LINEAR LIST COLORING OF SOME SPARSE GRAPHS

MING CHEN

College of Mathematics Physics and Information Engineering
Jiaxing University, Zhejiang 314001, China
e-mail: chen2001ming@163.com

YUSHENG LI

School of Mathematical Sciences, Tongji University Shanghai 200092, China
e-mail: li_yusheng@tongji.edu.cn

AND

LI ZHANG

School of Mathematical Sciences, Tongji University Shanghai 200092, China
e-mail: lizhang@tongji.edu.cn

Abstract

A linear $k$-coloring of a graph is a proper $k$-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. A graph $G$ is linearly $L$-colorable if there is a linear coloring $c$ of $G$ for a given list assignment $L = \{L(v) : v \in V(G)\}$ such that $c(v) \in L(v)$ for all $v \in V(G)$, and $G$ is linearly $k$-choosable if $G$ is linearly $L$-colorable for any list assignment with $|L(v)| \geq k$. The smallest integer $k$ such that $G$ is linearly $k$-choosable is called the linear list chromatic number, denoted by $lc_l(G)$. It is clear that $lc_l(G) \geq \lceil \Delta(G)^2 \rceil + 1$ for any graph $G$ with maximum degree $\Delta(G)$. The maximum average degree of a graph $G$, denoted by $mad(G)$, is the maximum of the average degrees of all subgraphs of $G$. In this note, we shall prove the following. Let $G$ be a graph, (1) if $mad(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$, then $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$; (2) if $mad(G) < \frac{14}{3}$ and $\Delta(G) \geq 5$, then $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$; (3) if $mad(G) < \frac{20}{3}$ and $\Delta(G) \geq 5$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

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\footnote{Corresponding author.}
1. Introduction

All graphs considered here are finite, simple and undirected. For a graph $G$, denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the vertex set, edge set, the minimum degree and the maximum degree, respectively. For a vertex $v \in V(G)$, let $N(v)$ and $d(v)$ be the neighborhood and the degree of $v$ in $G$, respectively. The closed neighborhood of a vertex $v \in V(G)$, denoted by $N[v]$, is defined to be $N(v) \cup v$.

A $k$-vertex ($k^-$-vertex and $k^+$-vertex, respectively) is a vertex with degree $k$ (at most $k$ and at least $k$, respectively). A 2-vertex $v \in V(G)$ is called an $(a, b)$-vertex if it is adjacent to an $a$-vertex and a $b$-vertex, and an $(a, b^+)$-vertex is defined similarly. The maximum average degree $\text{mad}(G)$ of a graph $G$ is defined as $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}$, where $H \subseteq G$ signified that $H$ is a subgraph of $G$.

A proper $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. A linear $k$-coloring of a graph is a proper $k$-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. The linear chromatic number $\text{lc}(G)$ of a graph $G$ is the smallest number $k$ such that $G$ has a linear $k$-coloring. A graph $G$ is linearly $L$-colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring $c$ of $G$ such that $c(v) \in L(v)$ for all $v \in V(G)$. If $G$ is linearly $L$-colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said to be linearly $k$-choosable. The smallest integer $k$ such that the graph $G$ is linearly $k$-choosable is called the linear list chromatic number, denoted by $\text{llc}(G)$. The concept of linear coloring was first introduced by Yuster [8], and linear list colorings were first investigated by Esperet, Montassier and Raspaud [4].

It is clear that the linear chromatic number $\text{lc}(G)$ of a graph $G$ with maximum degree $\Delta(G)$ has a trivial lower bound $\text{lc}(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $\text{lc}(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. Esperet et al. [4] proved that trees with maximum degree $\Delta(G)$ satisfy $\text{lc}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. This equality suggests that the linear list chromatic numbers of sparse graphs (with $\text{mad}(G) < 3$) might be close to the trivial lower bound. Cranston and Yu [1] asked: Does there exist a constant $C$ such that every sparse graph $G$ satisfies $\text{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? Some authors have proved that for the class of some sparse graphs, such constant $C$ exists and is close to or equal to 1. We list the currently known results about this subject as follows.

Theorem 1. Let $G$ be a graph.

(i) (Esperet et al. [4]) If $\text{mad}(G) < \frac{8}{3}$, then $\text{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.

(ii) (Wang and Wu [7]) If $\text{mad}(G) < \frac{14}{5}$, then $\text{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$. 


(iii) (Cranston and Yu [1]) If $\text{mad}(G) < 3$ and $\Delta(G) \geq 9$, then $\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

(iv) (Cranston and Yu [1]) If $\text{mad}(G) < \frac{12}{5}$ and $\Delta(G) \geq 3$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

A planar graph is a graph that can be drawn on the Euclidean plane such that its edges meet at their ends only. The girth of a graph $G$, denoted $g(G)$, is the length of a shortest cycle of $G$. For a planar graph $G$ with girth $g$, we have $\text{mad}(G) < \frac{2g}{g-2}$ by Euler’s formula. So we can get some results from above results. Li, Wang and Raspaud [5] also asked: Is there a constant $C$ such that every planar graph $G$ has $\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? About this question, there are some other results as follows.

**Theorem 2.** Let $G$ be a planar graph.

(i) (Cranston and Yu [1]) If $g(G) \geq 5$, then $\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 4$.

(ii) (Dong et al. [2]) If $g(G) \geq 6$, then $\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.

(iii) (Dong and Lin [3]) If $g(G) \geq 6$ and $\Delta(G) \geq 39$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

In this paper, we prove the following results.

**Theorem 3.** Let $G$ be a graph.

1. If $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

2. If $\text{mad}(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

3. If $\text{mad}(G) < \frac{20}{7}$ and $\Delta(G) \geq 5$, then $\text{lcl}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

Then the following results about planar graphs are implied immediately from Theorem 3(1) and (2), respectively.

**Theorem 4.** Let $G$ be a planar graph.

1. If $g(G) \geq 8$ and $\Delta(G) \geq 7$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

2. If $g(G) \geq 9$ and $\Delta(G) \geq 5$, then $\text{lcl}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

We will prove the three results of Theorem 3 by contradiction in the following three sections, respectively. For convenience, we introduce some notations that will be used. Let $c$ be a coloring of $G$; we use $c(v)$ to denote the color of $v$ in $c$, and $c(S) = \{c(v) : v \in S\}$ for $S \subset V(G)$. Let $c_i(v)$ be the set of colors appeared $i$ times in $N(v)$. For a vertex $v \in V(G)$, let $n_2(v)$ for clarity be the number of 2-vertices in $N(v)$. 


2. Graphs with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$

In order to prove Theorem 3(1), we prove the following result instead, which implies Theorem 3(1) immediately.

**Theorem 5.** Let $M \geq 7$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq M$, then $l_G(G) = \left\lceil \frac{M}{2} \right\rceil + 1$.

**Proof.** By contradiction, we suppose that Theorem 5 is false. Let $G$ be a counterexample with the fewest vertices, and $L$ the list assignment of size $\left\lceil \frac{M}{2} \right\rceil + 1$ such that $G$ has no linear $L$-coloring. Let $H$ be a proper subgraph of $G$. Clearly, $\text{mad}(H) < \frac{8}{3}$ and $\Delta(H) \leq M$. By the choice of $G$, we have $l_G(H) = \left\lceil \frac{M}{2} \right\rceil + 1$, while $l_G(G) > \left\lceil \frac{M}{2} \right\rceil + 1$. In the proof we need some structural lemmatas, Lemma 6 is well-known.

**Lemma 6.** The graph $G$ is connected, and $\delta(G) \geq 2$.

**Lemma 7** ([3] Lemma 2.2). Let $v$ be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \left\lceil \frac{M}{2} \right\rceil + 1$.

**Lemma 8.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then $v_1, v_2, v_3$ must be $(3, 6^+)$-vertices.

**Proof.** Assume that $v_1$ is a $(3, 5^-)$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$, where $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_2) \neq c(v_3)$, we can extend the linear $L$-coloring $c$ of $G'$ to $v_1$ since $|L(v_1) \{c(u_1), c_2(u_1)\}| \geq 2$. Then we can color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(v_3)\}$ when $c(v_1) \notin \{c(v_2), c(v_3)\}$, or $L(v) \{c(u_1), c(v_2), c(v_3)\}$ when $c(v_1) \in \{c(v_2), c(v_3)\}$. Clearly, there will be no bi-colored cycles created, and we get a linear list coloring of $G$. If $c(v_2) = c(v_3)$, we can extend the linear $L$-coloring $c$ of $G'$ to $v_1$ since $|L(v_1) \{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. Finally, we can color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(u_2), c(u_3)\}$, which ensure that no bi-colored cycle passes $v_2v_3u_2$ or $v_2v_3u_3$. Thus, we also get a linear list coloring of $G$ extended from the linear $L$-coloring of $G'$. A contradiction.

**Lemma 9.** Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If $v_1, v_2, v_3, v_4$ are $(5, 3)$-vertices, then $v_5$ must be a $(5, 4^+)$-vertex.

**Proof.** Suppose to the contrary, let $v_5$ be a $(5, 3)$-vertex, and $u_i$ be the neighbor of $v_i$ other than $v$ for $i \in \{1, 2, \ldots, 5\}$.

Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. There exist at least $|L(v_1) \{c(u_1), c(N(u_1))\}| \geq 2$ available colors for $v_1$. Since $|L(v_2) \{c(v_1), c(u_2), c(N(u_2))\}| \geq 1$ and $|L(v_3) \{c(v_1), c(v_2), c(u_3), c_2(u_3)\}| \geq 1$, we can extend the coloring $c$ of $G'$ to $v_1, v_2, v_3$ such that $|\{c(v_1), c(v_2), c(v_3)\}| = 3$. 
Notice that there will be no bi-colored cycle passing $v_1u_1$ or $v_2u_2$. Then we color $v_3$ with a color in $L(v_3) \{c(u_4), c(N(u_4))\}$, and no bi-colored cycle will pass $v_3u_4$. Finally, we extend the coloring $c$ to $v_5$ and $v$ in two different cases.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 4$, we can linearly color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(v_3), c(v_4)\}$, and color $v_5$ such that no bi-colored cycle passes $v_5u_5$ as $|L(v_5) \{c(u_5), c(v), c(N(u_5))\}| \geq 1$. So we get a linear $L$-coloring of $G$.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 3$, we can color $v_3$ such that no bi-colored cycle passing $v_3u_5$ since $|L(v_5) \{c(v_4), c(u_5), c(N(u_5))\}| \geq 1$, and color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(v_3), c(v_4)\}$. Thus, we get a linear $L$-coloring of $G$.

Therefore, we can extend the linear $L$-coloring $c$ of $G'$ to $G$, a contradiction.

**Lemma 10.** Let $v$ be a 7-vertex with $N(v) = \{v_1, v_2, \ldots, v_7\}$ and $n_2(v) = 7$. If $v_1, v_2, \ldots, v_5$ are $(7, 2)$-vertices, then at least one of $v_6$ and $v_7$ is a $(7, 4^+)$-vertex.

**Proof.** Assume that $v_5$ and $v_7$ are $(7, 3^-)$-vertices, and $u_4$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, \ldots, 7$. Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. First, we extend the linear $L$-coloring $c$ of $G'$ to $v_7$ and $v_6$ such that $c(v_6) \neq c(v_7)$ and no bi-colored cycle passes $v_7u_5$ or $v_6u_5$ since $|L(v_7) \{c(u_7), c(N(u_7))\}| \geq 2$ and $|L(v_6) \{c(u_6), c(N(u_6)), c(v_7)\}| \geq 1$. Next, we can color $v_3$ such that $c(v_3) \notin \{c(v_6), c(v_7)\}$ and no bi-colored cycle passes $v_3u_5$ since $|L(v_5) \{c(u_5), c(N(u_5)), c(v_6), c(v_7)\}| \geq 1$. Then we can color $v_4$ with $c(v_4) \notin \{c(v_5), c(v_6), c(v_7)\}$ since $|L(v_4) \{c(u_4), c(v_3), c(v_5), c(v_6), c(v_7)\}| \geq 1$. Notice that $|\{c(v_4), c(v_5), c(v_6), c(v_7)\}| = 4$. Then we color $v$ with a color in $L(v) \{c(v_7), c(v_5), c(v_6), c(v_4)\}$. Since $|L(v_3) \{c(u_3), c(v), c(N(u_3))\}| \geq 2$ and $|L(v_2) \{c(u_2), c(v), c_2(v), c(N(u_2))\}| \geq 1$ ($|c_2(v)| \leq 1$ now), we can color $v_3, v_2$ in order such that no bi-colored cycle passes $v_3u_3$ or $v_2u_2$. Finally, in order to avoid bi-colored cycles passing $v_1u_1$, we can color $v_1$ with a color in $L(v_1) \{c(u_1), c(v), c_2(v)\}$ ($|c_2(v)| \leq 2$ now) when $c(u_1) \neq c(v)$, or color $v_1$ with a color in $L(v_1) \{c(u_1), c_2(v), c(N(u_1))\}$ when $c(u_1) = c(v)$. Thus, we get a linear list coloring of $G$ extended from the linear list coloring $c$ of $G'$, a contradiction.

To complete our proof of Theorem 5, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function $\omega$ on $V(G)$ by $\omega(v) = d(v) - \frac{8}{3}$ for every $v \in V(v)$. Since $\text{mad}(G) < \frac{8}{3}$, the sum of the initial charge is negative. If we can make suitable discharging rules to redistribute charges among vertices so that the final charge $\omega'(v)$ of every vertex $v \in V(G)$ is nonnegative, then we get a contradiction. The discharging rules are as follows.

**R1.** Every $8^+$-vertex sends $\frac{2}{3}$ to each adjacent 2-vertex.

**R2.** Every 7-vertex sends $\frac{2}{3}$ to each adjacent $(7, 2)$-vertex, $\frac{5}{3}$ to each adjacent $(7, 3)$-vertex, and $\frac{1}{3}$ to each adjacent $(7, 4^+)$-vertex.

**R3.** Every 6-vertex sends $\frac{5}{9}$ to each adjacent 2-vertex.
R4. Every 5-vertex sends $\frac{1}{2}$ to each adjacent (5, 3)-vertex, $\frac{1}{3}$ to each adjacent (5, 4\textsuperscript{+})-vertex.

R5. Every 4-vertex sends $\frac{1}{3}$ to each adjacent 2-vertex.

R6. Every 3-vertex sends $\frac{1}{6}$ to each adjacent (3, 5)-vertex, and $\frac{1}{8}$ to each adjacent (3, 6\textsuperscript{+})-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V(G)$.

Let $v$ be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 7$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{1}{3} = 2 - \frac{5}{3} + \frac{2}{3} = 0$. If $d(x) = 3$, then $d(y) \geq 5$ by Lemma 7. Thus $\omega'(v) \geq \omega(v) + \frac{1}{5} + \frac{1}{2} = 2 - \frac{3}{5} + \frac{2}{3} = 0$ or $\omega'(v) \geq \omega(v) + \frac{1}{9} + \frac{2}{9} = 2 - \frac{5}{9} + \frac{2}{3} = 0$ by R6, R2, R3, and R4. Otherwise, $d(x) \geq 4$ and $d(y) \geq 5$, we have $\omega'(v) \geq \omega(v) + \frac{3}{9} + \frac{2}{9} = 2 - \frac{7}{9} + \frac{2}{3} = 0$ by R5, R2, R3, and R4.

Let $v$ be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be (3, 6\textsuperscript{+})-vertices by Lemma 8. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{3} = 3 - \frac{9}{9} = 0$ by R6. If $n_2(v) \leq 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{3} = 3 - \frac{5}{9} = 0$ by R6.

Let $v$ be a 4-vertex. Then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{3} = 4 - \frac{8}{3} = 0$ by R5.

Let $v$ be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{5} = 5 - \frac{8}{5} = 2 > 0$ by R4. If $n_2(v) = 5$, then there are at most four (3, 5)-vertices in $N(v)$ by Lemma 9. Thus $\omega'(v) = \omega(v) - 4 \times \frac{1}{5} = 5 - \frac{8}{5} = 0$ by R4.

Let $v$ be a 6-vertex. Then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{6} = 6 - \frac{6}{6} = 0$ by R3.

Let $v$ be a 7-vertex. If $n_2(v) \leq 6$, then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{7} = 7 - \frac{6}{7} = 0$ by R2. When $n_2(v) = 7$, if there are no more than four (7, 2)-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{2}{3} = 7 - \frac{8}{3} - \frac{13}{3} = 0$ by R2; if there are five (7, 2)-vertices in $N(v)$, then at least one of the other neighbors is a (7, 4\textsuperscript{+})-vertex from Lemma 10, and $\omega'(v) = \omega(v) - 6 \times \frac{2}{3} = 7 - \frac{8}{3} - \frac{12}{3} = 0$ by R2.

Finally, if $d(v) \geq 8$, then $\omega'(v) \geq \omega(v) - \frac{2}{3} \times d(v) = \frac{d(v)}{3} - \frac{8}{3} = \frac{d(v)-8}{3} \geq 0$ by R1.

Thus, we get the desired contradiction, and Theorem 5 is proved.

It is interesting that Cranston and Yu [1] cited an example $(\text{mad}(K_{2,3}) = \frac{12}{5}$ and $\text{lcl}(K_{2,3}) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2)$ to illustrate that the bound in Theorem 1(iv) is sharp. Similarly, the graph $K_{2,4}$ satisfies $\text{lcl}(K_{2,4}) \geq \left\lceil \frac{\Delta}{2} \right\rceil + 2$, $\Delta(K_{2,4}) = 4$ and $\text{mad}(K_{2,4}) = \frac{8}{3}$. So the hypothesis about $\Delta(G)$ in Theorem 3(1) is essential, and we suspect it can be replaced by $\Delta(G) \geq 5$.

3. Graphs with $\text{mad}(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$

For Theorem 3(2), we prove the following result instead.

**Theorem 11.** Let $M \geq 5$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{18}{7}$ and $\Delta(G) \leq M$, then $\text{lcl}(G) = \left\lceil \frac{M}{2} \right\rceil + 1$. 


**Proof.** By contradiction, we suppose that Theorem 11 is false. Let $G$ be a counterexample with the fewest vertices and $L$ be a list assignment of size $\lceil \frac{M}{2} \rceil + 1 \geq 4$ such that $G$ has no linear $L$-coloring. In the proof we need some structural lemmatas, and it is clear that Lemma 6 and Lemma 7 are also true.

**Lemma 12.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then $v_1, v_2, v_3$ must be $(3,5^+)$-vertices.

**Proof.** Assume that $v_1$ is a $(3,4^-)$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_2) \neq c(v_3)$, there exist at least $|L(v_1)\backslash\{c(u_1), c_2(u_1)\}| \geq 2$ colors available for $v_1$. If there is an available color $\alpha \notin \{c(v_2), c(v_3)\}$ for $v_1$, then let $c(v_1) = \alpha$ and $c(v) \in L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$. If the available colors for $v_1$ are exactly $c(v_2)$ and $c(v_3)$, then let $c(v_1) = c(v_2)$ and $c(v) \in L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$. It is similar for $c(v_1) = c(v_3)$. Thus we get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. If $c(v_2) = c(v_3)$, we can extend the linear list coloring $c$ of $G'$ to $v_1$ since $|L(v_1)\backslash\{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. There is at least $|L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}| \geq 1$ color available for $v$. Thus, we also get a linear list coloring of $G$. A contradiction.

**Lemma 13.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$. If $v_1$ and $v_2$ are $(3,3)$-vertices, then $v_3$ must be a $4^+$-vertex.

**Proof.** Assume that $v_3$ is a $3^-$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear $L$-coloring $c$ to $v_1$ such that no bi-colored cycle passes $v_1u_1$ since $|L(v_1)\backslash\{c(u_1), c(N(u_1))\}| \geq 1$.

If $c(v_1) = c(v_3)$. Since $|L(v_2)\backslash\{c(v_1), c(u_2), c_2(u_2)\}| \geq 1$, we can extend the coloring $c$ to $v_2$. Finally, we can color $v$ with a color in $L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$ when $|c_2(v_3)| = 1$, or in $L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$ when $|c_2(v_3)| = 0$. It is clear that no bi-colored cycle passes $v_1v_2v_3$. Then we get a linear list coloring of $G$.

If $c(v_1) \neq c(v_3)$. There is at least $|L(v)\backslash\{c(v_1), c(v_3), c_2(v_3)\}| \geq 1$ color available for $v$. Finally, we can color $v_2$ with a color in $L(v_2)\backslash\{c(v_1), c(v_2), c_2(u_2)\}$ when $c(v_1) \neq c_2(u_2)$, or in $L(v_2)\backslash\{c(v_2), c(N(u_2))\}$ when $c(v) = c(u_2)$. In this process, there will be no bi-colored cycle passing $v_2v_3$. Thus, we also get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. A contradiction.

**Lemma 14.** Let $v$ be a 4-vertex with $n_2(v) = 4$ in $G$. Then there are at most two $(4,3)$-vertices in $N(v)$.

**Proof.** Let $N(v) = \{v_1, \ldots, v_4\}$, and $u_i$ be the other neighbor of $v_i$ for $i = 1, \ldots, 4$. Assume that $v_1, v_2$ and $v_3$ are $(4,3)$-vertices. Let $G' = G - [v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear
\(L\)-coloring \(c\) of \(G'\) to \(v_4\) since \(|L(v_4)\{c(u_4), c_2(u_4)\}| \geq 1\). We can continue to extend to \(v_3\) with \(c(v_3) \neq c(v_4)\) since \(|L(v_3)\{c(u_3), c_2(u_3), c(v_4)\}| \geq 1\). Then we color \(v_2\) with a color in \(L(v_2)\{c(u_2), c(N(u_2))\}\). Notice that no bi-colored cycle passes \(vu_2u_2\) or \(v_3v_4\). This signifies that any bi-colored cycle in \(G\) if there will be must passes \(v_1\). Finally, we will extend the coloring \(c\) to \(v_1\) and \(v\) in two different cases.

If \(c(v_2) \notin \{c(v_3), c(v_4)\}\), we can choose a color from \(|L(v)\{c(v_2), c(v_3), c(v_4)\}|\) for \(v\). Then there is at least \(|L(v_1)\{c(v), c(u_1), c_2(u_1)\}| \geq 1\) when \(c(v) \neq c(u_1)\), or \(|L(v_1)\{c(u_3), c(N(u_1))\}| \geq 1\) when \(c(v) = c(u_1)\) color available for \(v_1\), which ensure no bi-colored cycle passes \(v_1u_1\). So we get a linear list coloring of \(G\).

If \(c(v_2) \in \{c(v_3), c(v_4)\}\), suppose \(c(v_2) = c(v_3)\) (similarly for \(c(v_2) = c(v_4)\)). If \(\{c(u_1)\} = 1\), we color \(v_1\) with a color in \(L(v_1)\{c(v_2), c(u_1), c_2(u_1)\}\), and no bi-colored cycle passes \(v_1u_1\). If \(\{c(u_1)\} = 0\), we color \(v_1\) with a color in \(L(v_1)\{c(v_2), c(u_1), c_2(v_4)\}\), which ensure that no bi-colored cycle passes \(v_1v_4\). Then we color \(v\) with a color in \(L(v)\{c(v_1), c(v_2), c(v_4)\}\). Notice that no bi-colored cycle passes \(v_1v_3\) since \(c(v_1) \neq c(v_3)\). Thus, we also get a linear list coloring \(c\) of \(G\). A contradiction.

**Lemma 15.** Let \(v\) be a 5-vertex with \(N(v) = \{v_1, \ldots, v_5\}\). If \(v_1, v_2, v_3, v_4\) are four \((5, 2)\)-vertices, then \(v_5\) must be a \(3^+\)-vertex.

**Proof.** Assume that \(v_5\) is a 2-vertex, and \(u_i\) is the neighbor of \(v_i\) other than \(v\) for \(i = 1, 2, \ldots, 5\). Let \(G' = G - N[v]\). Then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). We can extend the \(L\)-coloring \(c\) of \(G'\) to \(v_5\) since \(|L(v_5)\{c(u_3), c_2(u_3)\}| \geq 1\), and continue to \(v_4\) such that \(c(v_4) \neq c(v_5)\) and no bi-colored cycle passes \(v_4u_4\) since \(|L(v_4)\{c(u_4), c(N(u_4))\}| \geq 1\), then to \(v_3\) with \(c(v_3) \notin \{c(v_4), c(v_4)\}\) since \(|L(v_3)\{c(u_3), c_2(u_3), c_2(v_3)\}| \geq 1\). We can color \(v\) with a color in \(|L(v)\{c(v_3), c_3(v_3)\}|\) and color \(v_2\) such that no bi-colored cycle passes \(v_2u_2\) since \(|L(v_2)\{c(v), c_2(v_2), c(N(u_2))\}| \geq 1\). Finally, we can color \(v_1\) linearly since \(|L(v_1)\{c(v), c_2(v), c(u_1)\}| \geq 1\) when \(c(v) \neq c(u_1)\), or \(|L(v_1)\{c(v), c_2(v), c(N(u_1))\}| \geq 1\) when \(c(v) = c(u_1)\). Note that \(c(v_3) \neq c(v_5)\), there will be no bi-colored cycle created. Thus we can extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\). A contradiction.

**Lemma 16.** Let \(v\) be a 5-vertex with \(N(v) = \{v_1, \ldots, v_5\}\) and \(n_2(v) = 5\). If \(v_1, v_2, v_3\) are \((5, 2)\)-vertices, then at least one of \(v_4\) and \(v_5\) is a \((5, 4^+)\)-vertex.

**Proof.** Assume that \(v_1\) and \(v_3\) are \((5, 3^-)\)-vertices, and \(u_i\) is the neighbor of \(v_i\) other than \(v\) for \(i = 1, \ldots, 5\). Let \(G' = G - N[v]\). Then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\). We can extend the coloring \(c\) of \(G'\) to \(v_5\) such that no bi-colored cycle passes \(v_5u_5\) since \(|L(v_5)\{c(u_5), c(N(u_5))\}| \geq 1\), and continue to \(v_4\) such that \(c(v_4) \neq c(v_5)\) as \(|L(v_4)\{c(u_4), c(2(u_4), c(v_5))| \geq 1\), then to \(v_3\) with \(c(v_3) \notin \{c(v_5), c(v_4)\}\) since \(|L(v_3)\{c(u_3), c(v_4), c(v_5)\}| \geq 1\). Now we
can color \( v \) with a color in \( L(v) \setminus \{c(v_5), c(v_4), c(v_3)\} \), and color \( v_2 \) such that no bi-colored cycle passes \( vv_2u_2 \) since \( |L(v_2) \setminus \{c(v), c(u_2), c(N(u_2))\}| \geq 1 \). Finally, we can color \( v_1 \) linearly since \( |L(v_1) \setminus \{c(v), c_2(v), c(u_1)\}| \geq 1 \) when \( c(v) \neq c(u_1) \), or \( |L(v_1) \setminus \{c(v), c_2(v), c(N(u_1))\}| \geq 1 \) when \( c(v) = c(u_1) \). Note that \( c(v_3) \neq c(v_4) \), there will be no bi-colored cycle created. Thus, we can extend the linear \( L \)-coloring \( c \) of \( G' \) to \( G \). A contradiction.

We will derive a contradiction by a discharging procedure proceeded in \( G \) to complete the proof of Theorem 11. In the discharging procedure, the initial charge function \( \omega \) is defined as \( \omega(v) = d(v) - \frac{18}{7} \) for every vertex \( v \in V(G) \), and the discharging rules are as follows.

**R1.** Every \( 6^+ \)-vertex sends \( \frac{4}{7} \) to each adjacent 2-vertex or 3-vertex.

**R2.** Every 5-vertex sends \( \frac{1}{7} \) to each adjacent \((5,2)\)-vertex, \( \frac{2}{7} \) to each adjacent \((5,3)\)-vertex, \( \frac{1}{3} \) to each adjacent \((5,4^+)\)-vertex, \( \frac{1}{7} \) to each adjacent 3-vertex.

**R3.** Every 4-vertex sends \( \frac{3}{7} \) to each adjacent \((4,3)\)-vertex, \( \frac{2}{7} \) to each adjacent \((4,4^+)\)-vertex, \( \frac{1}{7} \) to each adjacent 3-vertex.

**R4.** Every 3-vertex sends \( \frac{2}{7} \) to each adjacent \((3,3)\)-vertex, \( \frac{1}{7} \) to each adjacent \((3,4^+)\)-vertex.

Now we are going to show that \( \omega'(v) \geq 0 \) for all \( v \in V \).

If \( d(v) \geq 6 \), then \( \omega'(v) \geq \omega(v) - \frac{4}{7} \times d(v) = \frac{3d(v)}{7} - \frac{18}{7} = \frac{3(d(v) - 18)}{7} \geq 0 \) by R1.

Let \( v \) be a 5-vertex. If \( n_2(v) \leq 4 \), then \( \omega'(v) \geq \omega(v) - 4 \times \frac{4}{7} - \frac{1}{7} = 5 - \frac{18}{7} - \frac{1}{7} = 0 \) by R2. When \( n_2(v) = 5 \), there are at most three \((2,5)\)-vertices in \( N(v) \) by Lemma 15. If there are two or less \((2,5)\)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{4}{7} - 3 \times \frac{3}{7} = 5 - \frac{18}{7} - \frac{9}{7} - \frac{9}{7} = 0 \) by R2. If there are three \((2,5)\)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 3 \times \frac{4}{7} - \frac{2}{7} = 5 - \frac{18}{7} - \frac{12}{7} - \frac{5}{7} = 0 \) by Lemma 16 and R2.

Let \( v \) be a 4-vertex. If \( n_2(v) \leq 3 \), then \( \omega'(v) \geq \omega(v) - 3 \times \frac{3}{7} - \frac{1}{7} = 4 - \frac{18}{7} - \frac{9}{7} - \frac{1}{7} = 0 \) by R3. If \( n_2(v) = 4 \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{3}{7} = 4 - \frac{18}{7} - \frac{9}{7} = 0 \) by Lemma 14 and R3.

Let \( v \) be a 3-vertex. If \( n_2(v) = 3 \), then the vertices in \( N(v) \) must be \((3,5^+)\)-vertices by Lemma 12. Thus \( \omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0 \) by R4. If \( n_2(v) = 2 \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{2}{7} + \frac{1}{7} = 3 - \frac{18}{7} - \frac{4}{7} + \frac{1}{7} = 0 \) by Lemma 13 and all discharging rules, or \( \omega'(v) \geq \omega(v) - \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0 \). If \( n_2(v) \leq 1 \), then \( \omega'(v) \geq \omega(v) - \frac{2}{7} = 3 - \frac{18}{7} - \frac{2}{7} > 0 \) by R4.

Finally, let \( v \) be a 2-vertex with \( N(v) = \{x, y\} \) and \( d(x) \leq d(y) \). If \( d(x) = 2 \), then \( d(y) \geq 5 \) by Lemma 7. By R1 and R2, \( \omega'(v) = \omega(v) + \frac{4}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0 \).

When \( d(x) = 3 \), we have \( \omega'(v) = \omega(v) + 2 \times \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0 \) if \( d(y) = 3 \), and \( \omega'(v) = \omega(v) + \frac{4}{7} + \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0 \) if \( d(y) \geq 4 \). Otherwise, \( d(x) \geq 4 \) and \( d(y) \geq 4 \), we have \( \omega'(v) \geq \omega(v) + \frac{2}{7} + \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0 \) by R1, R2, and R3. We get the desired contradiction, and Theorem 11 is proved.
Similarly, the condition $\Delta(G)$ in Theorem 3(2) must be $\Delta(G) \geq 4$.

4. Graphs with $\text{mad}(G) < \frac{20}{7}$ and $\Delta(G) \geq 5$

Cranston and Yu [1] conjectured that the hypothesis $\Delta(G) \geq 9$ of Theorem 1(iii) can be replaced by $\Delta(G) \geq 7$, even $\Delta(G) \geq 5$. Now, we prove Theorem 3(3) to support their conjecture. In order to prove Theorem 3(3), we prove the following theorem which implies Theorem 3(3) immediately.

**Theorem 17.** Let $M \geq 5$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{20}{7}$ and $\Delta(G) \leq M$, then $\text{lc}(G) \leq \lceil \frac{M}{2} \rceil + 2$.

**Proof.** Let $G$ be a counterexample of the fewest vertices with $\text{mad}(G) < \frac{20}{7}$ and $5 \leq \Delta(G) \leq 8$ (Theorem 17 is true for graphs $G$ with $\Delta(G) \geq 9$ by Theorem 1(iii)). There exists an assignment $L$ with $|L| \geq \lceil \frac{M}{2} \rceil + 2 \geq 5$ such that $G$ is not linearly $L$-choosable, but $H$ has a linear $L$-coloring, where $H$ is any proper subgraph of $G$. Clearly, $G$ is connected and $\delta(G) \geq 2$. In the proof we need some structural lemmas.

**Lemma 18.** Let $v$ be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\lceil \frac{d(v_1)}{2} \rceil + \lceil \frac{d(v_2)}{2} \rceil \geq \lceil \frac{M}{2} \rceil + 2$.

**Proof.** Assume $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 1$. Let $G' = G - v$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_1) \neq c(v_2)$, we can color $v$ with any color in $L(v) \setminus \{c(v_1), c(v_2), c_2(v_1), c_2(v_2)\}$. Then the number of available colors for $v$ is at least $\left\lceil \frac{M}{2} \right\rceil + 2 - \left(2 + \left\lceil \frac{d(v_1) - 1}{2} \right\rceil + \left\lceil \frac{d(v_2) - 1}{2} \right\rceil \right) = \left\lceil \frac{M}{2} \right\rceil + 2 - \left( \left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \right) \geq 1$. Clearly, there will be no bi-colored cycle created. So we extend the linear $L$-coloring $c$ of $G'$ to $G$. Now we suppose $c(v_1) = c(v_2)$. In order to color $v$ linearly and avoid bi-colored cycles created, the forbidden color set for $v$ contains the color $c(v_1)$, the colors appearing twice in $N(v_1)$ or $N(v_2)$, and the colors appearing in both $N(v_1)$ and $N(v_2)$. So at most $1 + |c_2(v_1) \cup c_2(v_2)| + |c_1(v_1) \cap c_1(v_2)| \leq \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \leq \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 1$ colors are forbidden for $v$. Thus, we also can get a linear $L$-coloring of $G$. A contradiction.

**Lemma 19.** Let $v$ be a 3-vertex of $G$ with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 2$, then $d(v_2) \geq 3$ and $\left\lceil \frac{d(v_2) + d(v_3)}{2} \right\rceil \geq \left\lceil \frac{M}{2} \right\rceil + 1$.

**Proof.** We prove $d(v_2) \geq 3$ first. To the contrary, we assume $d(v_2) = 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. The neighbors of $v_1$ and $v_2$ other than $v$ are denoted by $u_1$ and $u_2$, respectively. We can extend the coloring $c$ of $G'$ to $v_1$ such that $c(v_1) \neq c(v_3)$ since...
\[ |L(v_1) \setminus \{c(u_1), c_2(u_1), c(v_3)\}| \geq 1, \] which ensures that no bi-colored cycle passes \(v_1v_3\). We can continue to extend to \(v\) since \(|L(v) \setminus \{c(v_1), c_2(v_3), c(v_3)\}| \geq 1\). If \(c(v) \neq c(u_2)\), which means that no bi-colored cycle passes \(v_2v_2\), we can color \(v_2\) linearly since \(|L(v_2) \setminus \{c(v), c_2(u_2), c(v_3)\}| \geq 1\). When \(c(v) = c(u_2)\), the number of available colors for \(v_2\) is at least \(|L(v_2) \setminus \{c(u_2), c_2(u_2)\}| \geq 2\). If there is an available color \(\alpha \notin \{c(v_1), c(v_3)\}\) for \(v_2\), then we color \(v_2\) with \(\alpha\). Now we assume that the available colors for \(v_2\) are exactly \(c(v_1)\) and \(c(v_3)\). Notice that \(|c_2(u_2)| = \left\lceil \frac{M-1}{2} \right\rceil\) and \(|c_1(u_2)| \leq 1\) now. To avoid bi-colored cycle created, the number of forbidden colors for \(v_2\) is at most \(|\{c(u_2), c(N(u_2))\}| = 1 + |c_2(u_2)| + |c_1(u_2)| \leq 1 + \left\lceil \frac{M-1}{2} \right\rceil + 1 = \left\lceil \frac{M}{2} \right\rceil + 1\), so we can color \(v_2\) linearly. Thus, we get a linear \(L\)-coloring of \(G\) extended from the linear \(L\)-coloring \(c\) of \(G'\). A contradiction.

Now, we prove the inequality. Suppose to the contrary that, we have \(\left\lfloor \frac{d(v_2)+d(v_3)}{2} \right\rfloor \leq \left\lceil \frac{M}{2} \right\rceil\), and \(u_1\) is the neighbor of \(v_1\) other than \(v\). Let \(G' = G - v_1\), then \(G'\) has a linear \(L\)-coloring \(c\) by the minimality of \(G\).

**Case 1.** \(c(v_2) \neq c(v_3)\). If \(c(v) \neq c(u_1)\), then we can extend the coloring \(c\) to \(v_1\) to get a linear \(L\)-coloring of \(G\) since \(|L(v_1) \setminus \{c(v), c(u_1), c_2(u_1)\}| \geq 1\). If \(c(v) = c(u_1)\), the number of available colors for \(v_1\) is at least \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq 2\). If there is a color \(\alpha \notin \{c(v_2), c(v_3)\}\) available for \(v\), then we can extend \(c\) from \(G'\) to \(G\) by coloring \(v_1\) with \(\alpha\). Now we assume that \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| = \{c(v_2), c(v_3)\}\). Notice that \(|c_2(u_1)| = \left\lceil \frac{M-1}{2} \right\rceil\) now. Then \(c(v_2)\) and \(c(v_3)\) appears at most once in \(N(u_1)\), but both of them could not appear in \(N(u_1)\) at the same time (otherwise \(|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq \left\lceil \frac{M}{2} \right\rceil + 2 - (1 + \left\lceil \frac{M-3}{2} \right\rceil) \geq 3\). So we color \(v_1\) with \(c(v_2)\) if \(c(v_2)\) appears in \(N(u_1)\), otherwise color \(v_1\) with \(c(v_2)\). Then there will be no bi-colored cycle created. Thus, we get a linear \(L\)-coloring of \(G\) extended from the linear \(L\)-coloring \(c\) of \(G'\).

**Case 2.** \(c(v_2) = c(v_3)\). If \(c(v) = c(u_1)\), then we can color \(v_1\) linearly since \(|L(v_1) \setminus \{c(v_2), c(v), c_2(u_1)\}| \geq 1\), and no bi-colored cycle created. Now, suppose \(c(v) \neq c(u_1)\). We erase the color of \(v\) first, then we can extend the list coloring \(c\) to \(v_1\) since \(|L(v_1) \setminus \{c(v_2), c(u_1), c_2(u_1)\}| \geq 1\). To avoid bi-colored cycle created, the number of forbidden colors for \(v\) is at most \(2 + |c_2(v_2) \cup c_2(v_3)| + |c_1(v_2) \cap c_1(v_3)| \leq 2 + \frac{|d(v_2)+d(v_3)|}{2} = 1 + \left\lceil \frac{d(v_2)+d(v_3)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 1\). We also can extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\). A contradiction.

**Lemma 20.** Let \(v\) be a 4-vertex in \(G\). Then \(n_2(v) \leq 3\).

**Proof.** Let \(N(v) = \{v_1, \ldots, v_4\}\). Suppose to the contrary, let \(n_2(v) = 4\), and \(u_i\) be the neighbor of \(v_i\) other than \(v\) for \(i = 1, \ldots, 4\). Let \(G' = G - N[v]\). Then \(G'\) has a linear \(L\)-coloring by the minimality of \(G\). Since \(|L(v_1) \setminus \{c(u_1), c_2(u_1)\}| \geq 2\), we can color \(v_i\) linearly with at least two different colors for \(i = 1, 2, 3, 4\). Finally, we can color \(v\) with one color in \(L(v)\setminus c(N(v))\) if \(|c(N(v))| = 4\), or in
Let \( v \) be a 4-vertex with \( N(v) = \{ v_1, \ldots, v_4 \} \). If \( d(v_1) = d(v_2) = d(v_3) = 2 \) and \( v_4 \) is a \((4,5)\)-vertex, then \( v_4 \) must be a \(4^+\)-vertex.

**Proof.** Suppose to the contrary, let \( v_4 \) be a \(3^-\)-vertex, and \( u_i \) be the neighbor of \( v_i \) other than \( v \) for \( i = 1, 2, 3 \). Let \( G' = G - \{ v, v_1, v_2, v_3 \} \), then \( G' \) has a linear \( L \)-coloring \( c \) by the minimality of \( G \). We can extend the coloring \( c \) of \( G' \) to \( v_1 \) with \( c(v_1) \neq c(v_4) \) since \( |L(v_1) \{ c(u_1), c(u_1), c(v_4) \}| \geq 1 \). Then there are at least \( |L(v_2) \{ c(u_2), c_2(u_2) \}| \geq 2 \) colors available for \( v_2 \).

If there is a color \( \alpha \notin \{ c(v_1), c(v_4) \} \) available for \( v_2 \), let \( c(v_2) = \alpha \). If \( |c_2(v_4)| = 1 \), we can choose a color for \( v \) in \( L(v) \{ c(v_1), c(v_2), c(v_4), c_2(v_4) \} \), and there will be no bi-colored cycle created passing \( v_4 \). Then, we color \( v_3 \) with a color in \( L(v_3) \{ c(v), c(u_3), c_2(u_3) \} \) if \( c(v) \neq c(u_3) \). When \( c(v) = c(u_3) \), in order to color \( v_3 \) linearly (no bi-colored cycle created), we must forbidden \( c(v), c_2(N(u_3)) \) and \( \{ c(v_1), c(v_2), c(v_4) \} \cap c_1(N(u_3)) \). Notice that \( d(u_3) = 5 \), then \( |c_2(N(u_3)) \cup \{ c(v_1), c(v_2), c(v_4) \} \cap c_1(N(u_3))| \leq 3 \). So we can color \( v_3 \) linearly with a color in \( L(v_3) \{ c(v), c_2(N(u_3)), c(v_1), c(v_2), c(v_4) \} \cap c_1(N(u_3)) \). Thus, we get a linear list coloring of \( G \).

Suppose the available color set for \( v_2 \) is exactly \( \{ c(v_1), c(v_4) \} \). Notice that \( |c_2(v_2)| = \left\lceil \frac{M-1}{2} \right\rceil \) now. We color \( v_2 \) with \( c(v_4) \) first. If \( |c_2(u_3)| = 2 \), we color \( v_3 \) with a color in \( L(v_3) \{ c(v_4), c(u_3), c_2(u_3) \} \). If \( |c_2(u_3)| \leq 1 \), we color \( v_3 \) with a color in \( L(v_3) \{ c(v_1), c(v_2), c(u_3), c_2(u_3) \} \). Notice \( d(u_3) = 5 \), then no bi-colored cycle passes \( v_3u_3 \). Finally, we can color \( v \) with a color in \( L(v) \{ c(v_1), c(v_3), c(v_4), c_2(v_4) \} \) if \( |c_2(v_4)| = 1 \), or in \( L(v) \{ c(v_1), c(v_3), c(v_4), c(u_2) \} \) if \( |c_2(v_4)| = 0 \). Clearly, there will be no bi-colored cycle passing \( v_4 \). Then we extend the linear \( L \)-coloring of \( G' \) to \( G \), a contradiction. 

**Lemma 22.** Let \( v \) be a 4-vertex with \( N(v) = \{ v_1, \ldots, v_4 \} \). If \( d(v_1) = d(v_2) = d(v_3) = 2 \) and \( v_4, v_3 \) are \((4,5)\)-vertices, then \( v_4 \) must be a \(5^+\)-vertex.

**Proof.** Suppose to the contrary, let \( v_4 \) be a \(4^-\)-vertex, and \( u_i \) be the neighbor of \( v_i \) other than \( v \) for \( i = 1, 2, 3 \). Let \( G' = G - \{ v, v_1, v_2, v_3 \} \). Then \( G' \) has a linear \( L \)-coloring \( c \) by the minimality of \( G \). We can extend the coloring \( c \) of \( G' \) to \( v_1 \) with \( c(v_1) \neq c(v_4) \) since \( |L(v_1) \{ c(u_1), c(u_1), c(v_4) \}| \geq 1 \). Then there are at least \( |L(v_2) \{ c(u_2), c_2(u_2) \}| \geq 2 \) colors available for \( v_2 \).

If there is an available color \( \alpha \notin \{ c(v_1), c(v_4) \} \) for \( v_2 \), let \( c(v_2) = \alpha \). If \( |c_2(v_4)| = 1 \), we color \( v \) with a color in \( L(v) \{ c(v_1), c(v_2), c(v_4), c_2(v_4) \} \). Then, we color \( v_3 \) with a color in \( L(v_3) \{ c(v), c(u_3), c_2(u_3) \} \) if \( c(v) \neq c(u_3) \). If \( c(v) = c(u_3) \), we color \( v_3 \) with a color in \( L(v_3) \{ c(u_3), c_2(u_3) \} \), \( L(v_3) \{ c(u_3), c_1(u_3), c_2(u_3) \} \) or \( L(v_3) \{ c(u_3), c(v), c_2(u_3) \} \) when \( |c_2(u_3)| = 2 \), \( |c_2(u_3)| = 1 \) or \( |c_2(u_3)| = 0 \),
respectively. Notice that \( v_3 \) is a \((4,5)\)-vertex, it means \( d(u_3) = 5 \), then there will be no bi-colored cycle passing \( v_3u_3 \). Then we get a linear list coloring of \( G \). If \( |c_2(v_3)| = 0 \), we choose a color for \( v \) in \( L(v)\{c(v_1), c(v_2), c(v_3), c(u_3)\} \), then we can color \( v_3 \) linearly since \( |L(v_3)\{c(v), c(u_3), c_2(u_3)\}| \geq 1 \). We also get a linear list coloring of \( G \).

When the available color set for \( v_2 \) is exactly \( \{c(v_1), c(v_4)\} \) (notice that \( |c_2(v_2)| = 2 \), and there will be no bi-colored cycle passing \( v_2u_2 \), we can color \( v_2 \) with \( c(v_4) \). If \( |c_2(u_3)| = 2 \), we color \( v_3 \) with a color in \( L(v_3)\{c(v_4), c(u_3), c_2(u_3)\} \); if \( |c_2(u_3)| \leq 1 \), we color \( v_3 \) with a color in \( L(v_3)\{c(v_4), c(v_1), c(u_3), c_2(u_3)\} \). Notice \( d(u_3) = 5 \), there will be no bi-colored cycle passing \( v_3u_3 \). Finally, we can color \( v \) with a color in \( L(v)\{c(v_1), c(v_3), c(v_4), c_2(v_4)\} \) if \( |c_2(v_4)| = 1 \), or in \( L(v)\{c(v_1), c(v_3), c(v_4)\} \) if \( |c_2(v_4)| = 0 \). Then we extend the linear \( L \)-coloring \( c \) of \( G' \) to \( G \), a contradiction. ■

To complete our proof of Theorem 17, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function \( \omega \) on \( V(G) \) by

\[
\omega(v) = d(v) - \frac{20}{7}
\]

for every \( v \in V(G) \). The discharging rules are as follows.

**R1.** Every \( 5^+ \)-vertex sends \( \frac{d(v)-20}{d(v)} \) to each adjacent vertex.

**R2.** Every \( 4^- \)-vertex sends \( \frac{2}{7} \) to each adjacent \( (4,5) \)-vertex, \( \frac{1}{7} \) to each adjacent \( (4,6) \)-vertex, \( \frac{1}{7} \) to each adjacent \( 3^- \)-vertex;

**R3.** Every \( 3^- \)-vertex sends \( \frac{2}{7} \) to each adjacent \( 2^- \)-vertex (if it has one).

Now we are going to show that \( \omega'(v) \geq 0 \) for all \( v \in V(G) \). We only need to check the final charges of \( 4^- \)-vertices from the discharging rules.

Let \( v \) be a \( 4^- \)-vertex in \( G \). Then \( n_2(v) \leq 3 \) by Lemma 20. If \( n_2(v) \leq 2 \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - \frac{2}{7} - \frac{1}{7} = 0 \) by R2. When \( n_2(v) = 3 \), if there are three \( (4,6) \)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} - \frac{1}{7} = 0 \); if there is only one \( (4,5) \)-vertex in \( N(v) \), then \( \omega'(v) \geq \omega(v) - 2 \times \frac{1}{7} - \frac{3}{7} > 0 \) by Lemma 21 and R2; if there are two or more \( (4,5) \)-vertices in \( N(v) \), we have \( \omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} + \frac{3}{7} > 0 \) by Lemma 22 and R2.

Let \( v \) be a \( 3^- \)-vertex in \( G \). Then \( n_2(v) \leq 1 \) by Lemma 19. If \( n_2(v) = 0 \), then \( \omega'(v) = \omega(v) = 3 - \frac{20}{7} > 0 \). When \( n_2(v) = 1 \), if there is a \( 3^- \)-vertex in \( N(v) \), we have \( \omega'(v) \geq \omega(v) - \frac{6}{7} + \frac{2}{7} > 0 \) by Lemma 19 and R3; if there are two \( 4^- \)-vertices in \( N(v) \), then \( \omega'(v) \geq \omega(v) - \frac{3}{7} + 2 \times \frac{1}{7} = 0 \).

Let \( v \) be a \( 2^- \)-vertex with \( N(v) = \{x, y\} \) and \( d(x) \leq d(y) \). Clearly, \( d(x) \geq 3 \) by Lemma 18. If \( d(x) = 3 \), then \( d(y) \geq 5 \) by Lemma 19, so \( \omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \) by R3 and R1. If \( d(x) = 4 \), then \( d(y) \geq 5 \) by Lemma 18, so \( \omega'(v) \geq \omega(v) + \frac{1}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \), or \( \omega'(v) \geq \omega(v) + \frac{1}{7} + \frac{11}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \). Otherwise, \( d(x) \geq 5 \) and \( d(y) \geq 5 \), we have \( \omega'(v) \geq \omega(v) + \frac{2}{7} + \frac{2}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0 \).

In summary, the proof of Theorem 3 is completed. ■
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