LINEAR LIST COLORING OF SOME SPARSE GRAPHS

MING CHEN

College of Mathematics Physics and Information Engineering
Jiaxing University, Zhejiang 314001, China

e-mail: chen2001ming@163.com

YUSHENG LI

School of Mathematical Sciences, Tongji University Shanghai 200092, China

e-mail: li_yusheng@tongji.edu.cn

AND

LI ZHANG

School of Mathematical Sciences, Tongji University Shanghai 200092, China

e-mail: lizhang@tongji.edu.cn

Abstract

A linear k-coloring of a graph is a proper k-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. A graph G is linearly L-colorable if there is a linear coloring c of G for a given list assignment \( L = \{L(v) : v \in V(G)\} \) such that \( c(v) \in L(v) \) for all \( v \in V(G) \), and \( G \) is linearly k-choosable if \( G \) is linearly L-colorable for any list assignment with \( |L(v)| \geq k \). The smallest integer \( k \) such that \( G \) is linearly \( k \)-choosable is called the linear list chromatic number, denoted by \( lc_l(G) \). It is clear that \( lc_l(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) for any graph \( G \) with maximum degree \( \Delta(G) \). The maximum average degree of a graph \( G \), denoted by \( mad(G) \), is the maximum of the average degrees of all subgraphs of \( G \). In this note, we shall prove the following. Let \( G \) be a graph, (1) if \( mad(G) < \frac{8}{3} \) and \( \Delta(G) \geq 7 \), then \( lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \); (2) if \( mad(G) < \frac{14}{3} \) and \( \Delta(G) \geq 5 \), then \( lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \); (3) if \( mad(G) < \frac{20}{3} \) and \( \Delta(G) \geq 5 \), then \( lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

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1Corresponding author.
1. Introduction

All graphs considered here are finite, simple and undirected. For a graph $G$, denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the vertex set, edge set, the minimum degree and the maximum degree, respectively. For a vertex $v \in V(G)$, let $N(v)$ and $d(v)$ be the neighborhood and the degree of $v$ in $G$, respectively. The closed neighborhood of a vertex $v \in V(G)$, denoted by $N[v]$, is defined to be $N(v) \cup v$. A $k$-vertex ($k^-$-vertex and $k^+$-vertex, respectively) is a vertex with degree $k$ (at most $k$ and at least $k$, respectively). A 2-vertex $v \in V(G)$ is called an $(a, b)$-vertex if it is adjacent to an $a$-vertex and a $b$-vertex, and an $(a, b^+)$-vertex is defined similarly. The maximum average degree $mad(G)$ of a graph $G$ is defined as $\frac{2|E(H)|}{|V(H)|}$ whenever $H \subseteq G$, where $H \subseteq G$ signified that $H$ is a subgraph of $G$.

A proper $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. A linear $k$-coloring of a graph is a proper $k$-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. The linear chromatic number $lc(G)$ of a graph $G$ is the smallest number $k$ such that $G$ has a linear $k$-coloring. A graph $G$ is linearly $L$-colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring $c$ of $G$ such that $c(v) \in L(v)$ for all $v \in V(G)$. If $G$ is linearly $L$-colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said to be linearly $k$-choosable. The smallest integer $k$ such that the graph $G$ is linearly $k$-choosable is called the linear list chromatic number, denoted by $lc_l(G)$. The concept of linear coloring was first introduced by Yuster [8], and linear list colorings were first investigated by Esperet, Montassier and Raspaud [4].

It is clear that the linear chromatic number $lc(G)$ of a graph $G$ with maximum degree $\Delta(G)$ has a trivial lower bound $lc(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $lc_l(G) \geq \frac{\Delta(G)}{2} + 1$. Esperet et al. [4] proved that trees with maximum degree $\Delta(G)$ satisfy $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. This equality suggests that the linear list chromatic numbers of sparse graphs (with $mad(G) < 3$) might be close to the trivial lower bound. Cranston and Yu [1] asked: Does there exist a constant $C$ such that every sparse graph $G$ satisfies $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? Some authors have proved that for the class of some sparse graphs, such constant $C$ exists and is close to or equal to 1. We list the currently known results about this subject as follows.

**Theorem 1.** Let $G$ be a graph.

(i) (Esperet et al. [4]) If $mad(G) < \frac{8}{3}$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.

(ii) (Wang and Wu [7]) If $mad(G) < \frac{14}{5}$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$. 

(iii) (Cranston and Yu [1]) If \( \text{mad}(G) < 3 \) and \( \Delta(G) \geq 9 \), then \( \text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

(iv) (Cranston and Yu [1]) If \( \text{mad}(G) < \frac{12}{5} \) and \( \Delta(G) \geq 3 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

A planar graph is a graph that can be drawn on the Euclidean plane such that its edges meet at their ends only. The girth of a graph \( G \), denoted \( g(G) \), is the length of a shortest cycle of \( G \). For a planar graph \( G \) with girth \( g \), we have \( \text{mad}(G) < \frac{2g}{g-2} \) by Euler’s formula. So we can get some results from above results. Li, Wang and Raspaud [5] also asked: Is there a constant \( C \) such that every planar graph \( G \) has \( \text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C \)? About this question, there are some other results as follows.

**Theorem 2.** Let \( G \) be a planar graph.

(i) (Cranston and Yu [1]) If \( g(G) \geq 5 \), then \( \text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 4 \).

(ii) (Dong et al. [2]) If \( g(G) \geq 6 \), then \( \text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3 \).

(iii) (Dong and Lin [3]) If \( g(G) \geq 6 \) and \( \Delta(G) \geq 39 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

In this paper, we prove the following results.

**Theorem 3.** Let \( G \) be a graph.

1. If \( \text{mad}(G) < \frac{8}{3} \) and \( \Delta(G) \geq 7 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).
2. If \( \text{mad}(G) < \frac{18}{7} \) and \( \Delta(G) \geq 5 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).
3. If \( \text{mad}(G) < \frac{20}{7} \) and \( \Delta(G) \geq 5 \), then \( \text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

Then the following results about planar graphs are implied immediately from Theorem 3(1) and (2), respectively.

**Theorem 4.** Let \( G \) be a planar graph.

1. If \( g(G) \geq 8 \) and \( \Delta(G) \geq 7 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).
2. If \( g(G) \geq 9 \) and \( \Delta(G) \geq 5 \), then \( \text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

We will prove the three results of Theorem 3 by contradiction in the following three sections, respectively. For convenience, we introduce some notations that will be used. Let \( c \) be a coloring of \( G \); we use \( c(v) \) to denote the color of \( v \) in \( c \), and \( c(S) = \{c(v) : v \in S\} \) for \( S \subset V(G) \). Let \( c_i(v) \) be the set of colors appeared \( i \) times in \( N(v) \). For a vertex \( v \in V(G) \), let \( n_2(v) \) for clarity be the number of 2-vertices in \( N(v) \).
2. Graphs with \( \text{mad}(G) < \frac{8}{3} \) and \( \Delta(G) \geq 7 \)

In order to prove Theorem 3(1), we prove the following result instead, which implies Theorem 3(1) immediately.

**Theorem 5.** Let \( M \geq 7 \) be an integer. If \( G \) is a graph with \( \text{mad}(G) < \frac{8}{3} \) and \( \Delta(G) \leq M \), then \( l_1(G) = \lceil \frac{M}{2} \rceil + 1 \).

**Proof.** By contradiction, we suppose that Theorem 5 is false. Let \( G \) be a counterexample with the fewest vertices, and \( L \) the list assignment of size \( \lceil \frac{M}{2} \rceil + 1 \) such that \( G \) has no linear \( L \)-coloring. Let \( H \) be a proper subgraph of \( G \). Clearly, \( \text{mad}(H) < \frac{8}{3} \) and \( \Delta(H) \leq M \). By the choice of \( G \), we have \( l_1(H) = \lceil \frac{M}{2} \rceil + 1 \), while \( l_1(G) > \lceil \frac{M}{2} \rceil + 1 \). In the proof we need some structural lemmatas, Lemma 6 is well-known.

**Lemma 6.** The graph \( G \) is connected, and \( \delta(G) \geq 2 \).

**Lemma 7** ([3] Lemma 2.2). Let \( v \) be a 2-vertex with \( N(v) = \{v_1, v_2\} \). Then \( \left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 1 \).

**Lemma 8.** Let \( v \) be a 3-vertex with \( N(v) = \{v_1, v_2, v_3\} \) and \( n_2(v) = 3 \). Then \( v_1, v_2, v_3 \) must be \((3, 6^+)\)-vertices.

**Proof.** Assume that \( v_1 \) is a \((3, 5^-)\)-vertex, and \( u_i \) is the neighbor of \( v_i \) other than \( v \), where \( i = 1, 2, 3 \). Let \( G' = G - \{v, v_1\} \). Then \( G' \) has a linear \( L \)-coloring \( c \) by the minimality of \( G \). If \( c(v_2) \neq c(v_3) \), we can extend the linear \( L \)-coloring \( c \) of \( G' \) to \( v_1 \) since \( |L(v_1) \setminus \{c(u_1), c(u_2)\}| \geq 2 \). Then we can color \( v \) with a color in \( L(v) \setminus \{c(v_1), c(v_2), c(v_3)\} \) when \( c(v_1) \notin \{c(v_2), c(v_3)\} \), or \( L(v) \setminus \{c(u_1), c(u_2), c(v_3)\} \) when \( c(v_1) \in \{c(v_2), c(v_3)\} \). Clearly, there will be no bi-colored cycles created, and we get a linear list coloring of \( G \). If \( c(v_2) = c(v_3) \), we can extend the linear \( L \)-coloring \( c \) of \( G' \) to \( v_1 \) since \( |L(v_1) \setminus \{c(u_1), c(u_2), c(v_2)\}| \geq 1 \). Finally, we can color \( v \) with a color in \( L(v) \setminus \{c(v_1), c(v_2), c(u_2), c(u_3)\} \), which ensure that no bi-colored cycle passes \( v_2v_2u_2 \) or \( v_3v_3u_3 \). Thus, we also get a linear list coloring of \( G \) extended from the linear \( L \)-coloring of \( G' \). A contradiction.

**Lemma 9.** Let \( v \) be a 5-vertex with \( N(v) = \{v_1, \ldots, v_5\} \) and \( n_2(v) = 5 \). If \( v_1, v_2, v_3, v_4 \) are \((5, 3)\)-vertices, then \( v_5 \) must be a \((5, 4^+)\)-vertex.

**Proof.** Suppose to the contrary, let \( v_5 \) be a \((5, 3)\)-vertex, and \( u_i \) be the neighbor of \( v_i \) other than \( v \) for \( i = 1, 2, \ldots, 5 \).

Let \( G' = G - N[v] \). Then \( G' \) has a linear \( L \)-coloring \( c \) by the minimality of \( G \). There exist at least \( |L(v_1) \setminus \{c(u_1), c(N(u_1))\}| \geq 2 \) available colors for \( v_1 \). Since \( |L(v_2) \setminus \{c(v_1), c(u_2), c(N(u_2))\}| \geq 1 \) and \( |L(v_3) \setminus \{c(v_1), c(v_2), c(u_3), c(u_2)\}| \geq 1 \), we can extend the coloring \( c \) of \( G' \) to \( v_1, v_2, v_3 \) such that \( |\{c(v_1), c(v_2), c(v_3)\}| = 3 \).
Notice that there will be no bi-colored cycle passing \(v_1u_1\) or \(v_2u_2\). Then we color \(v_3\) with a color in \(L(v_3) \setminus \{c(u_2), c(N(u_3))\}\), and no bi-colored cycle will pass \(v_3u_4\). Finally, we extend the coloring \(c\) to \(v_5\) and \(v\) in two different cases.

If \(|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 4\), we can linearly color \(v\) with a color in \(L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_4)\}\), and color \(v_5\) such that no bi-colored cycle passes \(v_5u_5\) as \(|L(v_5) \setminus \{c(u_3), c(v), c(N(u_5))\}| \geq 1\). So we get a linear \(L\)-coloring of \(G\).

If \(|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 3\), we can color \(v_5\) such that no bi-colored cycle passing \(v_5u_5\) since \(|L(v_5) \setminus \{c(v_4), c(u_5), c(N(u_5))\}| \geq 1\), and color \(v\) with a color in \(L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_4)\}\). Thus, we get a linear \(L\)-coloring of \(G\).

Therefore, we can extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\), a contradiction.

**Lemma 10.** Let \(v\) be a 7-vertex with \(N(v) = \{v_1, v_2, \ldots, v_7\}\) and \(n_2(v) = 7\). If \(v_1, v_2, \ldots, v_5\) are \((7,2)\)-vertices, then at least one of \(v_6\) and \(v_7\) is a \((7,4^+)\)-vertex.

**Proof.** Assume that \(v_6\) and \(v_7\) are \((7,3^-)\)-vertices, and \(u_4\) is the neighbor of \(v_i\) other than \(v\) for \(i = 1, 2, \ldots, 7\). Let \(G = G - N[v]\). Then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). First, we extend the linear \(L\)-coloring \(c\) of \(G'\) to \(v_7\) and \(v_6\) such that \(c(v_6) \neq c(v_7)\) and no bi-colored cycle passes \(v_5u_5\) or \(v_5u_0\) since \(|L(v_7) \setminus \{c(v_7), c(N(u_7))\}| \geq 2\) and \(|L(v_6) \setminus \{c(u_6), c(N(u_6)), c(v_7)\}| \geq 1\). Next, we can color \(v_5\) such that \(c(v_5) \notin \{c(v_6), c(v_7)\}\) and no bi-colored cycle passes \(v_5u_5\) since \(|L(v_5) \setminus \{c(u_5), c(N(u_5)), c(v_6), c(v_7)\}| \geq 1\). Then we can color \(v_4\) with \(c(v_4) \notin \{c(v_5), c(v_6), c(v_7)\}\) since \(|L(v_4) \setminus \{c(u_4), c(v_5), c(v_6), c(v_7)\}| \geq 1\). Notice that \(|\{c(v_4), c(v_5), c(v_6), c(v_7)\}| = 4\). Then we color \(v\) with a color in \(L(v) \setminus \{c(v_7), c(u_3), c(v_5), c(v_4)\}\). Since \(|L(v_3) \setminus \{c(u_3), c(v), c(N(u_3))\}| \geq 2\) and \(|L(v_2) \setminus \{c(u_2), c(v), c_2(v), c(N(u_2))\}| \geq 1\) \((|c_2(v)| \leq 1\) now), we can color \(v_3, v_2\) in order such that no bi-colored cycle passes \(v_3u_3\) or \(v_2u_2\). Finally, in order to avoid bi-colored cycles passing \(v_1u_1\), we can color \(v_1\) with a color in \(L(v_1) \setminus \{c(u_1), c(v), c_2(v)\}\) \((|c_2(v)| \leq 2\) now) when \(c(u_1) \neq c(v)\), or \(v_1\) with a color in \(L(v_1) \setminus \{c(u_1), c_2(v), c(N(u_1))\}\) when \(c(u_1) = c(v)\). Thus, we get a linear list coloring of \(G\) extended from the linear list coloring \(c\) of \(G'\), a contradiction.

To complete our proof of Theorem 5, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function \(\omega\) on \(V(G)\) by \(\omega(v) = d(v) - \frac{8}{3}\) for every \(v \in V(v)\). Since \(mad(G) < \frac{8}{3}\), the sum of the initial charge is negative. If we can make suitable discharging rules to redistribute charges among vertices so that the final charge \(\omega'(v)\) of every vertex \(v \in V(G)\) is nonnegative, then we get a contradiction. The discharging rules are as follows.

**R1.** Every \(8^+\)-vertex sends \(\frac{2}{3}\) to each adjacent 2-vertex.

**R2.** Every 7-vertex sends \(\frac{2}{3}\) to each adjacent \((7,2)\)-vertex, \(\frac{2}{3}\) to each adjacent \((7,3)\)-vertex, and \(\frac{1}{2}\) to each adjacent \((7,4^+)\)-vertex.

**R3.** Every 6-vertex sends \(\frac{5}{6}\) to each adjacent 2-vertex.
R4. Every 5-vertex sends $\frac{1}{2}$ to each adjacent (5,3)-vertex, $\frac{1}{3}$ to each adjacent (5,4+)-vertex.

R5. Every 4-vertex sends $\frac{1}{3}$ to each adjacent 2-vertex.

R6. Every 3-vertex sends $\frac{1}{5}$ to each adjacent (3,5)-vertex, and $\frac{1}{6}$ to each adjacent (3,6+)-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V(G)$.

Let $v$ be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 7$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{2}{3} = 2 - \frac{5}{3} + \frac{2}{3} = 0$. If $d(x) = 3$, then $d(y) \geq 5$ by Lemma 7. Thus $\omega'(v) = \omega(v) + \frac{1}{3} + \frac{1}{2} = 2 - \frac{3}{3} + \frac{2}{3} = 0$ or $\omega'(v) \geq \omega(v) + \frac{1}{9} + \frac{2}{9} = 2 - \frac{3}{9} + \frac{2}{9} = 0$ by R6, R2, R3, and R4. Otherwise, $d(x) \geq 4$ and $d(y) \geq 5$, we have $\omega'(v) \geq \omega(v) + \frac{1}{3} + \frac{1}{3} = 2 - \frac{3}{3} + \frac{2}{3} = 0$ by R5, R2, R3, and R4.

Let $v$ be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3,6+)$-vertices by Lemma 8. Thus $\omega'(v) = \omega(v) - 3 \times \frac{1}{2} = 3 - \frac{3}{2} = 0$ by R6. If $n_2(v) \leq 3$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{2} = 3 - \frac{3}{2} = 0$ by R6.

Let $v$ be a 4-vertex. Then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{3} = 4 - \frac{4}{3} = 0$ by R5.

Let $v$ be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 5 - \frac{5}{2} = 0$ by R4. If $n_2(v) = 5$, then there are at most four $(3,5)$-vertices in $N(v)$ by Lemma 9. Thus $\omega'(v) = \omega(v) - 4 \times \frac{1}{3} = 5 - \frac{5}{3} = 0$ by R4.

Let $v$ be a 6-vertex. Then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{3} = 6 - \frac{6}{3} = 0$ by R3.

Let $v$ be a 7-vertex. If $n_2(v) \leq 6$, then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{3} = 7 - \frac{6}{3} = 4 > 0$ by R2. When $n_2(v) = 7$, if there are no more than four $(7,2)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{2}{3} = 7 - \frac{8}{3} = \frac{13}{3} = 0$ by R2; if there are five $(7,2)$-vertices in $N(v)$, then at least one of the other neighbors is a $(7,4+)$-vertex from Lemma 10, and $\omega'(v) = \omega(v) - 6 \times \frac{2}{3} = 7 - \frac{8}{3} = \frac{19}{3} = 0$ by R2.

Finally, if $d(v) \geq 8$, then $\omega'(v) \geq \omega(v) - \frac{2}{3} \times d(v) = \frac{d(v)}{3} - \frac{8}{3} \geq 0$ by R1.

Thus, we get the desired contradiction, and Theorem 5 is proved.

It is interesting that Cranston and Yu [1] cited an example ($\text{mad}(K_{2,3}) = \frac{12}{5}$ and $\text{lc}(K_{2,3}) \geq \left[\frac{\Delta(G)}{2}\right] + 2$) to illustrate that the bound in Theorem 1(iv) is sharp. Similarly, the graph $K_{2,4}$ satisfies $\text{lc}(K_{2,4}) \geq \left[\frac{\Delta}{2}\right] + 2$, $\Delta(K_{2,4}) = 4$ and $\text{mad}(K_{2,4}) = \frac{8}{3}$. So the hypothesis about $\Delta(G)$ in Theorem 3(1) is essential, and we suspect it can be replaced by $\Delta(G) \geq 5$.

3. Graphs with $\text{mad}(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$

For Theorem 3(2), we prove the following result instead.

**Theorem 11.** Let $M \geq 5$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{18}{7}$ and $\Delta(G) \leq M$, then $\text{lc}_1(G) = \left[\frac{M}{2}\right] + 1$. 

**Proof.** By contradiction, we suppose that Theorem 11 is false. Let $G$ be a counterexample with the fewest vertices and $L$ be a list assignment of size $\left\lceil \frac{M}{2} \right\rceil + 1 \geq 4$ such that $G$ has no linear $L$-coloring. In the proof we need some structural lemmatas, and it is clear that Lemma 6 and Lemma 7 are also true. ■

**Lemma 12.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then $v_1, v_2, v_3$ must be $(3, 5^+)$-vertices.

**Proof.** Assume that $v_1$ is a $(3, 4^-)$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_2) \neq c(v_3)$, there exist at least $|L(v)| \{c(u_1), c(u_2)\} \geq 2$ colors available for $v_1$. If there is an available color $\alpha \notin \{c(v_2), c(v_3)\}$ for $v_1$, then let $c(v_1) = \alpha$ and $c(v) \in L(v) \{c(v_1), c(v_2), c(v_3)\}$. If the available colors for $v_1$ are exactly $c(v_2)$ and $c(v_3)$, then let $c(v_1) = c(v_2)$ and $c(v) \in L(v) \{c(v_1), c(u_2), c(v_3)\}$. It is similar for $c(v_1) = c(v_3)$. Thus we get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. If $c(v_2) = c(v_3)$, we can extend the linear list coloring $c$ of $G'$ to $v_1$ since $|L(v)| \{c(u_1), c(u_2), c(v_2)\} \geq 1$. There is at least $|L(v)| \{c(v_1), c(v_2), c(v_3)\} \geq 1$ color available for $v$. Thus, we also get a linear list coloring of $G$. A contradiction. ■

**Lemma 13.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$. If $v_1$ and $v_2$ are $(3, 3)$-vertices, then $v_3$ must be a $4^+$-vertex.

**Proof.** Assume that $v_3$ is a $3^-$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear $L$-coloring $c$ to $v_1$ such that no bi-colored cycle passes $v_1, v_2$ since $|L(v)| \{c(u_1), c(N(u_1))\} \geq 1$.

If $c(v_1) = c(v_3)$. Since $|L(v)| \{c(v_1), c(u_2), c_2(v_3)\} \geq 1$, we can extend the coloring $c$ to $v_2$. Finally, we can color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(v_3)\}$ when $c_2(v_3) = 1$, or in $L(v) \{c(v_1), c(v_2), c(u_2)\}$ when $c_2(v_3) = 0$. It is clear that no bi-colored cycle passes $v_2v_3$. Then we get a linear list coloring of $G$.

If $c(v_1) \neq c(v_3)$. There is at least $|L(v)| \{c(v_1), c(v_3), c(v_2)\} \geq 1$ color available for $v$. Finally, we can color $v_3$ with a color in $L(v) \{c(v), c(u_2), c_2(v_2)\}$ when $c(v) \neq c(v_2)$, or in $L(v) \{c(v_2), c(N(u_2))\}$ when $c(v) = c(u_2)$. In this process, there will be no bi-colored cycle passing $v_2v_3$. Thus, we also get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. A contradiction. ■

**Lemma 14.** Let $v$ be a 4-vertex with $n_2(v) = 4$ in $G$. Then there are at most two $(4, 3)$-vertices in $N(v)$.

**Proof.** Let $N(v) = \{v_1, \ldots, v_4\}$, and $u_i$ be the other neighbor of $v_i$ for $i = 1, \ldots, 4$. Assume that $v_1, v_2$ and $v_3$ are $(4, 3)$-vertices. Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear
$L$-coloring $c$ of $G'$ to $v_4$ since $|L(v_4)\{c(u_4), c_2(u_4)\}| \geq 1$. We can continue to extend to $v_3$ with $c(v_3) \neq c(v_4)$ since $|L(v_3)\{c(u_3), c_2(u_3), c(v_4)\}| \geq 1$. Then we color $v_2$ with a color in $L(v_2)\{c(u_2), c(N(u_2))\}$. Notice that no bi-colored cycle passes $v_2u_2$ or $v_3v_4$. This signifies that any bi-colored cycle in $G$ if there will be must passes $v_1$. Finally, we will extend the coloring $c$ to $v_1$ and $v$ in two different cases.

If $c(v_2) \notin \{c(v_4), c(v_4)\}$, we can choose a color from $L(v)\{c(v_2), c(v_3), c(v_4)\}$ for $v$. Then there is at least $|L(v_1)\{c(v), c(u_1), c_2(u_1)\}| \geq 1$ when $c(v_1) = c_1(u_1)$, or $|L(v_1)\{c(u_1), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$ color available for $v_1$, which ensure no bi-colored cycle passes $v_1u_1$. So we get a linear list coloring of $G$.

If $c(v_2) \in \{c(v_3), c(v_4)\}$, suppose $c(v_2) = c(v_3)$ (similarly for $c(v_2) = c(v_4)$). If $|c_2(u_1)| = 1$, we color $v_1$ with a color in $L(v_1)\{c(v_2), c(u_1), c_2(u_1)\}$, and no bi-colored cycle passes $v_1u_1$. If $|c_2(u_1)| = 0$, we color $v_1$ with a color in $L(v_1)\{c(v_2), c(u_1), c(v_4)\}$, which ensure that no bi-colored cycle passes $v_1v_4$. Then we color $v$ with a color in $L(v)\{c(v_1), c(v_2), c(v_3)\}$. Notice that no bi-colored cycle passes $v_1v_3$ since $c(v_1) \neq c(v_3)$. Thus, we also get a linear list coloring $c$ of $G$. A contradiction.

\begin{lemma}
Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$. If $v_1, v_2, v_3, v_4$ are four (5,2)-vertices, then $v_5$ must be a 3$^+$-vertex.
\end{lemma}

\begin{proof}
Assume that $v_5$ is a 2-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, \ldots, 5$. Let $G' = G - [v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the $L$-coloring $c$ of $G'$ to $v_5$ since $|L(v_5)\{c(u_5), c_2(u_5)\}| \geq 1$, and continue to $v_4$ such that $c(v_4) \neq c(v_5)$ and no bi-colored cycle passes $v_4u_4$ since $|L(v_4)\{c(u_4), c(N(u_4))\}| \geq 1$, then to $v_3$ with $c(v_3) \notin \{c(v_4), c(v_5)\}$ since $|L(v_3)\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. We can color $v$ with a color in $L(v)\{c(v_5), c(v_4), c(v_3)\}$, and color $v_2$ such that no bi-colored cycle passes $v_2u_2$ since $|L(v_2)\{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color $v_1$ linearly since $|L(v_1)\{c(v), c_2(v), c(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_5)$, there will be no bi-colored cycle created. Thus we can extend the linear $L$-coloring $c$ of $G'$ to $G$. A contradiction.
\end{proof}

\begin{lemma}
Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If $v_1, v_2, v_3$ are (5,2)-vertices, then at least one of $v_4$ and $v_5$ is a (5,4$^+$)-vertex.
\end{lemma}

\begin{proof}
Assume that $v_3$ and $v_5$ are (5,3$^-$)-vertices, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, \ldots, 5$. Let $G' = G - [v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the coloring $c$ of $G'$ to $v_5$ such that no bi-colored cycle passes $v_5u_5$ since $|L(v_5)\{c(u_5), c(N(u_5))\}| \geq 1$, and continue to $v_4$ such that $c(v_4) \neq c(v_5)$ as $|L(v_4)\{c(u_4), c_2(u_4), c(v_5)\}| \geq 1$, then to $v_3$ with $c(v_3) \notin \{c(v_5), c(v_4)\}$ since $|L(v_3)\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. Now we
can color $v$ with a color in $L(v) \setminus \{c(v_5), c(v_4), c(v_3)\}$, and color $v_2$ such that no bi-colored cycle passes $vv_2u_2$ since $|L(v_2) \setminus \{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color $v_1$ linearly since $|L(v_1) \setminus \{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1) \setminus \{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_4)$, there will be no bi-colored cycle created. Thus, we can extend the linear $L$-coloring $c$ of $G'$ to $G$. A contradiction.

We will derive a contradiction by a discharging procedure proceeded in $G$ to complete the proof of Theorem 11. In the discharging procedure, the initial charge function $\omega$ is defined as $\omega(v) = d(v) - \frac{18}{7}$ for every vertex $v \in V(G)$, and the discharging rules are as follows.

**R1.** Every $6^+$-vertex sends $\frac{4}{7}$ to each adjacent 2-vertex or 3-vertex.

**R2.** Every 5-vertex sends $\frac{7}{9}$ to each adjacent $(5, 2)$-vertex, $\frac{4}{9}$ to each adjacent $(5, 3)$-vertex, $\frac{7}{9}$ to each adjacent $(5, 4^+)$-vertex, $\frac{4}{9}$ to each adjacent 3-vertex.

**R3.** Every 4-vertex sends $\frac{4}{9}$ to each adjacent $(4, 3)$-vertex, $\frac{7}{9}$ to each adjacent $(4, 4^+)$-vertex, $\frac{4}{9}$ to each adjacent 3-vertex.

**R4.** Every 3-vertex sends $\frac{4}{9}$ to each adjacent $(3, 3)$-vertex, $\frac{7}{9}$ to each adjacent $(3, 4^+)$-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V$.

If $d(v) \geq 6$, then $\omega'(v) \geq \omega(v) - \frac{4}{7} \times d(v) = \frac{3d(v)}{7} - \frac{18}{7} = \frac{3d(v) - 18}{7} \geq 0$ by R1.

Let $v$ be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{4}{7} - \frac{1}{7} = 5 - \frac{18}{7} - \frac{1}{7} = 0$ by R2. When $n_2(v) = 5$, there are at most three $(2, 5)$-vertices in $N(v)$ by Lemma 15. If there are two or less $(2, 5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{4}{7} - 3 \times \frac{3}{7} = 5 - \frac{18}{7} - \frac{9}{7} - \frac{2}{7} = 0$ by R2. If there are three $(2, 5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{4}{7} - \frac{2}{7} = 5 - \frac{18}{7} - \frac{12}{7} - \frac{5}{7} = 0$ by Lemma 16 and R2.

Let $v$ be a 4-vertex. If $n_2(v) \leq 3$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{3}{7} - \frac{1}{7} = 4 - \frac{18}{7} - \frac{9}{7} - \frac{1}{7} = 0$ by R3. If $n_2(v) = 4$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{2}{7} = 4 - \frac{18}{7} - \frac{4}{7} - \frac{4}{7} = 0$ by Lemma 14 and R3.

Let $v$ be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3, 5^+)$-vertices by Lemma 12. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0$ by R4. If $n_2(v) = 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{2}{7} + \frac{1}{7} = 3 - \frac{18}{7} - \frac{4}{7} + \frac{1}{7} = 0$ by Lemma 13 and all discharging rules, or $\omega'(v) \geq \omega(v) - \frac{3}{7} - \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0$. If $n_2(v) \leq 1$, then $\omega'(v) \geq \omega(v) - \frac{3}{7} = 3 - \frac{18}{7} - \frac{3}{7} > 0$ by R4.

Finally, let $v$ be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 5$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{4}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$. When $d(x) = 3$, we have $\omega'(v) = \omega(v) + 2 \times \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ if $d(y) = 3$, and $\omega'(v) = \omega(v) + \frac{1}{7} + \frac{4}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ if $d(y) \geq 4$. Otherwise, $d(x) \geq 4$ and $d(y) \geq 4$, we have $\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{4}{7} = 2 - \frac{18}{7} + \frac{3}{7} = 0$ by R1, R2, and R3. We get the desired contradiction, and Theorem 11 is proved.
Similarly, the condition $\Delta(G)$ in Theorem 3(2) must be $\Delta(G) \geq 4$.

4. Graphs with $\text{mad}(G) < \frac{20}{7}$ and $\Delta(G) \geq 5$

Cranston and Yu [1] conjectured that the hypothesis $\Delta(G) \geq 9$ of Theorem 1(iii) can be replaced by $\Delta(G) \geq 7$, even $\Delta(G) \geq 5$. Now, we prove Theorem 3(3) to support their conjecture. In order to prove Theorem 3(3), we prove the following theorem which implies Theorem 3(3) immediately.

**Theorem 17.** Let $M \geq 5$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{20}{7}$ and $\Delta(G) \leq M$, then $\text{lc}(G) \leq \left\lceil \frac{M}{2} \right\rceil + 2$.

**Proof.** Let $G$ be a counterexample of the fewest vertices with $\text{mad}(G) < \frac{20}{7}$ and $5 \leq \Delta(G) \leq 8$ (Theorem 17 is true for graphs $G$ with $\Delta(G) \geq 9$ by Theorem 1(iii)). There exists an assignment $L$ with $|L| \geq \left\lceil \frac{M}{2} \right\rceil + 2 \geq 5$ such that $G$ is not linearly $L$-choosable, but $H$ has a linear $L$-coloring, where $H$ is any proper subgraph of $G$. Clearly, $G$ is connected and $\delta(G) \geq 2$. In the proof we need some structural lemmatas.

**Lemma 18.** Let $v$ be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \left\lceil \frac{M}{2} \right\rceil + 2$.

**Proof.** Assume $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 2$. Let $G' = G - v$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_1) \neq c(v_2)$, we can color $v$ with any color in $L(v) \setminus \{c(v_1), c(v_2), c_1(v_1), c_2(v_2)\}$. Then the number of available colors for $v$ is at least $\left\lceil \frac{M}{2} \right\rceil + 2 - \left(2 + \left\lceil \frac{d(v_1) - 1}{2} \right\rceil + \left\lceil \frac{d(v_2) - 1}{2} \right\rceil \right) = \left\lceil \frac{M}{2} \right\rceil + 2 - \left(\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \right) \geq 1$. Clearly, there will be no bi-colored cycle created. So we extend the linear $L$-coloring $c$ of $G'$ to $G$. Now we suppose $c(v_1) = c(v_2)$. In order to color $v$ linearly and avoid bi-colored cycles created, the forbidden color set for $v$ contains the color $c(v_1)$, the colors appearing twice in $N(v_1)$ or $N(v_2)$, and the colors appearing in both $N(v_1)$ and $N(v_2)$. So at most $1 + |c_2(v_1) \cup c_2(v_2)| + |c_1(v_1) \cap c_1(v_2)| \leq \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \leq \left\lceil \frac{d(x)}{2} \right\rceil + \left\lceil \frac{d(y)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 1$ colors are forbidden for $v$. Thus, we also can get a linear $L$-coloring of $G$. A contradiction.

**Lemma 19.** Let $v$ be a 3-vertex of $G$ with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 2$, then $d(v_2) \geq 3$ and $\left\lceil \frac{d(v_2) + d(v_3)}{2} \right\rceil \geq \left\lceil \frac{M}{2} \right\rceil + 1$.

**Proof.** We prove $d(v_2) \geq 3$ first. To the contrary, we assume $d(v_2) = 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. The neighbors of $v_1$ and $v_2$ other than $v$ are denoted by $u_1$ and $u_2$, respectively. We can extend the coloring $c$ of $G'$ to $v_1$ such that $c(v_1) \neq c(v_3)$ since
\(|L(v_1) \setminus \{c(u_1), c_2(u_1), c(v_3)\}| \geq 1\), which ensures that no bi-colored cycle passes \(v_1v_3\). We can continue to extend to \(v\) since \(|L(v) \setminus \{c(v_1), c_2(v_3), c(v_3)\}| \geq 1\). If \(c(v) \neq c(u_2)\), which means that no bi-colored cycle passes \(v_1v_2\), we can color \(v_2\) linearly since \(|L(v_2) \setminus \{c(v), c(u_2), c_2(u_2)\}| \geq 1\). When \(c(v) = c(u_2)\), the number of available colors for \(v_2\) is at least \(|L(v_2) \setminus \{c(u_2), c_2(u_2)\}| \geq 2\). If there is an available color \(\alpha \notin \{c(v_1), c(v_3)\}\) for \(v_2\), then we color \(v_2\) with \(\alpha\). Now we assume that the available colors for \(v_2\) are exactly \(c(v_1)\) and \(c(v_3)\). Notice that \(|c_2(u_2)| = \left\lceil \frac{M-1}{2} \right\rceil\) and \(|c_1(u_2)| \leq 1\). To avoid bi-colored cycle created, the number of forbidden colors for \(v_2\) is at most \(|\{c(u_2), c(N(u_2))\}| = 1 + |c_2(u_2)| + |c_1(u_2)| \leq 1 + \left\lceil \frac{M-1}{2} \right\rceil + 1 = \left\lceil \frac{M}{2} \right\rceil + 1\), so we can color \(v_2\) linearly. Thus, we get a linear \(L\)-coloring of \(G\) extended from the linear \(L\)-coloring \(c\) of \(G\). A contradiction.

Now, we prove the inequality. Suppose to the contrary that, we have \(\left\lceil \frac{d(v_2)+d(v_3)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil\), and \(u_1\) is the neighbor of \(v_1\) other than \(v\). Let \(G' = G - v_1\), then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\).

Case 1. \(c(v_2) \neq c(v_3)\). If \(c(v) \neq c(u_1)\), then we can extend the coloring \(c\) to \(v_1\) to get a linear \(L\)-coloring of \(G\) since \(|L(v_1) \setminus \{c(v), c(u_1), c_2(u_1)\}| \geq 1\). If \(c(v) = c(u_1)\), the number of available colors for \(v_1\) is at least \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq 2\). If there is a color \(\alpha \notin \{c(v_2), c(v_3)\}\) available for \(v\), then we can extend \(c\) from \(G'\) to \(G\) by coloring \(v_1\) with \(\alpha\). Now we assume that \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| = \{c(v_2), c(v_3)\}\) available for \(v\). Notice that \(|c_2(u_1)| = \left\lceil \frac{M-1}{2} \right\rceil\) now. Then \(c(v_2)\) and \(c(v_3)\) appears at most once in \(N(u_1)\), but both of them could not appear in \(N(u_1)\) at the same time (otherwise \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq \left\lceil \frac{M}{2} \right\rceil + 2 - (1 + \left\lceil \frac{M-3}{2} \right\rceil) \geq 3\)). So we color \(v_1\) with \(c(v_1)\) if \(c(v_2)\) appears in \(N(u_1)\), otherwise color \(v_1\) with \(c(v_2)\). Then there will be no bi-colored cycle created. Thus, we get a linear \(L\)-coloring of \(G\) extended from the linear \(L\)-coloring \(c\) of \(G'\).

Case 2. \(c(v_2) = c(v_3)\). If \(c(v) = c(u_1)\), then we can color \(v_1\) linearly since \(|L(v_1) \setminus \{c(v_2), c(v_1), c_2(u_1)\}| \geq 1\), and no bi-colored cycle created. Now, suppose \(c(v) \neq c(u_1)\). We erase the color of \(v\) first, then we can extend the list coloring \(c\) to \(v_1\) since \(|L(v_1) \setminus \{c(v_2), c(u_1), c_2(u_1)\}| \geq 1\). To avoid bi-colored cycle created, the number of forbidden colors for \(v\) is at most \(2 + |c_2(v_2) \cup c_2(v_3)| + |c_1(v_2) \cap c_1(v_3)| \leq 2 + \frac{(d(v_2)+d(v_3)-1)}{2} = 1 + \frac{(d(v_2)+d(v_3)-1)}{2} \leq \left\lceil \frac{M}{2} \right\rceil + 1\). We also can extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\). A contradiction. \(\blacksquare\)

**Lemma 20.** Let \(v\) be a 4-vertex in \(G\). Then \(n_2(v) \leq 3\).

**Proof.** Let \(N(v) = \{v_1, \ldots, v_4\}\). Suppose to the contrary, let \(n_2(v) = 4\), and \(u_i\) be the neighbor of \(v_i\) other than \(v\) for \(i = 1, \ldots, 4\). Let \(G' = G - N[v]\). Then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). Since \(|L(v_1) \setminus \{c(u_i), c_2(u_i)\}| \geq 2\), we can color \(v_i\) linearly with at least two different colors for \(i = 1, 2, 3, 4\). Finally, we can color \(v\) with a color in \(L(v) \setminus c(N(v))\) if \(|c(N(v))| = 4\), or in
\(L(v)\{c(N(v)), c(u_i)\}\) if \(c(v_i) = c(v_j)\) for \(1 \leq i < j \leq 4\). And no bi-colored cycle appears in this process. Thus, we get a linear list coloring of \(G\) extended from the linear \(L\)-coloring \(c\) of \(G'\), a contradiction.

**Lemma 21.** Let \(v\) be a 4-vertex with \(N(v) = \{v_1, \ldots, v_4\}\). If \(d(v_1) = d(v_2) = d(v_3) = 2\) and \(v_4\) is a \(4^+\)-vertex, then \(v_4\) must be a \(4^+\)-vertex.

**Proof.** Suppose to the contrary, let \(v_4\) be a 3^-vertex, and \(u_i\) be the neighbor of \(v_i\) other than \(v\) for \(i = 1, 2, 3\). Let \(G' = G - \{v, v_1, v_2, v_3\}\), then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). We can extend the coloring \(c\) of \(G'\) to \(v\) with \(c(v_1) \neq c(v_4)\) since \(|L(v_1)\{c(u_1), c_2(u_1), c(v_4)\}| \geq 1\). Then there are at least \(|L(v_2)\{c(u_2), c_2(u_3)\}| \geq 2\) colors available for \(v_2\).

If there is a color \(\alpha \notin \{c(v_1), c(v_4)\}\) available for \(v_2\), let \(c(v_2) = \alpha\). If \(|c_2(v_4)| = 1\), we can choose a color for \(v\) in \(L(v)\{c(v_1), c(v_2), c(v_4), c_2(v_4)\}\), and there will be no bi-colored cycle created passing \(v_4\). Then, we color \(v_3\) with a color in \(L(v_3)\{c(v), c(v_3), c_2(v_3)\}\) if \(c(v) \neq c(u_3)\). When \(c(v) = c(u_3)\), in order to color \(v_3\) linearly (no bi-colored cycle created), we must forbidden \(c(v), c_2(N(u_3))\) and \(\{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))\). Notice that \(d(u_3) = 5\), then \(|c_2(N(u_3)) \cup \{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))| \geq 3\). So we can color \(v_3\) linearly with a color in \(L(v_3)\{c(v), c_2(N(u_3)), c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))\). Thus, we get a linear list coloring of \(G\).

Suppose the available color set for \(v_2\) is exactly \(\{c(v_1), c(v_4)\}\). Notice that \(|c_2(v_2)| = \left\lfloor \frac{M-1}{2} \right\rfloor\) now. We color \(v_2\) with \(c(v_4)\) first. If \(|c_2(u_3)| = 2\), we color \(v_3\) with a color in \(L(v_3)\{c(v_4), c_3(u_3), c_2(u_3)\}\). If \(|c_2(u_3)| \leq 1\), we color \(v_3\) with a color in \(L(v_3)\{c(v_4), c(v_1), c_3(u_3), c_2(u_3)\}\). Notice \(d(u_3) = 5\), then no bi-colored cycle passes \(v_3u_3\). Finally, we can color \(v\) with a color in \(L(v)\{c(v_1), c(v_3), c(v_4), c_2(v_4)\}\) if \(|c_2(v_4)| = 1\), or in \(L(v)\{c(v_1), c(v_3), c(v_4), c_2(v_2)\}\) if \(|c_2(v_4)| = 0\). Clearly, there will be no bi-colored cycle passing \(v_4\). Then we extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\), a contradiction.

**Lemma 22.** Let \(v\) be a 4-vertex with \(N(v) = \{v_1, \ldots, v_4\}\). If \(d(v_1) = d(v_2) = d(v_3) = 2\) and \(v_4\) are \(4^+\)-vertices, then \(v_4\) must be a \(5^+\)-vertex.

**Proof.** Suppose to the contrary, let \(v_4\) be a \(4^-\)-vertex, and \(u_i\) be the neighbor of \(v_i\) other than \(v\) for \(i = 1, 2, 3\). Let \(G' = G - \{v, v_1, v_2, v_3\}\). Then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). We can extend the coloring \(c\) of \(G'\) to \(v\) with \(c(v_1) \neq c(v_4)\) since \(|L(v_1)\{c(u_1), c_2(u_1), c(v_4)\}| \geq 1\). Then there are at least \(|L(v_2)\{c(u_2), c_2(u_3)\}| \geq 2\) colors available for \(v_2\).

If there is an available color \(\alpha \notin \{c(v_1), c(v_4)\}\) for \(v_2\), let \(c(v_2) = \alpha\). If \(|c_2(v_4)| = 1\), we color \(v\) with a color in \(L(v)\{c(v_1), c(v_2), c(v_4), c_2(v_4)\}\). Then, we color \(v_3\) with a color in \(L(v_3)\{c(v), c(v_3), c_2(v_3)\}\) if \(c(v) \neq c(u_3)\). If \(c(v) = c(u_3)\), we color \(v_3\) with a color in \(L(v_3)\{c(v_3), c_2(u_3)\}\), \(L(v_3)\{c(v_3), c_1(u_3), c_2(u_3)\}\) or \(L(v_3)\{c(v_3), c(v_1), c(v_2), c(v_4)\}\) when \(|c_2(u_3)| = 2\), \(|c_2(u_3)| = 1\) or \(|c_2(u_3)| = 0\),
respectively. Notice that \( v_3 \) is a \((4, 5)\)-vertex, it means \( d(u_3) = 5 \), then there will be no bi-colored cycle passing \( v_3u_3 \). Then we get a linear list coloring of \( G \). If \(|c_2(v_4)| = 0\), we choose a color for \( v \) in \( L(v)\{c(v_1), c(v_2), c(v_3), c(u_3)\} \), then we can color \( v_3 \) linearly since \(|L(v_3)\{c(v), c(u_3), c_2(u_3)\}| \geq 1\). We also get a linear list coloring of \( G \).

When the available color set for \( v_2 \) is exactly \( \{c(v_1), c(v_4)\} \) (notice that \( |c_2(u_3)| = 2 \), and there will be no bi-colored cycle passing \( v_2u_2 \), we can color \( v_2 \) with \( c(v_4) \). If \( |c_2(u_3)| = 2 \), we color \( v_3 \) with a color in \( L(v_3)\{c(v_4), c(u_3), c_2(u_3)\} \); if \( |c_2(u_3)| \leq 1 \), we color \( v_3 \) with a color in \( L(v_3)\{c(v_4), c(v_1), c(u_3), c_2(u_3)\} \).

Notice \( d(u_3) = 5 \), there will be no bi-colored cycle passing \( v_3u_3 \). Finally, we can color \( v \) with a color in \( L(v)\{c(v_1), c(v_3), c(v_4), c_2(v_4)\} \) if \( |c_2(v_4)| = 1 \), or in \( L(v)\{c(v_1), c(v_3), c(v_4)\} \) if \( |c_2(v_4)| = 0 \). Then we extend the linear \( L\)-coloring \( c \) of \( G' \) to \( G \), a contradiction.

To complete our proof of Theorem 17, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function \( \omega \) on \( V(G) \) by \( \omega(v) = d(v) - \frac{20}{7} \) for every \( v \in V(G) \). The discharging rules are as follows.

**R1.** Every \( 5^+ \)-vertex sends \( \frac{d(v) - 20}{d(v)} \) to each adjacent vertex.

**R2.** Every \( 4^- \)-vertex sends \( \frac{3}{7} \) to each adjacent \((4, 5)\)-vertex, \( \frac{1}{3} \) to each adjacent \((4, 6)\)-vertex, \( \frac{1}{7} \) to each adjacent \(3^-\)-vertex;

**R3.** Every \( 3^- \)-vertex sends \( \frac{3}{7} \) to each adjacent \(2^-\)-vertex (if it has one).

Now we are going to show that \( \omega'(v) \geq 0 \) for all \( v \in V(G) \). We only need to check the final charges of \( 4^- \)-vertices from the discharging rules.

Let \( v \) be a \( 4^- \)-vertex in \( G \). Then \( n_2(v) \leq 3 \) by Lemma 20. If \( n_2(v) \leq 2 \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{1}{7} = 0 \) by R2. When \( n_2(v) = 3 \), if there are three \((4, 6)\)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 3 \times \frac{1}{3} - \frac{1}{7} = 0 \); if there is only one \((4, 5)\)-vertex in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{1}{3} - \frac{3}{7} > 0 \) by Lemma 21 and R2; if there are two or more \((4, 5)\)-vertices in \( N(v) \), we have \( \omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} + \frac{3}{7} > 0 \) by Lemma 22 and R2.

Let \( v \) be a \( 3^- \)-vertex in \( G \). Then \( n_2(v) \leq 1 \) by Lemma 19. If \( n_2(v) = 0 \), then \( \omega'(v) = \omega(v) = 3 - \frac{20}{7} > 0 \). When \( n_2(v) = 1 \), if there is a \( 3^- \)-vertex in \( N(v) \), we have \( \omega'(v) \geq \omega(v) - \frac{3}{7} + \frac{3}{7} > 0 \) by Lemma 19 and R3; if there are two \( 4^- \)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) = \frac{3}{7} + 2 \times \frac{1}{7} = 0 \).

Let \( v \) be a \( 2^- \)-vertex with \( N(v) = \{x, y\} \) and \( d(x) \leq d(y) \). Clearly, \( d(x) \geq 3 \) by Lemma 18. If \( d(x) = 3 \), then \( d(y) \geq 5 \) by Lemma 19, so \( \omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \) by R3 and R1. If \( d(x) = 4 \), then \( d(y) \geq 5 \) by Lemma 18, so \( \omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \), or \( \omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{11}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \). Otherwise, \( d(x) \geq 5 \) and \( d(y) \geq 5 \), we have \( \omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \).

In summary, the proof of Theorem 3 is completed.
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