NEIGHBOR SUM DISTINGUISHING TOTAL CHOOSABILITY OF IC-PLANAR GRAPHS

WEN-YAO SONG, LIAN-YING MIAO

School of Mathematics
China University of Mining and Technology
Xuzhou 221116, P.R. China

e-mail: songwenyao@cumt.edu.cn
miaolianying@cumt.edu.cn

AND

YUAN-YUAN DUAN

School of Mathematics and Statistics
Zaozhuang University
Zaozhuang 277160, P.R. China

e-mail: duanyy0827@sina.com

Abstract

Two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane such that every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short. A proper total-$k$-coloring of a graph $G$ is a mapping $c : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_e(c(v))$ denote the sum of the color of a vertex $v$ and the colors of all incident edges of $v$. A total-$k$-neighbor sum distinguishing-coloring of $G$ is a total-$k$-coloring of $G$ such that for each edge $uv \in E(G)$, $\sum_e(c(u)) \neq \sum_e(c(v))$. The least number $k$ needed for such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. In this paper, it is proved that if $G$ is an IC-planar graph with maximum degree $\Delta(G)$, then $\text{ch}''_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$, where $\text{ch}''_{\Sigma}(G)$ is the neighbor sum distinguishing total choosability of $G$.

Keywords: neighbor sum distinguishing total choosability, maximum degree, IC-planar graph, Combinatorial Nullstellensatz.

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1. Introduction

All graphs considered are finite, simple and undirected. Let $G$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For planar graph $G$, $F(G)$ denotes its face set, $d(v)$ denotes the degree of a vertex $v$ in $G$. The length or degree of a face $f$, denoted by $d(f)$, is the length of the boundary walk of $f$ in $G$. We call $v$ a $k$-vertex, or a $k^+$-vertex, or a $k^-$-vertex if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call $f$ a $k$-face, or a $k^+$-face, or a $k^-$-face if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A proper total-$k$-coloring of a graph $G$ is a mapping $c: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two adjacent elements in $V(G) \cup E(G)$ receive different colors. Let $\sum_c(v)$ be the sum of the color of a vertex $v$ and the colors of all edges incident with $v$. If for each edge $uv \in E(G)$, $\sum_c(u) \neq \sum_c(v)$, then we say such total-$k$-coloring a neighbor sum distinguishing total-$k$-coloring, denoted by $\text{tnsd}-k$-coloring for short. The least number $k$ needed for such a coloring of $G$ is the neighbor sum distinguishing total chromatic number, denoted by $\chi^n_{\Sigma}(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture proposed by Pilśniak and Woźniak [11].

**Conjecture 1.** For any graph $G$, $\chi^n_{\Sigma}(G) \leq \Delta(G) + 3$.

Loeb and Tang [10] proved that this bound was asymptotically correct by showing that $\chi^n_{\Sigma}(G) \leq \Delta(G)(1 + o(1))$. Pilśniak and Woźniak [11] proved that Conjecture 1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. With the Combinatorial Nullstellensatz, neighbor sum distinguishing total coloring have been studied widely, see [4–6, 8, 9, 12, 19].

For a given graph $G$, let $L_x (x \in V \cup E)$ be a set of lists of real numbers and each of size $k$. The neighbor sum distinguishing total choosability of $G$ is the least number $k$ for which for any specified collection of such lists, there exists a neighbor sum distinguish total coloring with colors from $L_x$ for each $x \in V \cup E$, and we denote it by $\text{chn}_{\Sigma}(G)$. We call such a coloring of $G$ list neighbor sum distinguish total-$k$-coloring and denote it by $\text{ltnsd}-k$-coloring. Ding et al. [4] proved that for any graph $G$, $\text{chn}_{\Sigma}(G) \leq 2\Delta(G) + \text{col}(G) - 1$, where $\text{col}(G)$ is the coloring number of $G$. Later Ding et al. [5] improved the bound to $\text{chn}_{\Sigma}(G) \leq 2\Delta(G) + \text{col}(G) - 2$. Recently, Lu et al. [20] improved the bound to $\text{chn}_{\Sigma}(G) \leq \max\{\Delta(G) + \left\lfloor \frac{3\text{col}(G)}{2} \right\rfloor - 1, 3\text{col}(G) - 2\}$. The list neighbor sum distinguish total-$k$-coloring of some special classes of graphs were also investigated. Graphs with bounded maximum average degree (Yao and Kong [16]); $d$-degenerate graphs (Yao et al. [18]); planar graphs (Qu et al. [13], Wang et al. [15]).

In this paper, we consider IC-planar graphs and prove the following result.
Theorem 2. Let $G$ be an IC-planar graph with maximum degree $\Delta(G)$. Then $\text{ch}^\prime_{\Sigma}(G) \leq \max\{\Delta(G) + 3, 17\}$.

An IC-plane graph is a topological graph where every edge is crossed at most once and no two crossed edges share a vertex, i.e., two distinct crossings are independent if the end-vertices of the crossed pair of edges are mutually different. If a graph $G$ has a drawing in the plane in which every two crossings are independent, then we call $G$ a plane graph with independent crossings or IC-planar graph for short throughout this paper. This definition of IC-planar graph was introduced by Albertson [1] in 2008. Setting a conjecture of Albertson [1], Král and Stacho [7] showed that every IC-planar graph is 5-colorable. Obviously, every IC-planar graph also is a 1-planar graph. We call $G$ a 1-planar graph if it can be drawn on a plane such that each edge is crossed by at most one other edge.

2. Preliminaries

Every IC-planar graph $G$ in this paper has been embedded on a plane such that all its crossings are independent and the number of crossings is as small as possible. In other words, we call $G$ an IC-plane graph. The associated plane graph $G^\times$ of $G$ is obtained by turning all crossings of $G$ into new 4-vertices on a plane. For convenience, a vertex in $G^\times$ is called false if it is not a vertex of $G$ and real otherwise. For a vertex $v \in V(G)$, we use $d_i(v)$ to denote the number of $i$-vertices which are adjacent to $v$. One can see that every real vertex in $G^\times$ is adjacent to at most one false vertex and incident with at most two false faces in $G^\times$.

Lemma 3 [17]. Let $G$ be a 1-plane graph and $G^\times$ be its associated plane graph. If $d_G(u) = 3$ and $v$ is a crossing vertex in $G^\times$, then either $uv \notin E(G^\times)$ or $uv$ is not incident with two 3-faces.

We define that a graph $G'$ is smaller than a graph $G$ if $|E(G')| < |E(G)|$. We call a graph minimal for a property when no smaller graph satisfies it. Let from now on $G = (V,E)$ be a minimal counterexample to Theorem 2. We set $k = \max\{\Delta(G) + 3, 17\}$. For each 5-vertex $v \in V(G)$, it is obvious that $v$ has at most five neighbors and five incident edges, so $v$ has at most 15 forbidden colors. Since $k \geq 17$, we can first erase the color of vertex $v$ and finally recolor it after arguing. In other words, we may omit the coloring for all 5-vertices of $G$ in the following discussion.

Theorem 4 (Combinatorial Nullstellensatz [2]). Let $\mathbb{F}$ be an arbitrary field, and let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose the degree $\text{deg}(P)$ of $P$ equals $\sum_{i=1}^{n} k_i$, where each $k_i$ is a nonnegative integer, and suppose the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ in $P$ is non-zero. Then if $S_1, S_2, \ldots, S_n$
are subsets of $F$ with $|S_i| > k_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.

**Lemma 5** [14]. If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\leq s_1 + s_2 + \cdots + s_n$, where $s_1, s_2, \ldots, s_n$ are nonnegative integers, then

$$\left(\frac{\partial}{\partial x_1}\right)^{s_1}\left(\frac{\partial}{\partial x_2}\right)^{s_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{s_n} P(x_1, x_2, \ldots, x_n)$$

$$= \sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \left(\frac{s_1}{x_1}\right) \cdots (-1)^{s_n+x_n} \left(\frac{s_n}{x_n}\right) P(x_1, x_2, \ldots, x_n).$$

**Lemma 6** [13]. Let $L_i$ be the sets of real numbers, with $|L_i| = l_i$, where $i = 1, 2, \ldots, p$. Let $L = \left\{ \sum_{i=1}^{p} x_i \mid x_i \in L_i \text{ and } \prod_{1 \leq i < j \leq p} (x_i - x_j) \neq 0 \right\}$. Then $|L| \geq \sum_{i=1}^{p} (l_i - p + 1) - (p - 1) = \sum_{i=1}^{p} l_i - p^2 + 1$.

3. Proof of Theorem 2

**3.1. Unavoidable configurations**

In the following, we will often delete some edges to get a proper subgraph $G'$ of $G$, then by the minimality of $G$, there exists an ltnsd-$k$-coloring $c$ of $G'$. Let $W_G(v) = \sum_{e \ni v, e \in E(G)} c(e) + c(v)$. We may extend this coloring $c$ to the whole graph $G$. For any $x \in V(G) \cup E(G)$, the available colors are the remaining colors after excluding the colors of its adjacent edges and vertices in $G'$ from $L_x$.

**Claim 7.** For any vertex $v \in V(G)$, it holds that

$$\sum_{j=1}^{t} [d_j(v)(\Delta(G) + 4 - d(v) - j)] \leq d(v) - 1, \quad (1 \leq t \leq 5).$$

**Claim 8.** For any vertex $v \in V(G)$, $d_{2-}(v) \leq \frac{d_{d+}(v)-1}{\Delta(G)-d(v)+1}$. Moreover, if $d(v) = \Delta(G)$, then $d_{2-}(v) \leq d_{d+}(v) - 1$.

The proof of Claim 7 and 8 are similar to that of Claim 3.1 and Claim 3.2 in [13], we omit it here. By Claim 7, we can easily get the following Corollaries.

**Corollary 9.** If $d(v) = 8$, then $d_{5-}(v) \leq 1$.

**Corollary 10.** If $d(v) = 9$, then $d_{5-}(v) \leq 2$.

**Corollary 11.** If $d(v) = 10$, then $d_{5-}(v) \leq 3$.

**Claim 12.** If $d(v) = 11$ and $d_{6+}(v) \leq 6$, then $d_{3-}(v) \leq 1$. 
**Proof.** Suppose to the contrary that \( v \) is adjacent to two 3-vertices. Without loss of generality, we assume that \( N(v) = \{v_1, v_2, \ldots, v_{11}\} \), \( d(v_1) = d(v_2) = 3 \) and \( d(v_j) \geq 6, (6 \leq j \leq 11) \). Consider \( G' = G - vv_1 - vv_2 \), then \( G' \) admits an ltnsd-\( k \)-coloring \( c \). Now we will color the edges \( vv_1, vv_2 \) and recolor vertices \( v_1, v_2 \). Let \( S_1, S_2 \) be the sets of available colors for \( vv_1, vv_2 \), respectively. It is easy to obtain that \( |S_1| = 17 - 12 = 5 \), \( |S_2| = 17 - 12 = 5 \). We can choose a pair, say \((x, y) \in S_1 \times S_2 \) with \( x \neq y \), such that \( x + y \notin \{W_G(v_j) - W_G(v) | 6 \leq j \leq 11\} \). Finally, we can recolor \( v_1, v_2 \) to get an ltnsd-\( k \)-coloring of \( G \), a contradiction.

**Claim 13.** If \( d(v) = 12 \) and \( d_{6+}(v) \leq 6 \), then \( d_{3-}(v) \leq 2 \).

**Proof.** Suppose to the contrary that \( v \) is adjacent to three 3-vertices. Without loss of generality, we assume that \( N(v) = \{v_1, v_2, \ldots, v_{12}\} \), \( d(v_1) = d(v_2) = d(v_3) = 3 \) and \( d(v_j) \geq 6, (7 \leq j \leq 12) \). Consider \( G' = G - \{v_i | i = 1, 2, 3\} \), then \( G' \) admits an ltnsd-\( k \)-coloring \( c \). Now we will color the edges \( vv_1, vv_2, vv_3 \) and recolor vertices \( v_1, v_2, v_3 \). Let \( S_1, S_2, S_3 \) be the sets of available colors for \( vv_1, vv_2, vv_3 \), respectively. It is easy to obtain that \( |S_1| = 17 - 11 = 6, (1 \leq i \leq 3) \). By Lemma 6, \( |L| \geq |S_1| + |S_2| + |S_3| - 9 + 1 = 7 > 6 \). We can choose a triple, say \((x, y, z) \in S_1 \times S_2 \times S_3 \) with \( x, y, z \) distinct colors, such that \( x + y + z \notin \{W_G(v_j) - W_G(v) | 7 \leq j \leq 12\} \). Finally, we can recolor \( v_1, v_2, v_3 \) to get an ltnsd-\( k \)-coloring of \( G \), a contradiction.

By Lemma 5, if \( P(x_1, x_2, \ldots, x_n) \) is a polynomial with \( \text{deg}(P) = n, k_1, k_2, \ldots, k_m \) are non-negative integers with \( \sum_{i=1}^m k_i = n \) and \( \text{cp} \left( x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \right) \) is the coefficient of \( x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \) in \( P \), then \( \frac{\partial^{\alpha} P}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} = \text{cp} \left( x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \right) \prod_{i=1}^m k_i! \). In the following, we use MATLAB to calculate the coefficients of specific monomials. Moreover, we will list the codes in Appendix.

**Claim 14.** Every 5-vertex is not adjacent to 7-vertex in \( G \).

**Proof.** Suppose to the contrary that there exists a 5-vertex \( u \) adjacent to a 7-vertex \( v \). Without loss of generality, we assume that \( d(u) = 5, d(v) = 7 \), \( N(u) = \{v, u_1, \ldots, u_4\} \), \( N(v) = \{u, v_1, \ldots, v_6\} \). Consider \( G' = G - uv \), then \( G' \) admits an ltnsd-\( k \)-coloring \( c \). Now we will recolor the vertices \( u, v \) and color the edge \( uv \). Let \( S_1, S_2, S_3 \) be the sets of available colors for \( u, uv, v \), respectively. Notice that the colors in \( \{c(u_i) | 1 \leq i \leq 4\} \cup \{c(u_i) | 1 \leq i \leq 4\} \) are forbidden for \( u \), the colors in \( \{c(u_i) | 1 \leq i \leq 4\} \cup \{c(v_i) | 1 \leq i \leq 6\} \) are forbidden for \( uv \), and the colors in \( \{c(v_i) | 1 \leq i \leq 6\} \cup \{c(v_i) | 1 \leq i \leq 6\} \) are forbidden for \( v \). Thus, \( |S_1| = 17 - 8 = 9 > 8, |S_2| = 17 - 10 = 7 > 6, |S_3| = 17 - 12 = 5 > 4 \). We associate that \( u, uv, v \) with the variables \( x_1, x_2, x_3 \), respectively. Then we
consider the following polynomial.

\[
P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)
\left( x_2 - x_3 + \sum_{i=1}^{4} (x_i + \sum_{l=1}^{4} c(uu_l) - x_3 - \sum_{k=1}^{6} c(vv_k) \right)
\]

\[
\prod_{i=1}^{4} \left( x_1 + x_2 + \sum_{l=1}^{4} c(uu_l) - W(u_i) \right)
\]

\[
\prod_{j=1}^{6} \left( x_2 + x_3 + \sum_{k=1}^{6} c(vv_k) - W(v_j) \right).
\]

We have \( cp(x_1^6 x_2^4 x_3^4) = -25 \). According to Theorem 4, there exists \( x_i \in S_i \), \((1 \leq i \leq 3)\) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-\( k \)-coloring of the graph \( G \), a contradiction. \( \blacksquare \)

**Claim 15.** Every 6\(^{-}\)-vertex is not adjacent to 6\(^{-}\)-vertex in \( G \).

**Proof.** Suppose to the contrary that there exists a 6\(^{-}\)-vertex \( u \) adjacent to a 6\(^{-}\)-vertex \( v \). Without loss of generality, we assume that \( d(u) = 6, d(v) = 6, N(u) = \{v, u_1, \ldots, u_5\}, N(v) = \{u, v_1, \ldots, v_5\} \). Consider \( G' = G - uv \), then \( G' \) admits an ltnsd-\( k \)-coloring \( c \). Now we will recolor the vertices \( u, v \) and color the edge \( uv \). Let \( S_1, S_2, S_3 \) be the sets of available colors for \( u, uv, v \), respectively. Notice that the colors in \{\( c(uu_i) | 1 \leq i \leq 5 \} \cup \{\( c(u_i) | 1 \leq i \leq 5 \} \) are forbidden for \( u \), the colors in \{\( c(uu_i) | 1 \leq i \leq 5 \} \cup \{\( c(vv_i) | 1 \leq i \leq 5 \} \) are forbidden for \( uv \), and the colors in \{\( c(vv_i) | 1 \leq i \leq 5 \} \cup \{\( c(v_i) | 1 \leq i \leq 5 \} \) are forbidden for \( v \). Thus, \( |S_1| = 17 - 10 = 7 > 6, |S_2| = 17 - 10 = 7 > 6, |S_3| = 17 - 10 = 7 > 6 \).

We associate that \( u, uv, v \) with the variables \( x_1, x_2, x_3 \), respectively. Then we consider the following polynomial.

\[
P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)
\left( x_3 + \sum_{i=1}^{5} c(uu_i) - x_3 - \sum_{k=1}^{5} c(vv_k) \right)
\]

\[
\prod_{i=1}^{5} \left( x_1 + x_2 + \sum_{l=1}^{5} c(uu_l) - W(u_i) \right)
\]

\[
\prod_{j=1}^{5} \left( x_2 + x_3 + \sum_{k=1}^{5} c(vv_k) - W(v_j) \right).
\]

We have \( cp(x_1^6 x_2^4 x_3^4) = -20 \). According to Theorem 4, there exists \( x_i \in S_i \), \((1 \leq i \leq 3)\) such that \( P(x_1, x_2, x_3) \neq 0 \). We color \( u, uv, v \) correspondingly. Finally, we can get an ltnsd-\( k \)-coloring of the graph \( G \), a contradiction. \( \blacksquare \)

**Claim 16.** Let \( d(v) = 13 \) and \( d_6 + (v) \leq 6 \), then \( d_3 - (v) \leq 5 \). Moreover, if \( d_2 - (v) \geq 1 \), then \( d_3 - (v) \leq 4 \).
Proof. Suppose to the contrary that there exists a 13-vertex \( v \) adjacent to six 3\(^{-}\)-vertices. Without loss of generality, assume that \( N(v) = \{v_1, v_2, \ldots, v_{13}\} \), \( d(v_1) = 3, (1 \leq i \leq 6) \) and \( d(v_j) \geq 6, (8 \leq j \leq 13) \). Consider \( G' = G - \{vv_i \mid i = 1, 2, \ldots, 6\} \), then \( G' \) admits an ltnsd-k-coloring \( c \). Now we will color the edges \( vv_i \) and recolor vertices \( v_i, (1 \leq i \leq 6) \). Let \( S_i, (1 \leq i \leq 6) \) be the sets of available colors for \( v_i, (1 \leq i \leq 6) \), respectively. It is easy to obtain that \( |S_i| = 17 - 7 - 1 - 2 = 7 > 6, (1 \leq i \leq 6) \). We associate that \( vv_i, (1 \leq i \leq 6) \) with the variables \( x_i, (1 \leq i \leq 6) \), respectively. Then we consider the following polynomial.

\[
P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{t=1}^{6} x_t + \sum_{l=7}^{13} c(vv_t) + c(v) - W(v_t) \right).
\]

We have \( cp(x_1^6x_2^5x_3^4x_4^3x_5^2x_6^1) = 1 \). According to Theorem 4, there exists \( x_i \in S_i, (1 \leq i \leq 6) \) such that \( P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0 \). We color \( vv_i, (1 \leq i \leq 6) \) correspondingly. Finally, we can recolor vertices \( v_i, (1 \leq i \leq 6) \) to get an ltnsd-k-coloring of the graph \( G \), a contradiction.

Moreover, if \( d(v_1) = 2, \ d(v_2) = 3, (2 \leq i \leq 5) \) and \( d(v_j) \geq 6, (8 \leq j \leq 13) \). Consider \( G' = G - \{vv_i \mid i = 1, 2, \ldots, 5\} \), then \( G' \) admits an ltnsd-k-coloring \( c \). Now we will color the edges \( vv_i \) and recolor vertices \( v_i, (1 \leq i \leq 5) \). Let \( S_i, (1 \leq i \leq 5) \) be the sets of available colors for \( vv_i, (1 \leq i \leq 5) \), respectively. It is easy to obtain that \( |S_i| = 17 - 8 - 1 - 1 = 7 > 6, |S_i| = 17 - 8 - 1 - 2 = 6 > 5, (2 \leq i \leq 5) \). We associate that \( vv_i, (1 \leq i \leq 5) \) with the variables \( x_i, (1 \leq i \leq 5) \), respectively. Then we consider the following polynomial.

\[
P(x_1, x_2, x_3, x_4, x_5) = \prod_{1 \leq i < j \leq 5} (x_i - x_j) \prod_{k=8}^{13} \left( \sum_{t=1}^{5} x_t + \sum_{l=6}^{13} c(vv_t) + c(v) - W(v_t) \right).
\]

We have \( cp(x_1^6x_2^4x_3^3x_4^2x_5^1) = -5 \). According to Theorem 4, there exists \( x_i \in S_i, (1 \leq i \leq 5) \) such that \( P(x_1, x_2, x_3, x_4, x_5) \neq 0 \). We color \( vv_i, (1 \leq i \leq 5) \) correspondingly. Finally, we can recolor vertices \( v_i, (1 \leq i \leq 5) \) to get an ltnsd-k-coloring of the graph \( G \), a contradiction.

Claim 17. Let \( d(v) = \Delta(G) \geq 14 \) and \( d_{6+}(v) \leq 6 \). If \( d_{2-}(v) \geq 1, \) then \( d_{3-}(v) \leq 5 \).

Proof. Let \( d(v) = d \). Suppose to the contrary that there exists a \( d \)-vertex \( v \) adjacent to six 3\(^{-}\)-vertices. Without loss of generality, assume that \( N(v) = \{v_1, v_2, \ldots, v_6\} \), \( d(v_1) = 2, \ d(v_i) = 3, (2 \leq i \leq 6) \) and \( d(v_j) \geq 6, (d - 5 \leq j \leq d) \). Consider \( G' = G - \{vv_i \mid i = 1, 2, \ldots, 6\} \), then \( G' \) admits an ltnsd-k-coloring \( c \). Now we will color the edges \( vv_i \) and recolor vertices \( v_i, (1 \leq i \leq 6) \). Let \( S_i, (1 \leq i \leq 6) \) be the sets of available colors for \( vv_i (1 \leq i \leq 6) \), respectively.
It is easy to obtain that $|S_1| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 - 1 = 7 > 6$, $|S_i| = (\Delta(G) + 3) - (\Delta(G) - 6) - 1 - 2 = 6 > 5$, $(2 \leq i \leq 6)$. We associate that $vv_i$, $(1 \leq i \leq 6)$ with the variables $x_i$, $(1 \leq i \leq 6)$, respectively. Then we consider the following polynomial.

\[
P(x_1, x_2, x_3, x_4, x_5, x_6) = \prod_{1 \leq i < j \leq 6} (x_i - x_j) \prod_{k=d-5}^{d} \left( \sum_{t=1}^{6} x_t + \sum_{l=7}^{d} c(vv_l) + c(v) - W(v_k) \right).
\]

We have $x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6^1 = 1$. According to Theorem 4, there exists $x_i \in S_i$, $(1 \leq i \leq 6)$ such that $P(x_1, x_2, x_3, x_4, x_5, x_6) \neq 0$. We color $vv_i$, $(1 \leq i \leq 6)$ correspondingly. Finally, we can recolor vertices $vv_i$, $(1 \leq i \leq 6)$ to get an ltnsd-$k$-coloring of the graph $G$, a contradiction.

\section{Discharging process}

Let $T$ be the graph obtained by removing all $2^-$-vertices from the graph $G$ and $T^\times$ be the associated plane graph of $T$. We have $d_T(v) = d(v) - d_{2^+}(v)$.

\begin{corollary}
For any vertex $v$ with $d(v) \geq 7$, it holds that $d_T(v) \geq 7$.
\end{corollary}

\begin{proof}
If $7 \leq d(v) \leq 10$, we can easily get $d_T(v) \geq 7$ by Claim 14 and Corollaries 9–11. When $d(v) > 10$, we just consider the situation $d_{2^+}(v) \leq 6$. By Claim 8, $d_T(v) = d(v) - d_{2^+}(v) \geq d(v) - \frac{d_{2^+}(v) - 1}{\Delta(G) - d(v) + 1} \geq 11 - \frac{5}{14 - 4} \geq 9$.
\end{proof}

We apply the discharging method on associated plane graph $T^\times$ of $T$ and complete the proof by contradiction. Since $T^\times$ is a plane graph, we have

\[
\sum_{v \in V(T^\times)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)
\]

\[
= \sum_{v \in V(T)} (d_T(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)
\]

\[
= \sum_{v \in V(T)} (d_T(v) - d_{2^+}(v) - 6) + \sum_{v \in V(T^\times) \setminus V(T)} (d_{T^\times}(v) - 6) + \sum_{f \in F(T^\times)} (2d_{T^\times}(f) - 6)
\]

\[
= -12.
\]

Now we define the initial charge function $ch(x)$ of $x \in V(T^\times) \cup F(T^\times)$. Let $ch(v) = d_T(v) - 6 = d(v) - d_{2^+}(v) - 6$ if $v \in V(T)$, $ch(v) = d_{T^\times}(v) - 6$ if $v \in V(T^\times) \setminus V(T)$ and $ch(f) = 2d_{T^\times}(f) - 6$ if $f \in F(T^\times)$. Then we define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(T^\times) \cup F(T^\times)$ such that $ch'(x) \geq 0$ for all $x \in V(T^\times) \cup F(T^\times)$. Notice that our discharging rules only move charge around and do not affect the sum. Thus we have $0 \leq \sum_{x \in V(T^\times) \cup F(T^\times)} ch'(x) =$
\[
\sum_{x \in V(T^+) \cup F(T^+)} ch(x) = -12, \text{ a contradiction. Since for every vertex } v \in V(T), \\
ch(v) = d_G(v) - d_{2^-}(v) - 6, \text{ in the discharging process, we use } d_G(v) \text{ instead of } d_T(v). \text{ Similarly, for every vertex } v \in V(T), \text{ when check } ch'(v) \geq 0, \text{ we split the proof into cases depending on the size of } d_G(v).
\]

For \( v \in V(T^+) \) and \( f \in F(T^+) \), we define the discharging rules as follows. Note that within all the degree of a real vertex shall refer to its degree in \( G \) and the faces and their degrees correspond to the graph \( T^+ \).

(R1): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 3 \), then \( u \) sends 1 to \( v \).
(R2): If the edge \( uv \) belongs to exactly one 3-face and \( d(v) = 3 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).
(R3): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 4 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).
(R4): If the edge \( uv \) belongs to two 3-faces and \( d(v) = 5 \), then \( u \) sends \( \frac{1}{3} \) to \( v \).
(R5): Every 4-face sends 1 to each incident real 5-vertex in \( T^+ \).
(R6): Every 5+-face sends 2 to each incident real 5^-vertex in \( T^+ \).
(R7): Let \( v \) be a false vertex crossed by edge \( uw \) and \( xy \) in \( T^+ \). If \( d(u) \geq 7 \), then \( u \) sends 1 to \( v \). Moreover, if \( d(w) = 3 \), then \( u \) sends \( \frac{1}{2} \) to \( v \).

By Corollary 18 and the discharging rules, we obtain the following facts easily.

**Fact 1.** For any \( f \in F(T^+) \), \( f \) is incident with at most \( \left\lfloor \frac{d(f)}{2} \right\rfloor \) real 5^-vertices in \( T^+ \).

**Fact 2.** Each vertex \( v \) gives at most \( \frac{d_{2^+}(v)}{2} + 1 \) away.

Let \( f \) be a face of \( T^+ \). Clearly, if \( d(f) = 3 \), then \( ch'(f) = ch(f) = 2d(f) - 6 = 0 \). If \( d(f) = 4 \), then \( ch'(f) \geq ch(f) - 2 = 0 \) by Fact 1 and (R5). If \( d(f) \geq 5 \), then \( ch'(f) \geq ch(f) - \left\lfloor \frac{d(f)}{2} \right\rfloor \times 2 = 0 \) by Fact 1 and (R6).

We next check the final charge of the vertex \( v \in V(T^+) \). Obviously, \( d(v) \geq 3 \). Recall that \( v \) has an initial weight of \( d(v) - d_{2^-}(v) - 6 \).

Suppose \( d(v) = 3 \). If \( v \) is incident with three 3-faces, then \( ch'(v) \geq ch(v) + 3 = 0 \) by (R1). If \( v \) is incident with two 3-faces, then \( ch'(v) \geq ch(v) + 1 + \frac{1}{2} \times 2 + 1 = 0 \) by (R1), (R2), (R5) and (R6). If \( v \) is incident with one 3-face, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 2 = 0 \) by (R2), (R5) and (R6). Otherwise, \( v \) is incident with three 4+-faces, then \( ch'(v) \geq ch(v) + 1 \times 3 = 0 \) by (R5) and (R6).

Suppose \( d(v) = 4 \) and \( v \) is a real vertex. We have \( d_{2^-}(v) = 0 \). If \( v \) is incident with four 3-faces, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 4 = 0 \) by (R3). If \( v \) is incident with three 3-faces, then \( ch'(v) \geq ch(v) + \frac{1}{2} \times 2 + 1 = 0 \) by (R3), (R5) and (R6). If \( v \) is incident with at most two 3-faces, then \( ch'(v) \geq ch(v) + 1 \times 2 = 0 \) by (R5) and (R6).

Suppose \( d(v) = 4 \) and \( v \) is a false vertex crossed by edge \( uw \) and \( xy \). By Claim 14 and 15, \( v \) is adjacent to at most two 6^-vertices.
If \( d_{6}^{-}(v) = 2 \), without loss of generality, we assume that \( d(u) \leq 6 \) and \( d(x) \leq 6 \). By Claim 15, if \( 4 \leq d(u) \leq 6 \) and \( 4 \leq d(x) \leq 6 \), then \( wx \notin E(T) \), \( v \) gives no weight away by (R3) and (R4). By the same claim, \( v \) is also adjacent to two \( 7^{+} \)-vertices. So \( v \) receives at least \( 1 \times 2 = 2 \) from its \( 7^{+} \)-neighbors by (R7). Thus, we have \( ch'(v) \geq ch(v) + 2 = 0 \). If one of the vertices \( x, u \) is a 3-vertex, without loss of generality, we assume that \( d(u) = 3 \). Then, by Claim 14, \( w \) is a \( 8^{+} \)-vertex. \( v \) may receives \( \frac{3}{2} \) from vertex \( w \) and 1 from vertex \( y \) by (R7) and gives at most \( \frac{3}{2} \) away by (R3) and (R4). Thus, we have \( ch'(v) \geq ch(v) + \frac{3}{2} + 1 - \frac{1}{2} = 0 \). Otherwise, \( d(u) = d(x) = 3 \). By Claim 14, \( v \) receives at least \( \frac{3}{2} \times 2 = 3 \) from its \( 8^{+} \)-neighbors by (R7). And \( v \) gives at most \( \frac{1}{2} \times 2 = 1 \) away by Lemma 3, Claim 15 and (R2). Thus, we have \( ch'(v) \geq ch(v) + 3 - 1 = 0 \).

If \( d_{6}^{-}(v) = 1 \), without loss of generality, we assume that \( d(u) \leq 6 \). Then \( v \) is adjacent to three \( 7^{+} \)-vertices. So \( v \) receives at least \( 1 \times 3 = 3 \), and \( v \) gives at most \( \frac{1}{2} \) away by Lemma 3 and (R2). Thus, we have \( ch'(v) \geq ch(v) + 3 - \frac{1}{2} > 0 \).

If \( v \) is adjacent to four \( 7^{+} \)-vertices, \( v \) receives at least \( 1 \times 4 = 4 \) from its \( 7^{+} \)-neighbors by (R7) and gives no weight away. So we have \( ch'(v) \geq ch(v) + 1 \times 4 > 0 \).

Suppose \( d(v) = 5 \). If \( v \) is not incident with any \( 4^{+} \)-faces, then by (R4), \( ch'(v) \geq ch(v) + \frac{5}{2} \times 5 = 0 \). Otherwise, if \( v \) is incident with at least one \( 4^{+} \)-faces, then by (R5) and (R6), \( ch'(v) \geq ch(v) + 1 = 0 \).

Suppose \( d(v) = 6 \). \( v \) gives no weight away to any other vertex by the discharging rules. So \( ch'(v) = ch(v) = 0 \).

Suppose \( d(v) = 7 \). \( v \) gives at most 1 to the false neighbor in \( T^x \) by (R7), then \( ch'(v) = ch(v) - 1 = 0 \).

Suppose \( d(v) = 8 \). By Corollary 9, \( ch'(v) = ch(v) - \max \{2, \frac{3}{2} \} \geq 0 \) by (R1)–(R4) and (R7).

Suppose \( d(v) = 9 \). By Corollary 10, \( ch'(v) = ch(v) - \max \{3, 1 + \frac{3}{2} \} \geq 0 \) by (R1)–(R4) and (R7).

Suppose \( d(v) = 10 \). By Corollary 11, \( ch'(v) = ch(v) - \max \{4, 2 + \frac{3}{2} \} \geq 0 \) by (R1)–(R4) and (R7).

Next we check the final charge of the vertices with \( d(v) \geq 11 \). Let \( w \) be a false vertex crossed by edge \( uw \) and edge \( xy \). According to the discharging rules, if \( d(u) \leq 5 \), then \( v \) gives at most \( d_{5}^{-}(v) + \frac{1}{2} \) away. Otherwise, \( v \) gives at most \( d_{5}^{-}(v) + 1 \) away. Therefore, for every vertex \( v \) with \( d_{6}^{-}(v) \geq 7 \), \( ch'(v) \geq 0 \). In the following discussion, we only consider the vertex with \( d(v) \geq 11 \) and \( d_{6}^{-}(v) \leq 6 \).

Suppose \( d(v) = 11 \). By Claim 12, we have that \( d_{3}^{-}(v) \leq 1 \). If \( d_{2}^{-}(v) = 0 \), then \( d_{3}(v) \leq 1 \). We have \( ch'(v) = ch(v) - \max \{1 + d_{3}(v) + (\lfloor \frac{11-1}{2} \rfloor - d_{3}(v)) \times \frac{1}{2}, \frac{3}{2} + \lfloor \frac{11-1}{2} \rfloor \times \frac{1}{2} \} = 5 - \max \{ \frac{11}{2}, 4 \} > 0 \) by (R1)–(R4) and (R7). If \( d_{2}^{-}(v) = 1 \), then \( d_{3}(v) = 0 \). We have \( ch'(v) = ch(v) - 1 - \lfloor \frac{11-1}{2} \rfloor \times \frac{1}{2} = 3 - \frac{5}{2} > 0 \) by (R3) and (R7).

Suppose \( d(v) = 12 \). By Claim 13, we have that \( d_{3}^{-}(v) \leq 2 \).

If \( d_{2}^{-}(v) = 0 \), then \( d_{3}(v) \leq 2 \). If \( d_{3}(v) \geq 1 \), we have \( ch'(v) = ch(v) -
max \{1 + d_3(v) + \left(\left\lfloor \frac{12-1}{2} \right\rfloor - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{12-1}{2} \right\rfloor - (d_3(v) - 1)\right) \times \frac{1}{2} \} = 6 - 7 + d_3(v) > 0 \text{ by (R1)-(R4) and (R7). If } d_3(v) = 0, \text{ then we have } ch'(v) = ch(v) - 1 - \left\lfloor \frac{12-1}{2} \right\rfloor \times \frac{1}{2} > 0 \text{ by (R3) and (R7).}

If \( d_2- (v) \geq 1 \) and \( d_3(v) \geq 1 \), we have \( ch'(v) = ch(v) - \max \left\{1 + d_3(v) + \left(\left\lfloor \frac{12-d_2(v)-1}{2} \right\rfloor - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{12-d_2(v)-1}{2} \right\rfloor - (d_3(v) - 1)\right) \times \frac{1}{2} \right\} \geq 9 - 3d_3(v) > 0 \text{ by (R1)-(R4) and (R7). Otherwise } d_3(v) = 0, \text{ then } d_2- (v) \leq 2. \text{ So } ch'(v) = ch(v) - 1 - \left\lfloor \frac{12-d_2(v)-1}{2} \right\rfloor \times \frac{1}{2} \geq \frac{9-3d_2(v)}{4} > 0 \text{ by (R1)-(R4) and (R7).}

Suppose \( d(v) = 13 \). By Claim 16, we have that \( d_3- (v) \leq 5 \). Moreover, if \( d_2- (v) \geq 1 \), then \( d_3- (v) \leq 4 \).

If \( d_2- (v) = 0 \), then \( d_3(v) \leq 5 \). If \( d_3(v) \geq 1 \), then we have \( ch'(v) = ch(v) - \max \left\{1 + d_3(v) + \left(\left\lfloor \frac{13-d_2(v)-1}{2} \right\rfloor - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{13-d_2(v)-1}{2} \right\rfloor - (d_3(v) - 1)\right) \times \frac{1}{2} \right\} \geq 3 - \frac{3d_3(v)}{4} + \frac{d_3(v)}{4} > 0 \text{ by (R1)-(R4) and (R7). Otherwise } d_3(v) = 0, \text{ then } d_2- (v) \leq 4. \text{ So } ch'(v) = ch(v) - 1 - \left(\left\lfloor \frac{13-1-d_2(v)}{2} \right\rfloor \times \frac{1}{2} \right) \geq 3 - \frac{3d_3(v)}{4} \geq 0 \text{ by (R1)-(R4) and (R7).}

Suppose \( d(v) = \Delta(G) \geq 14 \). If \( d_2- (v) = 0 \), then by Fact 2, we have \( ch'(v) = ch(v) - \frac{d_3(v)}{2} \geq 1 \geq 0 \text{ by (R1)-(R4) and (R7).}

If \( d_2- (v) \geq 1 \), then by Claim 17, \( d_3- (v) \leq 5 \). If \( d_3(v) \geq 1 \), then we have \( ch'(v) = ch(v) - \max \left\{1 + d_3(v) + \left(\left\lfloor \frac{14-d_2(v)-1}{2} \right\rfloor - d_3(v)\right) \times \frac{1}{2}, \frac{3}{2} + (d_3(v) - 1) + \left(\left\lfloor \frac{14-d_2(v)-1}{2} \right\rfloor - (d_3(v) - 1)\right) \times \frac{1}{2} \right\} \geq 15 - 3d_3(v) + \frac{d_3(v)}{4} > 0 \text{ by (R1)-(R4) and (R7). Otherwise } d_3(v) = 0, \text{ then } d_2- (v) \leq 5. \text{ So } ch'(v) = ch(v) - d_2- (v) - 1 - \frac{14-1-d_2(v)}{2} \times \frac{1}{2} \geq \frac{15-3d_2(v)}{4} \geq 0 \text{ by (R1)-(R4) and (R7).}

This completes the proof.

4. Remark

By the definition of IC-planar graphs, we know that every planar graphs are special IC-planar graphs. In [13], the authors proved that \( ch''_G(G) \leq \max \{\Delta(G) + 3, 16\} \). So we can easily obtain the following question.

**Question 1.** Is it true that \( ch''_G(G) \leq \Delta(G) + 3 \) for IC-planar graphs with \( \Delta = 13 \)?
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Appendix A

%%% The m.file of Matlab to compute the coefficients.
% INPUT
function coefficients ()
syms x1 x2 x3 x4 x5 x6 x7 % Variables used in the following.

% Claim 3.7 %To calculate the coefficient of x1^6 x2^4 x3^4
P=(x1−x2)*(x2−x3)*(x1−x3)^2*(x1+x2)^4*(x2+x3)^6; % The polynomial
cp1=diff(diff(diff(P,x1,6),x2,4),x3,4)/factorial(6)/factorial(4)
 /factorial(4)

% Claim 3.8
P=(x1−x2)*(x2−x3)*(x3−x4)^2*(x1+x2)^5*(x2+x3)^6;

% Claim 3.9
P=(x1−x2)*(x1−x3)* (x1−x4)* (x1−x5)* (x1−x6)* (x2−x3)* (x2−x4)* (x2−x5)
 * (x2−x6)* (x3−x4)* (x3−x5)* (x3−x6)* (x4−x5)* (x4−x6)* (x5−x6)
* (x6−x7); % The polynomial

% Claim 3.10
P=(x1−x2)*(x1−x3)* (x1−x4)* (x1−x5)* (x1−x6)* (x2−x3)* (x2−x4)* (x2−x5)
* (x2−x6)* (x3−x4)* (x3−x5)* (x3−x6)* (x4−x5)* (x4−x6)* (x5−x6)
* (x6−x7); % The polynomial

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