text{ is incident with one 3-face, then } ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 \times 2 = 0 \text{ by (R2), (R5) and (R6). Otherwise, } v \text{ is incident with three 4^-faces, then } ch'(v) \geq ch(v) + 1 \times 3 = 0 \text{ by (R5) and (R6).}

Suppose } d(v) = 4 \text{ and } v \text{ is a real vertex. We have } d_{2^-}(v) = 0. \text{ If } v \text{ is incident with four 3-faces, then } ch'(v) \geq ch(v) + \frac{1}{2} \times 4 = 0 \text{ by (R3). If } v \text{ is incident with three 3-faces, then } ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 = 0 \text{ by (R3), (R5) and (R6). If } v \text{ is incident with at most two 3-faces, then } ch'(v) \geq ch(v) + 1 \times 2 = 0 \text{ by (R5) and (R6).}

Suppose } d(v) = 4 \text{ and } v \text{ is a false vertex crossed by edge } uw \text{ and } xy. \text{ By Claim 14 and 15, } v \text{ is adjacent to at most two 6^-vertices.
If $d_{6-}(v) = 2$, without loss of generality, we assume that $d(u) \leq 6$ and $d(x) \leq 6$. By Claim 15, if $4 \leq d(u) \leq 6$ and $4 \leq d(x) \leq 6$, then $ux \notin E(T)$, $v$ gives no weight away by (R3) and (R4). By the same claim, $v$ is also adjacent to two $7^+$-vertices. So $v$ receives at least $1 \times 2 = 2$ from its $7^+$-neighbors by (R7). Thus, we have $ch'(v) \geq ch(v) + 2 = 0$. If one of the vertices $x, u$ is a $3$-vertex, without loss of generality, we assume that $d(u) = 12$. By Claim 13, we have that $d_{3-}(v) \leq 2$. If $d_{2-}(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) - ch(u) \geq ch(v) - ch(u) + 1 = 0$. 

If $d_{6-}(v) = 1$, without loss of generality, we assume that $d(u) \leq 6$. Then $v$ is adjacent to three $7^+$-vertices. So $v$ receives at least $1 \times 3 = 3$, and $v$ gives at most $\frac{1}{2}$ away by Lemma 3 and (R2). Thus, we have $ch'(v) \geq ch(v) + 3 - \frac{1}{2} > 0$.

If $v$ is adjacent to four $7^+$-vertices, $v$ receives at least $1 \times 4 = 4$ from its $7^+$-neighbors by (R7) and gives no weight away. So we have $ch'(v) \geq ch(v) + 1 \times 4 > 0$.

Suppose $d(v) = 5$. If $v$ is not incident with any $4^+$-faces, then by (R4), $ch'(v) \geq ch(v) + \frac{5}{2} \times 5 = 0$. Otherwise, if $v$ is incident with at least one $4^+$-face, then by (R5) and (R6), $ch'(v) \geq ch(v) + 1 = 0$.

Suppose $d(v) = 6$. $v$ gives no weight away to any other vertex by the discharging rules. So $ch'(v) = ch(v) = 0$.

Suppose $d(v) = 7$. $v$ gives at most 1 to the false neighbor in $T^x$ by (R7), then $ch'(v) = ch(v) - 1 = 0$.

Suppose $d(v) = 8$. By Corollary 9, $ch'(v) = ch(v) - \max \{2, \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 9$. By Corollary 10, $ch'(v) = ch(v) - \max \{3, 1 + \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Suppose $d(v) = 10$. By Corollary 11, $ch'(v) = ch(v) - \max \{4, 2 + \frac{3}{2} \} \geq 0$ by (R1)–(R4) and (R7).

Next we check the final charge of the vertices with $d(v) \geq 11$. Let $w$ be a false vertex crossed by edge $uv$ and edge $xy$. According to the discharging rules, if $d(u) \leq 5$, then $v$ gives at most $d_{5-}(v) + \frac{1}{2}$ away. Otherwise, $v$ gives at most $d_{5-}(v) + 1$ away. Therefore, for every vertex $v$ with $d_{5-}(v) \geq 7$, $ch'(v) \geq 0$. In the following discussion, we only consider the vertex with $d(v) \geq 11$ and $d_{5-}(v) \leq 6$.

Suppose $d(v) = 11$. By Claim 12, we have that $d_{3-}(v) \leq 1$. If $d_{2-}(v) = 0$, then $d_3(v) \leq 1$. We have $ch'(v) = ch(v) - \max \{1 + d_3(v) + \left(\frac{11-1}{2}\right) - d_3(v)\} \times \frac{1}{2}$, $\frac{3}{2} + \left[\frac{11-1}{2}\right] \times \frac{1}{2} = 5 - \max \left\{\frac{2 + d_3(v)}{2}, \frac{1}{2}\right\} > 0$ by (R1)–(R4) and (R7). If $d_{2-}(v) = 1$, then $d_3(v) = 0$. We have $ch'(v) = ch(v) - 1 - \left[\frac{11-1}{2}\right] \times \frac{1}{2} = 3 - \frac{5}{2} > 0$ by (R3) and (R7).

Suppose $d(v) = 12$. By Claim 13, we have that $d_{3-}(v) \leq 2$.

If $d_{2-}(v) = 0$, then $d_3(v) \leq 2$. If $d_3(v) \geq 1$, we have $ch'(v) = ch(v) -
max \{1 + d_3(v) + \left(\frac{12}{2} - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\frac{12}{2} - (d_3(v) - 1)\right) \times \frac{1}{2}\} = 6 - 7 + d_3(v) > 0 \text{ by (R1)-(R4) and (R7). If } d_3(v) = 0, \text{ then we have } c_h'(v) = ch(v) - \max\left\{1 + d_3(v) + \left(\frac{12}{2} - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\frac{12}{2} - (d_3(v) - 1)\right) \times \frac{1}{2}\right\} > 0 \text{ by (R3) and (R7).}

If \(d_2(v) \geq 1\) and \(d_3(v) \geq 1\), we have \(c_h'(v) = ch(v) - \max\left\{1 + d_3(v) + \left(\frac{12 - d_2(v) - 1}{2} - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\frac{12 - d_2(v) - 1}{2} - (d_3(v) - 1)\right) \times \frac{1}{2}\right\} \geq \frac{9 - 3d_2(v) + d_3(v)}{4} > 0 \text{ by (R1)-(R4) and (R7). Otherwise } d_3(v) = 0, \text{ then } d_2(v) \leq 2. \text{ So } c_h'(v) = ch(v) - 1 - \left[\frac{12 - d_2(v) - 1}{2}\right] \times \frac{1}{2} \geq \frac{9 - 3d_2(v)}{4} > 0 \text{ by (R1)-(R4) and (R7).}

Suppose \(d(v) = 13\). By Claim 16, we have that \(d_3(v) \leq 5\). Moreover, if \(d_2(v) \geq 1\), then \(d_3(v) \leq 4 \leq 2. \text{ So we can easily obtain the following question.}

\textbf{Question 1.} Is it true that } ch''_{Σ}(G) \leq \Delta(G) + 3 \text{ for IC-planar graphs with } \Delta = 13?\n
4. \textbf{Remark}

By the definition of IC-planar graphs, we know that every planar graphs are special IC-planar graphs. In [13], the authors proved that \(ch''_{Σ}(G) \leq \max\{\Delta(G) + 3, 16\}. \text{ So we can easily obtain the following question.}

\textbf{Question 1.} Is it true that } ch''_{Σ}(G) \leq \Delta(G) + 3 \text{ for IC-planar graphs with } \Delta = 13?
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APPENDIX A

%% The m.file of Matlab to compute the coefficients.
% INPUT
function coefficients ()

syms x1 x2 x3 x4 x5 x6 x7 % Variables used in the following.

% Claim 3.7 % To calculate the coefficient of $x_1^6 x_2^4 x_3^4$
P=(x1−x2)*(x2−x3)*(x1−x3)ˆ2*(x1+x2)ˆ4*(x2+x3)ˆ6; % The polynomial

% Claim 3.8
P=(x1−x2)*(x2−x3)*(x1−x3)ˆ2*(x1+x2)ˆ5*(x2+x3)ˆ5;

% Claim 3.9
P=(x1−x2)*(x1−x3)*(x1−x4)*(x1−x5)*(x1−x6)*(x2−x3)*(x2−x4)*(x2−x5)
*(x2−x6)*(x3−x4)*(x3−x5)*(x3−x6)*...*(x4−x5)*(x4−x6)*(x5−x6)
*(x1+x2+x3+x4+x5+x6)ˆ6;

% Claim 3.10
P=(x1−x2)*(x1−x3)*(x1−x4)*(x1−x5)*(x1−x6)*(x2−x3)*(x2−x4)*(x2−x5)
*(x2−x6)*(x3−x4)*(x3−x5)*(x3−x6)*...*(x4−x5)*(x4−x6)*(x5−x6)
*(x1+x2+x3+x4+x5+x6)ˆ6;

P=(x1−x2)*(x1−x3)*(x1−x4)*(x1−x5)*(x1−x6)*(x2−x3)*(x2−x4)*(x2−x5)
*(x2−x6)*(x3−x4)*(x3−x5)*(x3−x6)*...*(x4−x5)*(x4−x6)*(x5−x6)
*(x1+x2+x3+x4+x5+x6)ˆ6;

cp1=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)/factorial(4)
/cp2=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)/factorial(4)
/cp3=diff(diff(diff(diff(diff(P,x1,6),x2,4),x3,4),x4,3),x5,2)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
/cp4=diff(diff(diff(diff(diff(diff(P,x1,6),x2,4),x3,4),x4,3),x5,2),x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)
/cp5=diff(diff(diff(diff(diff(diff(diff(P,x1,6),x2,4),x3,4),x4,3),x5,2),x6,1)/factorial(6)/factorial(5)/factorial(4)/factorial(3)

\[ \text{References} \]


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