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2-SPANNING CYCLABILITY PROBLEMS OF SOME GENERALIZED PETERSEN GRAPHS

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Abstract

A graph G is called *r-spanning cyclable* if for every r distinct vertices $v_1, v_2, \ldots, v_r$ of G, there exists r cycles $C_1, C_2, \ldots, C_r$ in G such that v_i is on C_i for every i, and every vertex of G is on exactly one cycle C_i. In this paper, we consider the 2-spanning cyclable problem for the generalized Petersen graph $GP(n, k)$. We solved the problem for $k \leq 4$. In addition, we provide an additional observation for general k as well as stating a conjecture.

Keywords: Petersen graph, spanning cyclable.

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1. Introduction and Preliminaries

Hamiltonicity is a well-studied and important concept. A number of variations have been developed, including pancyclicity [6, 12], super spanning connectivity [1, 19, 20], Hamiltonian decompositions [3, 21, 22], and many other areas. Until the 1970's, the main interest in Hamiltonian cycles is due to their relationship with the 4-color problem. More recently, the study of Hamiltonian cycles in general graphs has been motivated by its applicability to the study of complexity and practical applications. In particular, having Hamiltonian-like property is a major requirement in designing good interconnection networks. The Hamiltonian condition can be adjusted in a number of ways. On the one hand, one can strengthen the condition to include a prescribed set of r vertices in a specific order, this is the r-ordered Hamiltonian problem [9, 10, 13, 16, 18, 23, 24]. On the other hand, one can relax the Hamiltonian condition to a union of disjoint cycles. In this paper, we study a relaxation/generalization of the Hamiltonian property. We allow the graph to be spanned by a prescribed number of disjoint cycles. However, each must contain a prescribed vertex. This concept can be applied to the problem of identifying faulty processors and other related issues in interconnection networks [8, 11, 14, 17].

Throughout this paper we use standard graph theory terminology as in [15]. A graph G is *Hamiltonian* if it contains a Hamiltonian cycle, that is, a cycle containing all vertices of G. Let k be a positive integer. A graph G is *r-spanning cyclable* if for every r distinct vertices $v_1, v_2, \ldots, v_r$ of G, there exists r cycles $C_1, C_2, \ldots, C_r$ in G such that v_i is on C_i for every i, and every vertex of G is on exactly one cycle C_i. Throughout the paper, we refer these r-disjoint cycles $C_1, C_2, \ldots, C_r$ whose union spans G as r-spanning cycles. (We note that one can generalize this concept by prescribing disjoint sets $A_1, A_2, \ldots, A_r$ of vertices and insisting that C_i must contain all the elements of A_i for every i. However, further restrictions on the A_i's must be placed; otherwise, it may be impossible for $r \geq 2$. For example, we may simply pick two vertices u and v and set A_1, A_2 to be $\{u, v\}$ and $V(G) \setminus \{u, v\}$, respectively.) If $r = 1$, then this is the usual Hamiltionian problem. An obvious question is whether this property is nested, that is, if a graph is r-spanning cyclable, does it imply that it is $(r-1)$-spanning cyclable or vice versa? The answer is no. The Petersen graph is not 1-spanning cyclable but it is 2-spanning cyclable. For the other direction, a graph being r-spanning cyclable also does not imply that it is $(r+1)$-spanning cyclable. An n-cycle is 1-spanning cyclable but it is not 2-spanning cyclable.

Although we will not discuss the r-ordered Hamiltonian problem here, we will briefly mention this concept for the purpose of illustrating the inherent difficulties of any Hamiltonian related problems. A graph G is called *r-ordered* if for any sequence of r distinct vertices of G, there exists a cycle in G containing these

r vertices in the specified order. It is *r-ordered-Hamiltonian* if, in addition, the required cycle is Hamiltonian in G. This concept was introduced in [24], and the following open problem was posed: Find an infinite class of 3-regular 4-ordered-Hamiltonian graphs. This problem remained open for many years and it was solved only recently [13, 16]. On the other hand, there are many papers on its sufficient conditions; in particular, [9] provides a comprehensive survey. So it is reasonable to expect that the r-spanning cyclability problem to be "difficult." Since the motivation is related to interconnection networks, we naturally restrict our attention to regular graphs. In particular, we want to find classes with this property where $r \geq 2$. In this paper, we show that such examples can be found in the class of generalized Petersen graphs.

The Petersen graph is an important graph in graph theory and there are several generalizations of it. One such generalization is the class of generalized Petersen graphs introduced in [28], which has attracted much research throughout the years. Some recent research include [4, 5, 26, 27, 29]. The *generalized Petersen graph* $GP(n,k)$, where $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, has $\{u_i, v_i : 0 \leq i < n\}$ as its vertex set. There are three types of edges. The first is of the form (u_i, u_{i+1}) (with $i+1$ computed modulo n) for $0 \leq i < n$. The second is of the form (v_i, v_{i+k}) (with $i+k$ computed modulo n) for $0 \leq i < n$. The third is of the form (u_i, v_i) for $0 \leq i < n$, which will be called *columns*. We call the edges in the first case the *outer edges*, the edges in the second case the *inner edges* and the edges in the third case the *columns*. We also call the u_i's *outer vertices* and the v_i's *inner vertices*.

It is clear that $GP(n,k)$ is 3-regular. We note that the subgraph induced by the vertices u_i, $0 \leq i < n$, form an n-cycle, and the subgraph induced by the vertices v_i, $0 \leq i < n$, form essentially a circulant graph. So $GP(5,2)$ is the Petersen graph. (We remark that in [4], the authors defined the $GP(n,k)$ for the range $1 \leq k < n$. The two definitions are equivalent except for the case $k = n/2$ when n is even. If $n/2 < k < n$, then $GP(n,k)$ is isomorphic to $GP(n, n-k)$. If $k = n/2$, then the resulting graph is not trivalent.) One major task was to determine which of these graphs are Hamiltonian. There were incremental results in various papers [7, 25]. The complete classification was finally solved by Alspach [2]: $GP(n,k)$ is Hamiltonian except for $GP(n,2)$ for $n \equiv 5 \pmod 6$. We refer the reader to Alspach [2] for the history, motivation and development of this problem and its solution. For the related problem in classifying which of these graphs are "Hamiltonian-connected/Hamiltonian-laceable," it is still unsolved. Alspach conjectured over twenty years ago that if $GP(n,k)$ is not isomorphic to $GP(6m+5, 2)$, and n and k are relatively prime, then $GP(n,k)$ is Hamiltonian-connected unless it is bipartite, in which case it is Hamiltonian-laceable. We note that $GP(n,k)$ is bipartite if and only if n is even and k is odd. Alspach [4] commented that this condition on n and k is not well understood, and further commented that this condition may be misleading after proving that the conjecture is true for $k = 1, 2, 3$ although the relatively prime condition is far from necessary. A refinement of this conjecture was proposed in [13]. This problem has only been solved for small k. It turns out that the generalized Petersen graphs also form a rich class of examples for the k-ordered problem. Again, one can consider the corresponding classification problem and it is only solved for $k = 2$ and $k = 3$. Given these research, it is reasonable to expect that $GP(n, k)$ will provide good examples of k-spanning cyclability and the corresponding classification problem would be "difficult." We start with the following observation.

Proposition 1. *If a graph G is r-spanning cyclable, then every vertex has degree at least $r + 1$.*

Proof. Let u be a vertex of G with minimum degree, and let its neighbors be $v_1, v_2, \ldots, v_r$. Then G cannot be r-spanning cyclable. Otherwise, we pick the r prescribed vertices to be $v_1, v_2, \ldots, v_r$. Since u must be on some cycle, such a cycle must contain two of $v_1, v_2, \ldots, v_r$. ■

The above observation tells us that for cubic graphs and hence generalized Petersen graphs, the best we can hope for is 2-spanning cyclability. The next observation shows that such graphs have girth at least 4.

Proposition 2. *If a cubic graph G is 2-spanning cyclable, then G has girth at least 4.*

Proof. Suppose G has a 3-cycle with vertices v_1, v_2, v_3. Choose v_1 and v_2 to be the two prescribed vertices. Since G is 2-spanning cyclable, there exist valid cycles C_1 and C_2 such that v_1 is on C_1 and v_2 is on C_2. Since v_2 is not on C_1 and G is cubic, v_3 must be on C_1. Similarly v_3 must be on C_2. This is a contradiction. ■

Theorem 3. *Let $k \geq 2$ and $n = rk+1$ where $r \geq 2$. Then $GP(n, k)$ is 2-spanning cyclable.*

Proof. By the definition of generalized Petersen graphs, the outer edges form an n-cycle. It follows from the assumption of n that the inner edges form an n-cycle. Then we are done if exactly one of the two prescribed vertices is an inner vertex (and the other one is an outer vertex) as we may choose the two n-cycles formed by the outer edges and the inner edges, respectively. Thus we may assume both prescribed vertices are inner or both outer. We will now construct two spanning cycles. Consider the outer vertices

$$(u_1, u_2, \ldots, u_k), (u_{k+1}, u_{k+2}, \ldots, u_{2k}), \ldots, (u_{(r-2)k+1}, u_{(r-2)k+2}, \ldots, u_{(r-1)k}),$$

$$(u_{(r-1)k+1}, u_{(r-1)k+2}, \ldots, u_{rk}), u_0.$$

The parentheses in the list are inserted to highlight how we group the vertices. The first cycle C_1 is constructed using the path with the outer vertices $u_k, u_{k+1}, u_{k+2}, \ldots, u_{rk}$ followed by the edges

$$(u_{rk}, v_{rk}), (v_{rk}, v_{(r-1)k}), (v_{(r-1)k}, v_{(r-2)k}), \ldots, (v_{2k}, v_k), (v_k, u_k).$$

For the second cycle, we start with

$$(u_0, v_0), (v_0, v_{(r-1)k+1}), (v_{(r-1)k+1}, v_{(r-2)k+1}), \ldots, (v_{k+1}, v_1),$$

then

$$(v_1, v_{(r-1)k+2}), (v_{(r-1)k+2}, v_{(r-2)k+2}), \ldots, (v_{k+2}, v_2),$$

which will be followed by

$$(v_2, v_{(r-1)k+3}), (v_{(r-1)k+3}, v_{(r-2)k+3}), \ldots, (v_{k+3}, v_3).$$

Continue in this process, we will use

$$(v_{k-2}, v_{(r-1)k+(k-1)}), (v_{(r-1)k+(k-1)}, v_{(r-2)k+(k-1)}), \ldots, (v_{k+(k-1)}, v_{k-1}).$$

Then we use the edge (v_{k-1}, u_{k-1}) followed by the path $(u_{k-1}, u_{k-2}, \ldots, u_1, u_0)$. See Figure 1 for an illustration of the general case and Figure 2 for the specific case when $r = 5$ and $k = 4$. (We remark that to avoid clutters, there are dangling edges on both sides of a graph but it is clear how the edges on one side continue to the other side. We will use this convention throughout the paper.)

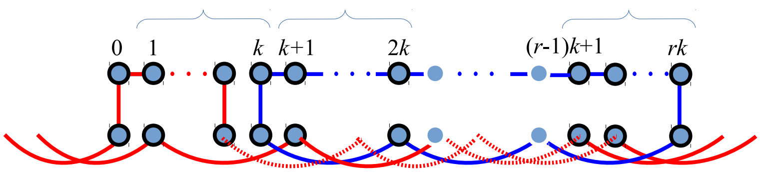


Figure 1. The two spanning cycles of $GP(rk + 1, k)$.

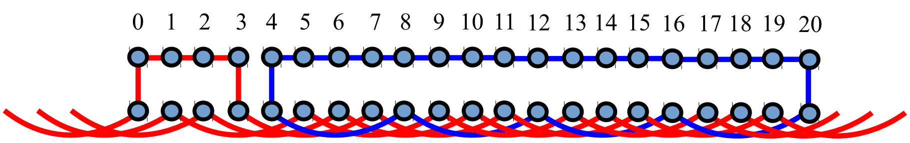


Figure 2. The two spanning cycles of $GP(21, 4)$.

We first suppose the two prescribed vertices x and y are outer vertices. We may assume one of them is $x = u_0$. Since $r \geq 2$, we may assume that y is not one of $u_1, u_2, \ldots, u_{k-1}$. Thus C_1 and C_2 give the desired two cycles with x on C_2 and y on C_1. Now assume both x and y are inner vertices. We may assume that $x = v_k$. Then we are done unless $y \in \{v_{2k}, v_{3k}, \ldots, v_{rk}\}$ as $v_k, v_{2k}, v_{3k}, \ldots, v_{rk}$ are on C_1 and all the other inner vertices are on C_2. So assume that x and y are prescribed as such. We shift our reference point and we may assume that $x = v_0$ and hence $y \in \{v_k, v_{2k}, \ldots, v_{(r-1)k}\}$. But then x is on C_2 and y is on C_1. ∎

Theorem 3 shows that for every $k \geq 2$, there are infinitely many $GP(n, k)$ that are 2-spanning cyclable. It would be interesting to see exactly which $GP(n, k)$ have this property. We first consider $k = 1$, which can easily be solved.

Proposition 4. *Let $n \geq 3$. Then $GP(n, 1)$ is 2-spanning cyclable if and only if $n \neq 3$.*

Proof. We first note that $GP(n, 1)$ is the Cartesian product of an n-cycle and the complete graph K_2, that is, it can be obtained by taking two copies of an n-cycle, and puting an edge between every pair of corresponding vertex of the first cycle and the corresponding vertex of the second cycle. If follows from Proposition 2 that for $n = 3$ $GP(n, 1)$ is not 2-spanning cyclable. Let $n \geq 4$. As in the proof of Theorem 3, we may assume the two prescribed vertices x and y are either both outer vertices or both inner vertices. In this case, they are equivalent; so assume both are outer vertices. We may assume that $x = u_0$ and $y \neq u_1$. Then let $C_1 = (u_0, u_1, v_1, v_0, u_0)$, $C_2 = (u_2, u_3, \ldots, u_{n-2}, u_{n-1}, v_{n-1}, v_{n-2}, \ldots, v_3, v_2, u_2)$, and we are done. ∎

Unfortunately, for $k \geq 2$, the classification is not as simple. We will study $k = 2, 3, 4$ in this paper.

2. $GP(n, 2)$

In this section, we determine the n for which $GP(n, 2)$'s are 2-spanning cyclable. By definition, $n \geq 5$. This result here is more interesting than $GP(n, 1)$ as there are infinitely many graphs in the set that are not 2-spanning cyclable. Essentially, it is 2-spanning cyclable if and only if n is odd.

Theorem 5. *Let $n \geq 5$. Then $GP(n, 2)$ is 2-spanning cyclable if and only if n is odd.*

Proof. If n is odd and $n \geq 5$, then $GP(n, 2)$ is 2-spanning cyclable by Theorem 3. It remains to show that if n is even, then $GP(n, 2)$ is not 2-spanning cyclable. We first observe that $GP(n, 2)$ is planar. Note that the inner edges form two disjoint $n/2$-cycles. The graph can be embedded so that it has three "rings." The middle ring is an n-cycle (using the outer edges). Figure 3(a) shows an example. The outside ring and the inside ring are both $n/2$-cycles using the inner edges. Essentially, we flip one of the $n/2$-cycles to the outside. In this embedding, every face is of size 5, except two, each having a size of $n/2$. We will use Grinberg's condition; more precisely, the following corollary. If a planar graph can be embedded in a way such that every face except one has size 2 modulo 3, and the exceptional face has size not congruent to 2 modulo 3, then the graph is not Hamiltonian. We consider three cases.

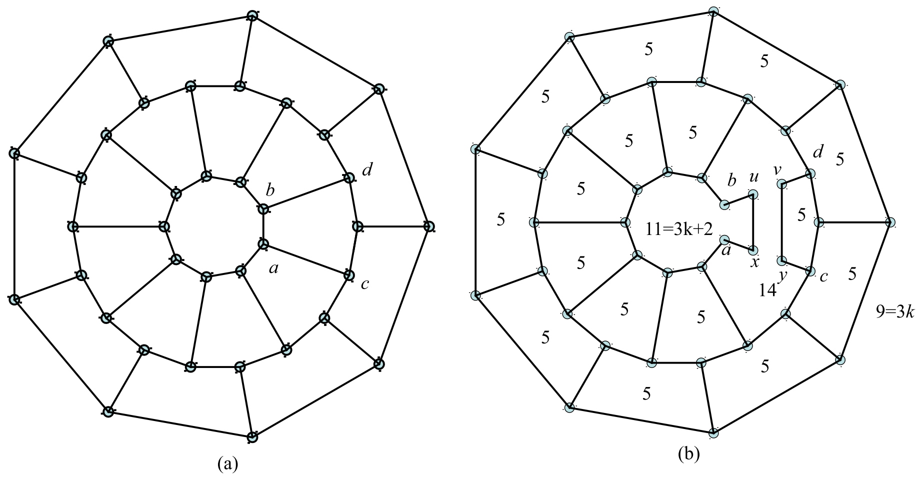
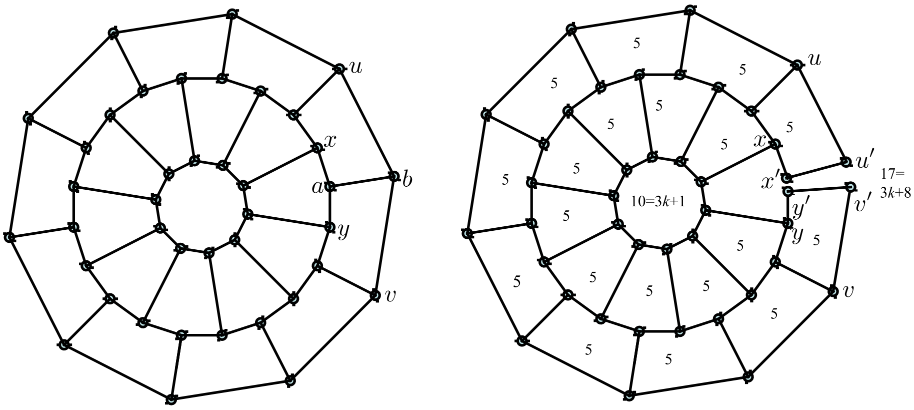


Figure 3. The graph $GP(6k,2)$ for $k=3$.

Case 1. $n=6k$. We pick the two exceptional vertices a and b as indicated in Figure 3(a). Suppose two desired cycles exist. Since these two vertices are adjacent and the graph is 3-regular, this implies that the two edges incident to a, other than (a,b), must be on the cycle C_1 containing a. Similarly, the two edges incident to b, other than (a,b), must be on the cycle C_2 containing b. Now C_1 and C_2 can be "merged" into a Hamiltonian cycle for a new graph adjusted from $GP(n,2)$. (Delete (a,b) from $GP(n,2)$, then take (a,c) on C_1 and (b,d) on C_2. Replace (a,c) by (a,x,y,c) and (b,d) by (b,u,v,d) where x,y,u,v are new vertices. Now replace (x,y) and (u,v) by (x,u) and (y,v) to obtain the new graph.) See Figure 3(b) for an example. This new graph is planar such that every face is of size 5, except three with size 14, $3k$ and $3k+2$. By construction, this graph is Hamiltonian. However, by the corollary to Grinberg's condition, it should not be Hamiltonian.

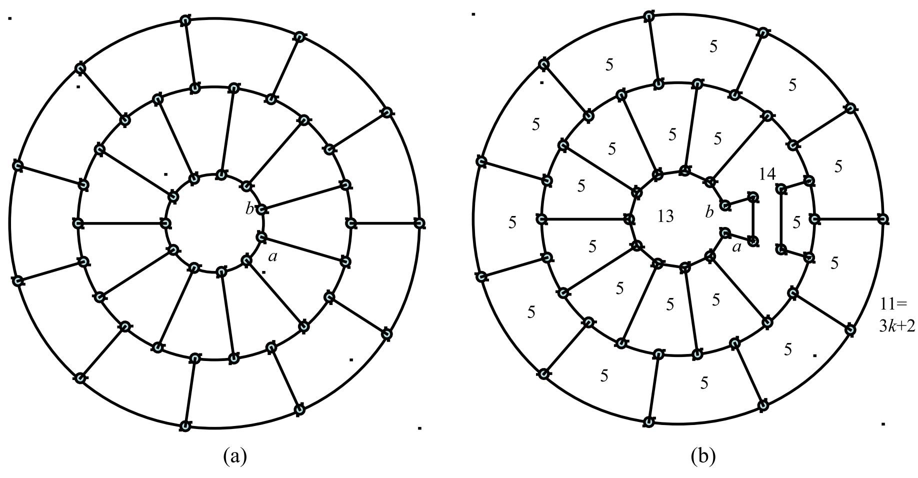
Case 2. $n=6k+2$. We pick the two exceptional vertices a and b as indicated in Figure 4(a). Again a and b are adjacent and we can determine which two edges



(a) (b)

Figure 4. The graph $GP(6k+2,2)$ for $k=3$.

incident to a belong to C_1, a cycle containing a, and which two edges incident to b belong to C_2, a cycle containing b. Using similar construction (see Figure 4(b)) to arrive at a planar graph with the property that every face is of size 5, except two with size $3k+1$ and $3k+8$, respectively. (Since a and b are adjacent, C_1 containing a must use (x,a) and (y,a), and C_2 containing b must use (u,b) and (v,b). Now delete (a,b) from $GP(n,2)$, replace (x,a) and (u,b) by (x,x',u',u) and replace (y,a) and (v,b) by (y,y',v',v), where x',u',y' and v' are new vertices.) Again, it gives a contradiction as it is Hamiltonian by "merging" C_1 and C_2 but it is not Hamiltonian by the corollary to Grinberg's condition.



(a) (b)

Figure 5. The graph $GP(6k+4,2)$ for $k=3$.

Case 3. $n = 6k+4$. We pick the two exceptional vertices a and b as indicated in Figure 5(a). Again a and b are adjacent and we can determine which two edges incident to a belong to C_1, a cycle containing a, and which two edges incident to b belong to C_2, a cycle containing b. Using similar construction (see Figure 5(b)) to arrive at a planar graph with the property that every face is of size 5, except three with size 14, $3k+2$ and $3k+4$, respectively. Again, it gives a contradiction as it is Hamiltonian by "merging" C_1 and C_2 but it is not Hamiltonian by the corollary to Grinberg's condition. ■

3. $GP(n, 3)$

In this section, we determine the n for which $GP(n, 3)$'s are 2-spanning cyclable. By definition, $n \geq 7$. Given the classification of $GP(n, 2)$, one may expect a similar result. However, this is not the case. We note that the inner edges form an n-cycle if and only if n is not divisible by 3. If n is divisible by 3, then they induce three $n/3$-cycles. Thus $GP(9, 3)$ contains a 3-cycle and hence not 2-spanning cyclable by Proposition 2.

Theorem 6. *Let $n \geq 7$. Then $GP(n, 3)$ is 2-spanning cyclable if and only if $n \neq 9$.*

Proof. If n is congruent to 1 modulo 3, then the claim is true by Theorem 3. We now suppose n is congruent to 2 modulo 3. As usual, we may assume the two prescribed vertices x and y are either both outer vertices or both inner vertices. Similar to the construction given in the proof of Theorem 3, we can construct another set of two cycles. Rather than listing the cycles as we have done in the proof of Theorem 3, we will present an example that clearly generalizes, to avoid such complicated notations, see Figure 6. Note that one cycle contains all the outer vertices except u_0, u_1, u_2. Since $n \geq 7$, we may assume $x = u_0$ and $y \notin \{u_1, u_2\}$. Thus we have the two desired cycles. Now assume both are inner vertices. We may assume that $x = v_0$. It follows from our construction that we are done unless $y \in \{v_2, v_5, v_8, \ldots, v_{3(k-1)+2}\}$. So assume that x and y are prescribed as such. This is covered by an alternate set of 2 cycles as shown in Figure 7. (Again it is clear that it generalizes.) We note that we actually do not need this alternate solution as we may simply label $v_1, v_2, \ldots, v_{3k+1}$ in reverse to $v_{3k+1}, \ldots, v_2, v_1$.

We now consider the case when $n = 3r$ where $r \geq 4$. Since the inner edges no longer form one n-cycle, we have to consider the case when x is outer and y is inner. Consider the set of two cycles given in Figure 8(a) (Again it is clear that it generalizes.) Here one cycle C_1 contains all the outer vertices with some inner vertices and another cycle C_2 contains exactly $v_0, v_3, v_6, \ldots, v_{3(r-1)}$. Thus

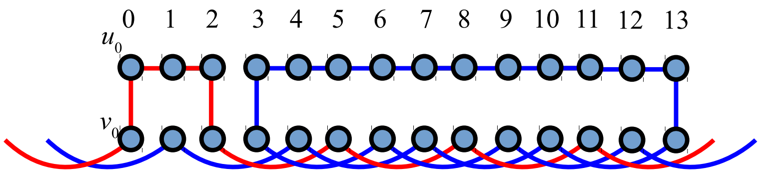


Figure 6. Two spanning cycles for $GP(3k+2,3)$ for $k=4$.

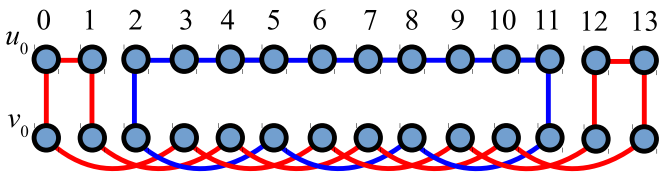


Figure 7. Another two spanning cycles for $GP(3k+2,3)$ for $k=4$.

we can assume $y=v_3$ and hence x is on C_1 and y is on C_2. To complete the proof, we need another set of two cycles. Here the first cycle C_1' is obtained by use the path $(u_1,u_2,u_3,\ldots,u_{3r-8})$, followed by the edges

$$(u_{3r-8},v_{3r-8}),(v_{3r-8},v_{3r-11}),(v_{3r-11},v_{3r-14}),\ldots,(v_7,v_4),(v_4,v_1),(v_1,u_1).$$

The second cycle C_2' contains the rest of the vertices. Indeed the cycle is forced. See Figure 8(b), (c), (d) for $r=4,5,6$. Since C_1' contains $u_1,u_2,u_3,\ldots,u_{3r-8}$ and C_2' contains $u_{3r-7},u_{3r-6},\ldots,u_{3r-2},u_{3r-1},u_0$, clearly we may assume x is one of $u_1,u_2,u_3,\ldots,u_{3r-8}$ and y is one of $u_{3r-7},u_{3r-6},\ldots,u_{3r-2},u_{3r-1},u_0$ if both are outer vertices. (We may let $y=u_0$, then by looking at either direction, we may assume x is one of $u_1,u_2,u_3,\ldots,u_{3r-8}$ unless $r=4$ and (x,y) is one of $(u_5,u_0),(u_6,u_0),(u_7,u_0)$. But this is equivalent to $(u_6,u_1),(u_7,u_1),(u_8,u_1)$, respectively.) Now consider both x and y are inner vertices. Without loss of generality, we may assume that $x=v_{3r-8}$. So x is on C_1'. Then we are done unless $y\in\{v_1,v_4,v_7,\ldots,v_{3r-11}\}$, which are on C_1'. But this is equivalent to $x=v_{3r-5}$ and $y\in\{v_4,v_7,v_{10},\ldots,v_{3r-8}\}$ and hence x is on C_2' and y is on C_1'. ■

4. $GP(n,4)$

In this section, we determine the n for which $GP(n,4)$'s are 2-spanning cyclable. By definition, $n\geq 9$. We note that the inner edges of $GP(12,4)$ induce four 3-cycles; hence it is not 2-spanning cyclable. It turns out that $GP(10,4)$ is also not 2-spanning cyclable. Since it is a 3-regular graph with 20 vertices, it is a graph that is small enough to check by hand. Note that the inner edges form

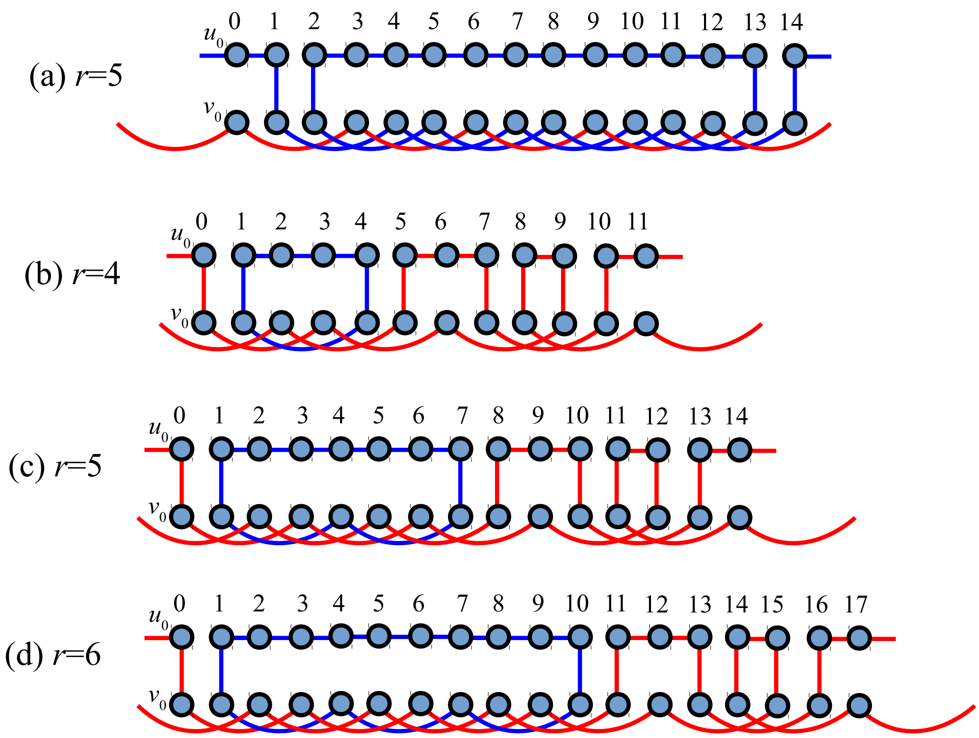


Figure 8. Two spanning cycles for $GP(3r, 3)$ for $r = 4, 5, 6$.

two 5-cycles. One can check that if the two prescribed vertices are both on such a 5-cycle, then there is no valid pair of cycles. We omit the details.

Theorem 7. *Let $n \geq 9$. Then $GP(n, 4)$ is 2-spanning cyclable if and only if $n \notin \{10, 12\}$.*

Proof. We have already concluded that $GP(10, 4)$ and $GP(12, 4)$ are not 2-spanning cyclable. We now show that these are the only exceptional cases. As usual, let x and y be the two prescribed vertices. If n is congruent to 1 modulo 4, then the claim is true by Theorem 3. We consider three additional cases.

Case 1. $n = 4r$ where $r \geq 4$. We construct C_1 by starting with the path $(u_2, u_3, v_3, v_7, v_{11}, \ldots, v_{4(r-1)+3})$ followed by $(v_{4(r-1)+3}, u_{4(r-1)+3})$, then by the path with only outer vertices, $(u_{4(r-1)+3}, u_{4(r-1)+2}, \ldots, u_7, u_6)$, followed by $(u_6, v_6, v_{10}, v_{14}, \ldots, v_{4(r-1)+2}, v_2, u_2)$. Similarly, one can construct C_2 using the remaining vertices, see Figure 9. We first suppose x is an outer vertex and y is an inner vertex. We may assume that $y = v_8$. Then C_1 and C_2 give the desired cycles unless $x \in \{u_0, u_1, u_4, u_5\}$. Then by changing our reference point, x being u_0, u_1, u_4, u_5 is equivalent to x being $u_{16}, u_{15}, u_{12}, u_{11}$, respectively. (This argument fails for

$n = 12$.) Now suppose both x and y are outer vertices. We may assume that $x = u_5$, then C_1 and C_2 give the desired cycles unless $y \in \{u_0, u_1, u_4\}$. Again we consider the other direction, that is, y being u_0, u_1, u_4 is equivalent to y being u_{10}, u_9, u_6, respectively. Finally assume both x and y are inner vertices. We first observe that v_i is on C_2 if and only if i is congruent to 0 or 1 modulo 4. We may assume that $x = v_0$. Then we are done unless $y = v_j$ where $j \neq 0$ is congruent to 0 or 1 modulo 4. Those that are congruent to 1 modulo 4 are vertices that are $1, 5, 9, \ldots$ away from v_0 with respect to the subscript. Again we can change our reference point by considering $x = v_1$ and this will cover those vertices that are $1, 5, 9, \ldots$ away from v_1 with respect to the subscript.

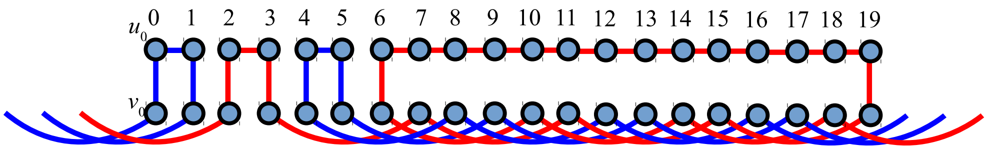


Figure 9. Two spanning cycles for $GP(4r, 4)$ for $r = 5$.

It remains to cover those that are $4, 8, 12, \ldots$ away from v_0 with respect to the subscript. For this, we consider another pair of cycles. This pair is actually more difficult to describe. So we describe it via an inductive argument. We consider $n \geq 20$. (The case $n = 16$ is not covered here.) See Figure 10 for the case $n = 20$ and $n = 24$. It is easy to see that the pair of cycles for $n = 24$ can be obtained from the pair for $n = 20$ by inserting 4 columns between column 16 and column 17 and extend appropriately. Moreover Figure 11 shows how to extend $GP(20, 4)$ to $GP(24, 4)$ and Figure 12 shows how to extend the pair of cycles. (We remark that Figure 11 and Figure 12 only show part of $GP(20, 4)$ and part of $GP(24, 4)$, so the dangling edges on the right do not correspond to the dangling edges on the left as in other pictures.) By an inductive argument, we obtain such pair for every $n = 4r$ where $r \geq 5$. We note that this construction produces two cycles C'_1 and C'_2 where the only inner vertices belonging to C'_1 are

$$\{v_2, v_3, v_7, v_{11}\} \cup \{v_i : 14 \leq i \leq 4r - 1 \text{ and } i \text{ is congruent to 2 or 3 modulo 4}\}.$$

Now change our point of reference and let $x = v_{10}$, which is on C'_2. But $v_{14}, v_{18}, \ldots, v_{4r-2}$ are on C'_1 as the subscripts are congruent to 2 modulo 4. Thus we are done if $y = v_{14}, v_{18}, \ldots, v_{4r-2}$. Now observe that $(x, y) = (v_{10}, v_6)$ is equivalent to $(x, y) = (v_{10}, v_{14})$ so $(x, y) = (v_{10}, v_6)$ is done. Now v_2 is on C'_1 and so $(x, y) = (v_{10}, v_2)$ is done. For $n = 16$, we still need to consider the cases when $(x, y) \in \{(v_0, v_4), (v_0, v_8), (v_0, v_{12})\}$. Now (v_0, v_4) and (v_0, v_{12}) are equivalent. Thus we only have to consider (v_0, v_8) and (v_0, v_{12}), which are solved by the two cycles given in Figure 13.

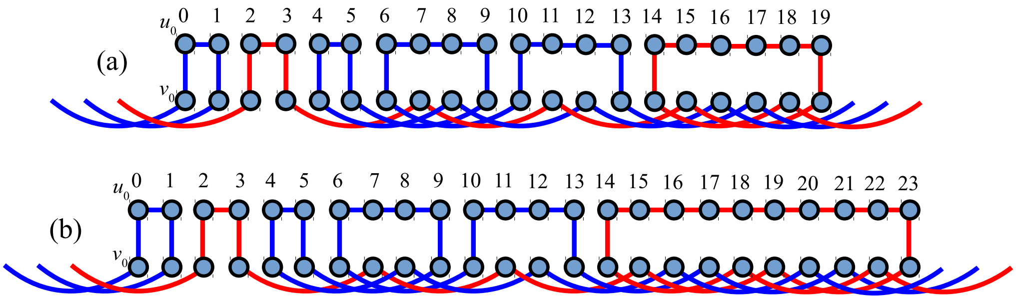


Figure 10. Another two spanning cycles for $GP(4r, 4)$ for $r = 5$ and $r = 6$.

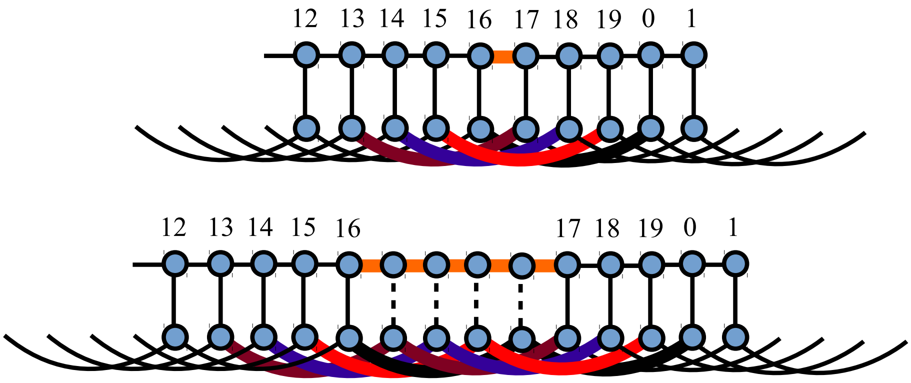


Figure 11. Expanding the graph from $GP(20, 4)$ to $GP(24, 4)$.

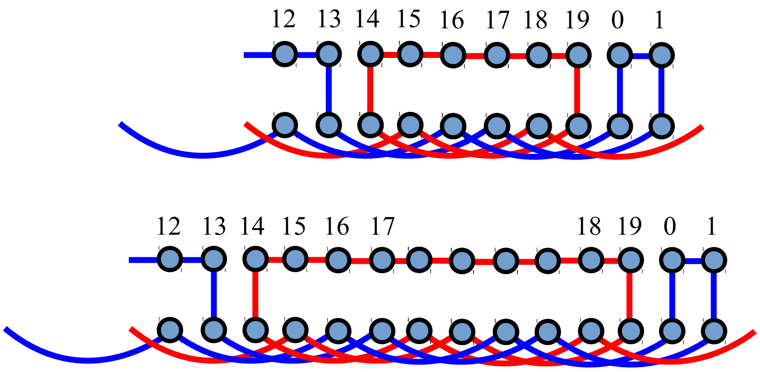


Figure 12. Expanding a pair of valid cycles from $GP(20, 4)$ to $GP(24, 4)$.

Case 2. $n = 4r + 2$ where $r \geq 3$. The case $n = 14$ will be covered separately. Henceforth, we assume $r \geq 4$. Construct C_1 by starting with the path

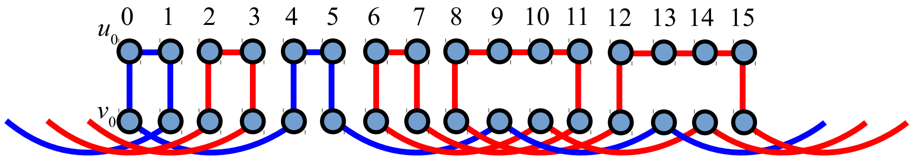


Figure 13. Two spanning cycles for $GP(4r, 4)$ for $r = 4$.

$(u_0, u_1, u_2, \ldots, u_{4r-12})$ with only outer vertices, followed by (u_{4r-12}, v_{4r-12}), then by $(v_{4r-12}, v_{4r-16}, \ldots, v_4, v_0)$, and finally by (v_0, u_0). It is not too difficult to see that the remaining vertices will be on a cycle C_2. This is essentially forced if we want the following five subpaths on C_2:

$$(v_{4r-11}, u_{4r-11}, u_{4r-10}, u_{4r-9}, u_{4r-8}, v_{4r-8}), (v_{4r-7}, u_{4r-7}, u_{4r-6}, u_{4r-5}, v_{4r-5}),$$

$$(v_{4r-4}, u_{4r-4}, u_{4r-3}, v_{4r-3}), (v_{4r-2}, u_{4r-2}, u_{4r-1}, v_{4r-1}), (v_{4r}, u_{4r}, u_{4r+1}, v_{4r+1}).$$

See Figure 14 for $n = 18$ and $n = 22$. Since the inner edges no longer form one n-cycle, we have to consider the case when x is outer and y is inner. We may assume $x = u_0$. Then C_1 and C_2 give the desired cycles unless $y \in \{v_0, v_4, v_8, v_{4r-16}, v_{4r-12}\}$. Then we change our reference point to $x = u_1$ and we only have to consider $y \in \{v_1, v_5, v_9, v_{4r-15}, v_{4r-11}\}$. We are done as u_1 is on C_1 and $v_1, v_5, v_9, v_{4r-15}, v_{4r-11}$ are on C_2.

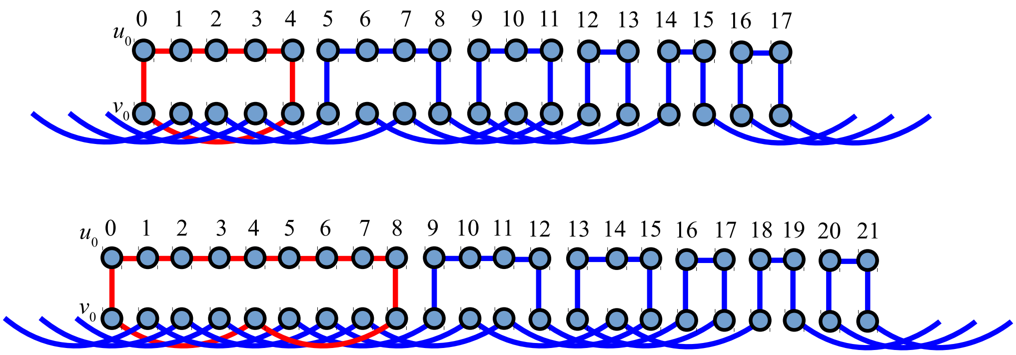


Figure 14. Two spanning cycles for $GP(4r + 2, 4)$ for $r = 4, 5$.

We now suppose both x are y are outer vertices. We may assume that $x = u_{4r+1}$. Then C_1 and C_2 give the desired cycles unless $y \in \{u_{4r-11}, u_{4r-10}, \ldots, u_{4r}\}$. We note that $\{u_{4r-11}, u_{4r-10}, \ldots, u_{4r}\}$ is of size 12. If $r \geq 6$, then $\{u_0, u_1, u_2, \ldots, u_{4r-12}\}$ is of size at least 12 and we can change our reference point to $x = u_{4r-11}$. For $r = 5$, the same argument eliminates all except $(x, y) \in \{(u_{21}, u_9), (u_{21}, u_{10}), (u_{21}, u_{11})\}$ which can be solved by considering $(x, y) \in \{(u_{20}, u_8), (u_{19}, u_8), (u_{18},$

$u_8)\}$. For $r = 4$, the same argument eliminate all except $(x, y) = (u_{17}, u_i)$ where $i = 5, 6, \ldots, 11$, which can be solved by considering $(x, y) = (u_i, u_4)$, where $i = 16, 15, \ldots, 10$.

Finally suppose both x are y are inner vertices. We may assume that $x = v_0$. Then C_1 and C_2 give the desired cycles unless $y = v_4, v_8, \ldots, v_{4r-12}$. Again we change our reference point by considering $x = v_{4r-8}$ and hence $(x, y) = (v_0, v_4)$ is equivalent to $(x, y) = (v_{4r-8}, v_{4r-12})$, $(x, y) = (v_0, v_8)$ is equivalent to $(x, y) = (v_{4r-8}, v_{4r-16})$ and so on ending with $(x, y) = (v_0, v_{4r-12})$ is equivalent to $(x, y) = (v_{4r-8}, v_4)$.

For $n = 14$, see Figure 15. Clearly these two cycles cover all cases.

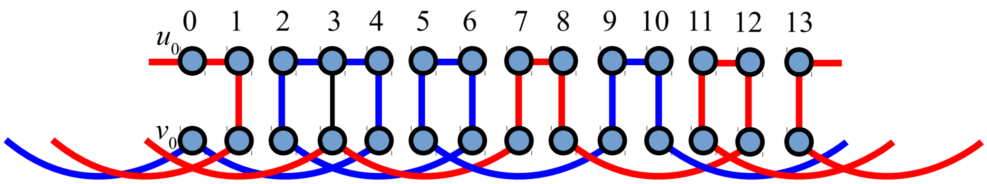


Figure 15. Two spanning cycles for $GP(4r + 2, 4)$ for $r = 3$.

Case 3. $n = 4r + 3$ where $r \geq 2$. We note that for this case, the inner edges form one n-cycle. We construct a pair of cycles similar to Case 2. Construct C_1 by starting with the path $(u_0, u_1, u_2, \ldots, u_{4r-4})$ with only outer vertices, followed by (u_{4r-4}, v_{4r-4}), then by $(v_{4r-4}, v_{4r-8}, \ldots, v_4, v_0)$, and finally by (v_0, u_0). It is not too difficult to see that the remaining vertices will be on a cycle C_2. This is essentially forced if we want the following three subpaths on C_2:

$$(v_{4r-3}, u_{4r-3}, u_{4r-2}, v_{4r-2}), (v_{4r-1}, u_{4r-1}, u_{4r}, v_{4r}), (v_{4r+1}, u_{4r+1}, u_{4r+2}, v_{4r+2}),$$

see Figure 16.

We now suppose both x are y are outer vertices. We may assume that $x = u_{4r+2}$. Then C_1 and C_2 give the desired cycles unless $y \in \{u_{4r-3}, u_{4r-2}, u_{4r-1}, u_{4r}, u_{4r+1}\}$. Since $r \geq 2$, the set $\{u_0, u_1, u_2, \ldots, u_{4r-4}\}$ is of size at least 5 and we can change our reference point to $x = u_{4r-5}$.

Finally suppose both x are y are inner vertices. We may assume that $x = v_0$. Then C_1 and C_2 give the desired cycles unless $y = v_4, v_8, \ldots, v_{4r-4}$. Again we change our reference point by considering $x = v_{4r}$ and hence $(x, y) = (v_0, v_4)$ is equivalent to $(x, y) = (v_{4r}, v_{4r-4})$, $(x, y) = (v_0, v_8)$ is equivalent to $(x, y) = (v_{4r}, v_{4r-8})$ and so on ending with $(x, y) = (v_0, v_{4r-4})$ is equivalent to $(x, y) = (v_{4r}, v_4)$. ■

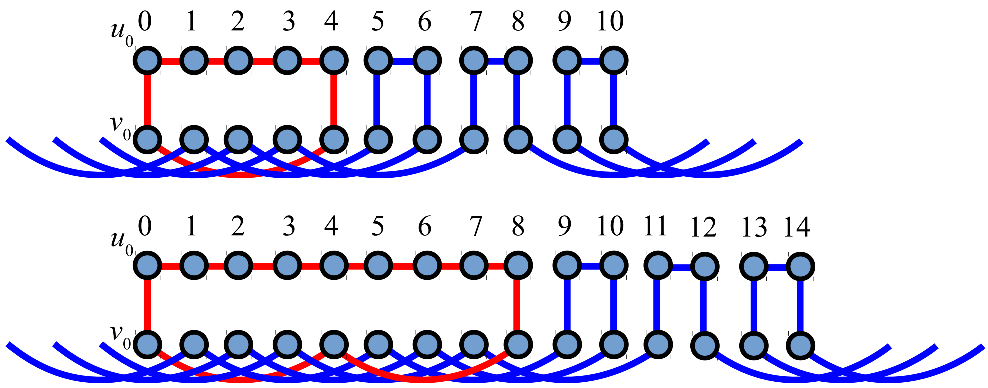


Figure 16. Two spanning cycles for $GP(4r+3,4)$ for $r=2,3$.

5. Conclusion

In this paper, we studied the 2-spanning cyclability problem for the generalized Petersen graphs. A typical way in proving that $GP(n,k)$ has certain property is by induction. For example, we have seen how $GP(n+4,4)$ can be obtained from $GP(n,4)$ by inserting four columns. We have also seen how a pair of cycles in $GP(n+4,4)$ can be obtained from $GP(n,4)$. Note that there are five special edges in Figure 11 and the extension of a pair of cycles depending on which of these five edges are being used. Thus there are 2^5 cases. In fact, often the extension is difficult and one has to "insert" more columns. Moreover, the more columns that we need to insert, the larger number of base cases that we have to check. Indeed, this was the approach that we had used (via a computer search) until we observed a pattern in the computer output and the induction step. Thus we are able to obtain a self-contained proof without the need of including a "computer proof" for the base cases. The case $k=2$ is more traceable as there are only two prescribed vertices. However, our computer solution does not show a pattern for the general k. Nevertheless, the result for $n=rk+1$ is relatively easy. We were hoping to obtain a result for $GP(n,k)$ when n and k are relatively prime but we were unsuccessful. We end this paper with the following conjecture. We remark that we do not have enough data to give an estimate of n with respect to k in the second part of the conjecture.

Conjecture 1. *Let $k \geq 1$ and $n \geq 2k+1$.*

1. *If n and k are relatively prime, then $GP(n,k)$ is 2-spanning cyclable.*
2. *If n and k are not relatively prime and $k \geq 3$, then $GP(n,k)$ is 2-spanning cyclable when n is sufficiently large.*

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