LINEAR LIST COLORING OF SOME SPARSE GRAPHS

Ming Chen
College of Mathematics Physics and Information Engineering
Jiaxing University, Zhejiang 314001, China
e-mail: chen2001ming@163.com

Yusheng Li
School of Mathematical Sciences, Tongji University Shanghai 200092, China
e-mail: li_yusheng@tongji.edu.cn

AND

Li Zhang
School of Mathematical Sciences, Tongji University Shanghai 200092, China
e-mail: lizhang@tongji.edu.cn

Abstract

A linear k-coloring of a graph is a proper k-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. A graph G is linearly L-colorable if there is a linear coloring c of G for a given list assignment L = {L(v) : v ∈ V(G)} such that c(v) ∈ L(v) for all v ∈ V(G), and G is linearly k-choosable if G is linearly L-colorable for any list assignment with |L(v)| ≥ k. The smallest integer k such that G is linearly k-choosable is called the linear list chromatic number, denoted by lc(G). It is clear that lc(G) ≥ ⌈Δ(G)/2⌉ + 1 for any graph G with maximum degree Δ(G). The maximum average degree of a graph G, denoted by mad(G), is the maximum of the average degrees of all subgraphs of G. In this note, we shall prove the following. Let G be a graph, (1) if mad(G) < 8/3 and Δ(G) ≥ 7, then lc(G) = ⌈Δ(G)/2⌉ + 1; (2) if mad(G) < 14/3 and Δ(G) ≥ 5, then lc(G) = ⌈Δ(G)/2⌉ + 1; (3) if mad(G) < 20/7 and Δ(G) ≥ 5, then lc(G) ≤ ⌈Δ(G)/2⌉ + 2.

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1Corresponding author.
1. Introduction

All graphs considered here are finite, simple and undirected. For a graph $G$, denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the vertex set, edge set, the minimum degree and the maximum degree, respectively. For a vertex $v \in V(G)$, let $N(v)$ and $d(v)$ be the neighborhood and the degree of $v$ in $G$, respectively. The closed neighborhood of a vertex $v \in V(G)$, denoted by $N[v]$, is defined to be $N(v) \cup v$. A $k$-vertex ($k^{-}$-vertex and $k^{+}$-vertex, respectively) is a vertex with degree $k$ (at most $k$ and at least $k$, respectively). A 2-vertex $v \in V(G)$ is called an $(a, b)$-vertex if it is adjacent to an $a$-vertex and a $b$-vertex, and an $(a, b^{+})$-vertex is defined similarly. The maximum average degree $mad(G)$ of a graph $G$ is defined as $mad(G) = \max \left\{ \frac{|E(H)|}{|V(H)|} : H \subseteq G \right\}$, where $H \subseteq G$ signified that $H$ is a subgraph of $G$.

A proper $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. A linear $k$-coloring of a graph is a proper $k$-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. The linear chromatic number $lc(G)$ of a graph $G$ is the smallest number $k$ such that $G$ has a linear $k$-coloring. A graph $G$ is linearly $L$-colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring $c$ of $G$ such that $c(v) \in L(v)$ for all $v \in V(G)$. If $G$ is linearly $L$-colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said to be linearly $k$-choosable. The smallest integer $k$ such that the graph $G$ is linearly $k$-choosable is called the linear list chromatic number, denoted by $lc_l(G)$. The concept of linear coloring was first introduced by Yuster [8], and linear list colorings were first investigated by Esperet, Montassier and Raspaud [4].

It is clear that the linear chromatic number $lc(G)$ of a graph $G$ with maximum degree $\Delta(G)$ has a trivial lower bound $lc(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $lc_l(G) \geq lc(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. Esperet et al. [4] proved that trees with maximum degree $\Delta(G)$ satisfy $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. This equality suggests that the linear list chromatic numbers of sparse graphs (with $mad(G) < 3$) might be close to the trivial lower bound. Cranston and Yu [1] asked: Does there exist a constant $C$ such that every sparse graph $G$ satisfies $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? Some authors have proved that for the class of some sparse graphs, such constant $C$ exists and is close to or equal to 1. We list the currently known results about this subject as follows.

**Theorem 1.** Let $G$ be a graph.

(i) (Esperet et al. [4]) If $mad(G) < \frac{8}{3}$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.

(ii) (Wang and Wu [7]) If $mad(G) < \frac{11}{7}$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$. 


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(iii) (Cranston and Yu [1]) If \( mad(G) < 3 \) and \( \Delta(G) \geq 9 \), then \( lcl(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

(iv) (Cranston and Yu [1]) If \( mad(G) < \frac{12}{5} \) and \( \Delta(G) \geq 3 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

A planar graph is a graph that can be drawn on the Euclidean plane such that its edges meet at their ends only. The girth of a graph \( G \), denoted \( g(G) \), is the length of a shortest cycle of \( G \). For a planar graph \( G \) with girth \( g \), we have \( mad(G) < \frac{3g}{g-2} \) by Euler’s formula. So we can get some results from above results. Li, Wang and Raspaud [5] also asked: Is there a constant \( C \) such that every planar graph \( G \) has \( lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C \)? About this question, there are some other results as follows.

**Theorem 2.** Let \( G \) be a planar graph.

(i) (Cranston and Yu [1]) If \( g(G) \geq 5 \), then \( lcl(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 4 \).

(ii) (Dong et al. [2]) If \( g(G) \geq 6 \), then \( lcl(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3 \).

(iii) (Dong and Lin [3]) If \( g(G) \geq 6 \) and \( \Delta(G) \geq 39 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

In this paper, we prove the following results.

**Theorem 3.** Let \( G \) be a graph.

1. If \( mad(G) < \frac{8}{3} \) and \( \Delta(G) \geq 7 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

2. If \( mad(G) < \frac{18}{7} \) and \( \Delta(G) \geq 5 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

3. If \( mad(G) < \frac{20}{7} \) and \( \Delta(G) \geq 5 \), then \( lcl(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \).

Then the following results about planar graphs are implied immediately from Theorem 3(1) and (2), respectively.

**Theorem 4.** Let \( G \) be a planar graph.

1. If \( g(G) \geq 8 \) and \( \Delta(G) \geq 7 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

2. If \( g(G) \geq 9 \) and \( \Delta(G) \geq 5 \), then \( lcl(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

We will prove the three results of Theorem 3 by contradiction in the following three sections, respectively. For convenience, we introduce some notations that will be used. Let \( c \) be a coloring of \( G \); we use \( c(v) \) to denote the color of \( v \) in \( c \), and \( c(S) = \{c(v) : v \in S\} \) for \( S \subseteq V(G) \). Let \( c_i(v) \) be the set of colors appeared \( i \) times in \( N(v) \). For a vertex \( v \in V(G) \), let \( n_2(v) \) for clarity be the number of 2-vertices in \( N(v) \).
2. Graphs with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$

In order to prove Theorem 3(1), we prove the following result instead, which implies Theorem 3(1) immediately.

**Theorem 5.** Let $M \geq 7$ be an integer. If $G$ is a graph with $\text{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq M$, then $\text{lc}(G) = \lceil \frac{M}{2} \rceil + 1$.

**Proof.** By contradiction, we suppose that Theorem 5 is false. Let $G$ be a counterexample with the fewest vertices, and $L$ the list assignment of size $\lceil \frac{M}{2} \rceil + 1$ such that $G$ has no linear $L$-coloring. Let $H$ be a proper subgraph of $G$. Clearly, $\text{mad}(H) < \frac{8}{3}$ and $\Delta(H) \leq M$. By the choice of $G$, we have $\text{lc}(H) = \lceil \frac{M}{2} \rceil + 1$, while $\text{lc}(G) > \lceil \frac{M}{2} \rceil + 1$. In the proof we need some structural lemmatas, Lemma 6 is well-known.

**Lemma 6.** The graph $G$ is connected, and $\delta(G) \geq 2$.

**Lemma 7** ([3] Lemma 2.2). Let $v$ be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then 
\[
\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 1.
\]

**Lemma 8.** Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then $v_1, v_2, v_3$ must be $(3, 6^+)$-vertices.

**Proof.** Assume that $v_1$ is a $(3, 5^-)$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$, where $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_2) \neq c(v_3)$, we can extend the linear $L$-coloring $c$ of $G'$ to $v_1$ since $|L(v_1)\{c(u_1), c_2(u_1)\}| \geq 2$. Then we can color $v$ with a color in $L(v)\{c(v_1), c(v_2), c(v_3)\}$ when $c(v_1) \notin \{c(v_2), c(v_3)\}$, or $L(v)\{c(u_1), c(v_2), c(v_3)\}$ when $c(v_1) \in \{c(v_2), c(v_3)\}$. Clearly, there will be no bi-colored cycles created, and we get a linear list coloring of $G$. If $c(v_2) = c(v_3)$, we can extend the linear $L$-coloring $c$ of $G'$ to $v_1$ since $|L(v_1)\{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. Finally, we can color $v$ with a color in $L(v)\{c(v_1), c(v_2), c(u_2), c(u_3)\}$, which ensure that no bi-colored cycle passes $v_2u_2$ or $vu_3u_3$. Thus, we also get a linear list coloring of $G$ extended from the linear $L$-coloring of $G'$. A contradiction.

**Lemma 9.** Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If $v_1$, $v_2$, $v_3$, $v_4$ are $(5, 3)$-vertices, then $v_5$ must be a $(5, 4^+)$-vertex.

**Proof.** Suppose to the contrary, let $v_5$ be a $(5, 3)$-vertex, and $u_i$ be the neighbor of $v_i$ other than $v$ for $i \in \{1, 2, \ldots, 5\}$.

Let $G'' = G - N[v]$. Then $G''$ has a linear $L$-coloring $c$ by the minimality of $G$. There exist at least $|L(v_1)\{c(u_1), c(N(u_1))\}| \geq 2$ available colors for $v_1$. Since $|L(v_2)\{c(v_1), c(u_2), c(N(u_2))\}| \geq 1$ and $|L(v_3)\{c(v_1), c(v_2), c(u_3), c_2(u_3)\}| \geq 1$, we can extend the coloring $c$ of $G''$ to $v_1, v_2, v_3$ such that $|\{c(v_1), c(v_2), c(v_3)\}| = 3$. Then $v_4$ must be a $(5, 4^+)$-vertex.
Notice that there will be no bi-colored cycle passing $v_1u_1$ or $v_2u_2$. Then we color $v_3$ with a color in $L(v_3) \setminus \{c(u_1), c(N(u_1))\}$, and no bi-colored cycle will pass $v_3v_4$. Finally, we extend the coloring $c$ to $v_5$ and $v$ in two different cases.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 4$, we can linearly color $v$ with a color in $L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_4)\}$, and color $v_5$ such that no bi-colored cycle passes $v_5u_3$ as $|L(v_5) \setminus \{c(u_3), c(v), c(N(u_5))\}| \geq 1$. So we get a linear $L$-coloring of $G$.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 3$, we can color $v_5$ such that no bi-colored cycle passing $v_5u_3$ since $|L(v_5) \setminus \{c(v_4), c(u_3), c(N(u_5))\}| \geq 1$, and color $v$ with a color in $L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_5)\}$. Thus, we get a linear $L$-coloring of $G$.

Therefore, we can extend the linear $L$-coloring $c$ of $G'$ to $G$, a contradiction. ■

**Lemma 10.** Let $v$ be a 7-vertex with $N(v) = \{v_1, v_2, \ldots, v_7\}$ and $n_2(v) = 7$. If $v_1, v_2, \ldots, v_5$ are $(7,2)$-vertices, then at least one of $v_6$ and $v_7$ is a $(7,4^+)$-vertex.

**Proof.** Assume that $v_6$ and $v_7$ are $(7,3^-)$-vertices, and $u_4$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, \ldots, 7$. Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. First, we extend the linear $L$-coloring $c$ of $G'$ to $v_7$ and $v_6$ such that $c(v_6) \neq c(v_7)$ and no bi-colored cycle passes $v_7u_7$ or $v_6u_6$ since $|L(v_7) \setminus \{c(v_7), c(N(u_7))\}| \geq 2$ and $|L(v_6) \setminus \{c(u_6), c(N(u_6)), c(v_7))| \geq 1$. Next, we can color $v_5$ such that $c(v_5) \notin \{c(v_6), c(v_7)\}$ and no bi-colored cycle passes $v_5u_5$ since $|L(v_5) \setminus \{c(u_5), c(N(u_5)), c(v_6), c(v_7)\}| \geq 1$. Then we can color $v_4$ with $c(v_4) \notin \{c(v_5), c(v_6), c(v_7)\}$ since $|L(v_4) \setminus \{c(u_4), c(v_5), c(v_6), c(v_7)\}| \geq 1$. Notice that $|\{c(v_4), c(v_5), c(v_6), c(v_7)\}| = 4$. Then we color $v$ with a color in $L(v) \setminus \{c(v_7), c(v_6), c(v_5), c(v_4)\}$. Since $|L(v_3) \setminus \{c(u_3), c(v), c(N(u_3))\}| \geq 2$ and $|L(v_2) \setminus \{c(u_2), c(v), c(N(u_2))\}| \geq 1$ (if $c_2(v) \leq 2$ now) when $c(u_1) \neq c(v)$, or color $v_2$ with a color in $L(v_1) \setminus \{c(u_1), c(v), c(N(u_1))\}$ when $c(u_1) = c(v)$. Thus, we get a linear list coloring of $G$ extended from the linear list coloring of $G'$, a contradiction. ■

To complete our proof of Theorem 5, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function $\omega$ on $V(G)$ by $\omega(v) = d(v) - \frac{8}{3}$ for every $v \in V(G)$. Since $mad(G) < \frac{8}{3}$, the sum of the initial charge is negative. If we can make suitable discharging rules to redistribute charges among vertices so that the final charge $\omega'(v)$ of every vertex $v \in V(G)$ is nonnegative, then we get a contradiction. The discharging rules are as follows.

**R1.** Every $8^+$-vertex sends $\frac{2}{3}$ to each adjacent $2$-vertex.

**R2.** Every 7-vertex sends $\frac{2}{3}$ to each adjacent $(7,2)$-vertex, $\frac{5}{6}$ to each adjacent $(7,3)$-vertex, and $\frac{1}{3}$ to each adjacent $(7,4^-)$-vertex.

**R3.** Every 6-vertex sends $\frac{5}{6}$ to each adjacent 2-vertex.
R4. Every 5-vertex sends $\frac{1}{2}$ to each adjacent $(5, 3)$-vertex, $\frac{1}{3}$ to each adjacent $(5, 4^+)$-vertex.

R5. Every 4-vertex sends $\frac{1}{3}$ to each adjacent 2-vertex.

R6. Every 3-vertex sends $\frac{1}{6}$ to each adjacent $(3, 5)$-vertex, and $\frac{1}{9}$ to each adjacent $(3, 6^+)$-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V(G)$.

Let $v$ be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 7$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{2}{3} = 2 - \frac{2}{3} + \frac{2}{3} = 0$. If $d(x) = 3$, then $d(y) \geq 5$ by Lemma 7. Thus $\omega'(v) = \omega(v) + \frac{1}{3} + \frac{1}{2} = 2 - \frac{2}{3} + \frac{2}{3} = 0$ or $\omega'(v) \geq \omega(v) + \frac{1}{6} + \frac{2}{3} = 2 - \frac{2}{3} + \frac{2}{3} = 0$ by R6, R2, R3, and R4. Otherwise, $d(x) \geq 4$ and $d(y) \geq 5$, we have $\omega'(v) \geq \omega(v) + \frac{1}{3} + \frac{1}{3} = 2 - \frac{2}{3} + \frac{2}{3} = 0$ by R5, R2, R3, and R4.

Let $v$ be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3, 6^+)$-vertices by Lemma 8. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{6} = 3 - \frac{2}{3} - \frac{1}{3} = 0$ by R6. If $n_2(v) \leq 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{6} = 3 - \frac{2}{3} - \frac{1}{3} = 0$ by R6.

Let $v$ be a 4-vertex. Then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{6} = 4 - \frac{2}{3} - \frac{4}{3} = 0$ by R5.

Let $v$ be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 5 - \frac{5}{3} - 2 > 0$ by R4. If $n_2(v) = 5$, then there are at most four $(3, 5)$-vertices in $N(v)$ by Lemma 9. Thus $\omega'(v) = \omega(v) - 4 \times \frac{1}{5} = 5 - \frac{4}{3} - 2 = 0$ by R4.

Let $v$ be a 6-vertex. Then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{6} = 6 - \frac{3}{3} - \frac{10}{3} = 0$ by R3.

Let $v$ be a 7-vertex. If $n_2(v) \leq 6$, then $\omega'(v) \geq \omega(v) - 6 \times \frac{1}{3} = 7 - \frac{5}{3} - 4 > 0$ by R2. When $n_2(v) = 7$, if there are no more than four $(7, 2)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{2}{3} - 3 \times \frac{3}{3} = 7 - \frac{5}{3} - \frac{13}{3} = 0$ by R2; if there are five $(7, 2)$-vertices in $N(v)$, then at least one of the other neighbors is a $(7, 4^+)$-vertex from Lemma 10, and $\omega'(v) = \omega(v) - 6 \times \frac{2}{3} - \frac{3}{3} = 7 - \frac{5}{3} - \frac{12}{3} = 0$ by R2.

Finally, if $d(v) \geq 8$, then $\omega'(v) \geq \omega(v) - \frac{2}{3} \times d(v) = \frac{d(v)}{3} - \frac{8}{3} = \frac{d(v) - 8}{3} \geq 0$ by R1.

Thus, we get the desired contradiction, and Theorem 5 is proved. ■

It is interesting that Cranston and Yu [1] cited an example $(mad(K_{2, 3}) = \frac{10}{3}$ and $lc(K_{2, 3}) \geq \lceil \frac{\Delta(G)}{2} \rceil + 2)$ to illustrate that the bound in Theorem 1(iv) is sharp. Similarly, the graph $K_{2, 4}$ satisfies $lc(K_{2, 4}) \geq \lceil \frac{\Delta(G)}{2} \rceil + 2$, $\Delta(K_{2, 4}) = 4$ and $mad(K_{2, 4}) = \frac{8}{3}$. So the hypothesis about $\Delta(G)$ in Theorem 3(1) is essential, and we suspect it can be replaced by $\Delta(G) \geq 5$.

### 3. Graphs with $mad(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$

For Theorem 3(2), we prove the following result instead.

**Theorem 11.** Let $M \geq 5$ be an integer. If $G$ is a graph with $mad(G) < \frac{18}{7}$ and $\Delta(G) \leq M$, then $lc(G) = \lceil \frac{M}{2} \rceil + 1$. 
Proof. By contradiction, we suppose that Theorem 11 is false. Let $G$ be a counterexample with the fewest vertices and $L$ be a list assignment of size $\lceil \frac{M}{2} \rceil + 1 \geq 4$ such that $G$ has no linear $L$-coloring. In the proof we need some structural lemmatas, and it is clear that Lemma 6 and Lemma 7 are also true.

Lemma 12. Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then $v_1, v_2, v_3$ must be $(3, 5^+)$-vertices.

Proof. Assume that $v_1$ is a $(3, 4^-)$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_2) \neq c(v_3)$, there exist at least $|L(v_1)\{c(u_1), c_2(u_1)\}| \geq 2$ colors available for $v_1$. If there is an available color $\alpha \notin \{c(v_2), c(v_3)\}$ for $v_1$, then let $c(v_1) = \alpha$ and $c(v) \in L(v)\{c(v_1), c(v_2), c(v_3)\}$. If the available colors for $v_1$ are exactly $c(v_2)$ and $c(v_3)$, then let $c(v_1) = c(v_2)$ and $c(v) \in L(v)\{c(v_1), c(u_2), c(v_3)\}$. It is similar for $c(v_1) = c(v_3)$. Thus we get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. If $c(v_2) = c(v_3)$, we can extend the linear list coloring $c$ of $G'$ to $v_1$ since $|L(v_1)\{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. There is at least $|L(v)\{c(u_1), c(v_2), c(v_3)\}| \geq 1$ color available for $v$. Thus, we also get a linear list coloring of $G$. A contradiction.

Lemma 13. Let $v$ be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$. If $v_1$ and $v_2$ are $(3, 3)$-vertices, then $v_3$ must be a $4^+$-vertex.

Proof. Assume that $v_3$ is a $3^-$-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear $L$-coloring $c$ to $v_3$ such that no bi-colored cycle passes $v_1v_3$ since $|L(v_1)\{c(u_1), c(N(u_1))\}| \geq 1$.

If $c(v_1) = c(v_3)$. Since $|L(v_2)\{c(v_1), c(u_2), c_2(u_2)\}| \geq 1$, we can extend the coloring $c$ to $v_2$. Finally, we can color $v$ with a color in $L(v)\{c(v_1), c(v_2), c_2(v_3)\}$ when $c_2(v_3) = 1$, or in $L(v)\{c(v_1), c(v_2), c_2(u_2)\}$ when $c_2(v_2) = 0$. It is clear that no bi-colored cycle passes $v_2v_3$. Then we get a linear list coloring of $G$.

If $c(v_1) \neq c(v_3)$. There is at least $|L(v)\{c(v_1), c(v_3), c_2(v_3)\}| \geq 1$ color available for $v$. Finally, we can color $v_2$ with a color in $L(v_2)\{c(v), c(u_2), c_2(u_2)\}$ when $c(v) \neq c(v_2)$, or in $L(v_2)\{c(v_2), c(N(u_2))\}$ when $c_2(v_2) = c(u_2)$. In this process, there will be no bi-colored cycle passing $v_2v_3$. Thus, we also get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. A contradiction.

Lemma 14. Let $v$ be a 4-vertex with $n_2(v) = 4$ in $G$. Then there are at most two $(4, 3)$-vertices in $N(v)$.

Proof. Let $N(v) = \{v_1, \ldots, v_4\}$, and $u_i$ be the other neighbor of $v_i$ for $i = 1, \ldots, 4$. Assume that $v_1, v_2$ and $v_3$ are $(4, 3)$-vertices. Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the linear
$L$-coloring $c$ of $G'$ to $v_4$ since $|L(v_4)\{c(u_4), c_2(u_4)\}| \geq 1$. We can continue to extend to $v_3$ with $c(v_3) \neq c(v_4)$ since $|L(v_3)\{c(u_3), c_2(u_3), c(v_4)\}| \geq 1$. Then we color $v_2$ with a color in $L(v_2)\{c(u_2), c(v_2)\}$. Notice that no bi-colored cycle passes $v_2u_2$ or $v_3v_4$. This signifies that any bi-colored cycle in $G$ if there will be must passes $v_1$. Finally, we will extend the coloring $c$ to $v_1$ and $v$ in two different cases.

If $c(v_2) \notin \{c(v_3), c(v_4)\}$, we can choose a color from $L(v)\{c(v_2), c(v_3), c(v_4)\}$ for $v$. Then there is at least $|L(v_1)\{c(v), c(u_1), c_2(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\{c(u_1), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$ color available for $v_1$, which ensure no bi-colored cycle passes $vv_1u_1$. So we get a linear list coloring of $G$.

If $c(v_2) \in \{c(v_3), c(v_4)\}$, suppose $c(v_2) = c(v_3)$ (similarly for $c(v_2) = c(v_4)$). If $|c_2(u_1)| = 1$, we color $v_1$ with a color in $L(v_1)\{c(v_2), c(u_1), c_2(u_1)\}$, and no bi-colored cycle passes $v_1v_4$. If $|c_2(u_1)| = 0$, we color $v_1$ with a color in $L(v_1)\{c(v_2), c(u_1), c(v_4)\}$, which ensure that no bi-colored cycle passes $v_1v_4$. Then we color $v$ with a color in $L(v)\{c(v_1), c(v_2), c(v_4)\}$. Notice that no bi-colored cycle passes $v_1v_3$ since $c(v_1) \neq c(v_3)$. Thus, we also get a linear list coloring $c$ of $G$. A contradiction. 

**Lemma 15.** Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$. If $v_1, v_2, v_3, v_4$ are four $(5,2)$-vertices, then $v_5$ must be a $3^+$-vertex.

**Proof.** Assume that $v_5$ is a 2-vertex, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, 2, \ldots, 5$. Let $G' = G - [v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the $L$-coloring $c$ of $G'$ to $v_5$ since $|L(v_5)\{c(u_3), c_2(u_3)\}| \geq 1$, and continue to $v_4$ such that $c(v_4) \neq c(v_5)$ and no bi-colored cycle passes $v_4u_4$ since $|L(v_4)\{c(u_4), c(N(u_4)), c(v_5)\}| \geq 1$, then to $v_3$ with $c(v_3) \notin \{c(u_4), c(v_4)\}$ since $|L(v_3)\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. We can color $v$ with a color in $L(v)\{c(v_5), c(v_4), c(v_3)\}$, and color $v_2$ such that no bi-colored cycle passes $v_2u_2$ since $|L(v_2)\{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color $v_1$ linearly since $|L(v_1)\{c(v), c_2(v), c(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_5)$, there will be no bi-colored cycle created. Thus we can extend the linear $L$-coloring $c$ of $G'$ to $G$. A contradiction.

**Lemma 16.** Let $v$ be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If $v_1, v_2, v_3$ are $(5,2)$-vertices, then at least one of $v_4$ and $v_5$ is a $(5,4^+)$-vertex.

**Proof.** Assume that $v_3$ and $v_5$ are $(5,3^-)$-vertices, and $u_i$ is the neighbor of $v_i$ other than $v$ for $i = 1, \ldots, 5$. Let $G' = G - [v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the coloring $c$ of $G'$ to $v_5$ such that no bi-colored cycle passes $v_5u_5$ since $|L(v_5)\{c(u_5), c(N(u_5))\}| \geq 1$, and continue to $v_3$ such that $c(v_4) \neq c(v_5)$ as $|L(v_4)\{c(u_4), c_2(u_4), c(v_5)\}| \geq 1$, then to $v_3$ with $c(v_3) \notin \{c(v_5), c(v_4)\}$ since $|L(v_3)\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. Now we
can color $v$ with a color in $L(v) \setminus \{c(v_5), c(v_4), c(v_3)\}$, and color $v_2$ such that no bi-colored cycle passes $v v_2 u_2$ since $|L(v_2) \setminus \{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color $v_1$ linearly since $|L(v_1) \setminus \{c(v), c_2(v), c(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1) \setminus \{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_4)$, there will be no bi-colored cycle created. Thus, we can extend the linear $L$-coloring $c$ of $G'$ to $G$. A contradiction.

We will derive a contradiction by a discharging procedure proceeded in $G$ to complete the proof of Theorem 11. In the discharging procedure, the initial charge function $\omega$ is defined as $\omega(v) = d(v) - \frac{18}{7}$ for every vertex $v \in V(G)$, and the discharging rules are as follows.

**R1.** Every $6^+$-vertex sends $\frac{4}{7}$ to each adjacent 2-vertex or 3-vertex.

**R2.** Every 5-vertex sends $\frac{4}{7}$ to each adjacent $(5, 2)$-vertex, $\frac{3}{7}$ to each adjacent $(5, 3)$-vertex, $\frac{1}{7}$ to each adjacent $(5, 4^+)$-vertex, $\frac{1}{7}$ to each adjacent 3-vertex.

**R3.** Every 4-vertex sends $\frac{3}{7}$ to each adjacent $(4, 3)$-vertex, $\frac{2}{7}$ to each adjacent $(4, 4^+)$-vertex, $\frac{2}{7}$ to each adjacent 3-vertex.

**R4.** Every 3-vertex sends $\frac{2}{7}$ to each adjacent $(3, 3)$-vertex, $\frac{1}{7}$ to each adjacent $(3, 4^+)$-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V$.

If $d(v) \geq 6$, then $\omega'(v) \geq \omega(v) - \frac{4}{7} \times d(v) = \frac{3d(v)}{7} - \frac{12}{7} \geq \frac{3d(v) - 18}{7} \geq 0$ by R1.

Let $v$ be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{4}{7} - \frac{1}{7} = 5 - \frac{16}{7} - \frac{1}{7} = 0$ by R2. When $n_2(v) = 5$, there are at most three $(2, 5)$-vertices in $N(v)$ by Lemma 15. If there are two or less $(2, 5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{4}{7} - 3 \times \frac{3}{7} = \frac{5}{7} - \frac{18}{7} - \frac{9}{7} - \frac{2}{7} = 0$ by R2. If there are three $(2, 5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{4}{7} - \frac{2}{7} = \frac{5}{7} - \frac{18}{7} - \frac{12}{7} - \frac{5}{7} = 0$ by Lemma 16 and R2.

Let $v$ be a 4-vertex. If $n_2(v) \leq 3$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{3}{7} - \frac{1}{7} = 4 - \frac{18}{7} - \frac{9}{7} - \frac{1}{7} = 0$ by R3. If $n_2(v) = 4$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{2}{7} = \frac{4}{7} - \frac{16}{7} - \frac{5}{7} = 0$ by Lemma 14 and R3.

Let $v$ be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3, 5^+)$-vertices by Lemma 12. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} = 3 - \frac{18}{7} - \frac{5}{7} = 0$ by R4. If $n_2(v) = 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{7} + \frac{1}{7} = 3 - \frac{18}{7} - \frac{4}{7} + \frac{1}{7} = 0$ by Lemma 13 and all discharging rules, or $\omega'(v) \geq \omega(v) - \frac{3}{7} - \frac{1}{7} = 3 - \frac{18}{7} - \frac{5}{7} = 0$. If $n_2(v) \leq 1$, then $\omega'(v) \geq \omega(v) - \frac{2}{7} = 3 - \frac{18}{7} - \frac{5}{7} = 0$ by R4.

Finally, let $v$ be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 5$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{4}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$. When $d(x) = 3$, we have $\omega'(v) = \omega(v) + 2 \times \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ if $d(y) = 3$, and $\omega'(v) = \omega(v) + \frac{4}{7} + \frac{1}{7} = 2 - \frac{18}{7} + \frac{5}{7} = 0$ if $d(y) \geq 4$. Otherwise, $d(x) \geq 4$ and $d(y) \geq 4$, we have $\omega'(v) \geq \omega(v) + \frac{2}{7} + \frac{2}{7} = 2 - \frac{18}{7} + \frac{5}{7} = 0$ by R1, R2, and R3. We get the desired contradiction, and Theorem 11 is proved.
Similarly, the condition $\Delta(G)$ in Theorem 3(2) must be $\Delta(G) \geq 4$.

4. **Graphs with $mad(G) < \frac{20}{7}$ and $\Delta(G) \geq 5$**

Cranston and Yu [1] conjectured that the hypothesis $\Delta(G) \geq 9$ of Theorem 1(iii) can be replaced by $\Delta(G) \geq 7$, even $\Delta(G) \geq 5$. Now, we prove Theorem 3(3) to support their conjecture. In order to prove Theorem 3(3), we prove the following theorem which implies Theorem 3(3) immediately.

**Theorem 17.** Let $M \geq 5$ be an integer. If $G$ is a graph with $mad(G) < \frac{20}{7}$ and $\Delta(G) \leq M$, then $le_2(G) \leq \lceil \frac{M}{2} \rceil + 2$.

**Proof.** Let $G$ be a counterexample of the fewest vertices with $mad(G) < \frac{20}{7}$ and $5 \leq \Delta(G) \leq 8$ (Theorem 17 is true for graphs $G$ with $\Delta(G) \geq 9$ by Theorem 1(iii)). There exists an assignment $L$ with $|L| \geq \lceil \frac{M}{2} \rceil + 2 \geq 5$ such that $G$ is not linearly $L$-choosable, but $H$ has a linear $L$-coloring, where $H$ is any proper subgraph of $G$. Clearly, $G$ is connected and $\delta(G) \geq 2$. In the proof we need some structural lemmatas. □

**Lemma 18.** Let $v$ be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 2$.

**Proof.** Assume $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \leq \lceil \frac{M}{2} \rceil + 1$. Let $G' = G - v$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. If $c(v_1) \neq c(v_2)$, we can color $v$ with any color in $L(v) \setminus \{c(v_1), c(v_2), c_1(v_1), c_2(v_2)\}$. Then the number of available colors for $v$ is at least $\left\lceil \frac{M}{2} \rceil + 2 - \left(2 + \left\lceil \frac{d(v_1) - 1}{2} \right\rceil + \left\lceil \frac{d(v_2) - 1}{2} \right\rceil \right) = \left\lceil \frac{M}{2} \rceil + 2 - \left( \left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \right)$. Clearly, there will be no bi-colored cycle created. So we extend the linear $L$-coloring $c$ of $G'$ to $G$. Now we suppose $c(v_1) = c(v_2)$. In order to color $v$ linearly and avoid bi-colored cycles created, the forbidden color set for $v$ contains the color $c(v_1)$, the colors appearing twice in $N(v_1)$ or $N(v_2)$, and the colors appearing in both $N(v_1)$ and $N(v_2)$. So at most $1 + |c_2(v_1) \cup c_2(v_2)| + |c_1(v_1) \cap c_1(v_2)| \leq \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \leq \left\lfloor \frac{d(x)}{2} \right\rfloor + \left\lfloor \frac{d(y)}{2} \right\rfloor + \left\lceil \frac{M}{2} \rceil + 1$ colors are forbidden for $v$. Thus, we also can get a linear $L$-coloring of $G$. A contradiction. □

**Lemma 19.** Let $v$ be a 3-vertex of $G$ with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 2$, then $d(v_2) \geq 3$ and $\left\lceil \frac{d(v_2) + d(v_3)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 1$.

**Proof.** We prove $d(v_2) \geq 3$ first. To the contrary, we assume $d(v_2) = 2$. Let $G' = G - \{v, v_1, v_2\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. The neighbors of $v_1$ and $v_2$ other than $v$ are denoted by $u_1$ and $u_2$, respectively. We can extend the coloring $c$ of $G'$ to $v_1$ such that $c(v_1) \neq c(v_3)$ since...
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$L(v_1) \setminus \{c(u_1), c_2(u_1), c(v_3)\} \geq 1$, which ensures that no bi-colored cycle passes $v_1v_3$. We can continue to extend to $v$ since $|L(v) \setminus \{c(v_1), c(v_3), c_2(v_3)\}| \geq 1$. If $c(v) \neq c(u_2)$, which means that no bi-colored cycle passes $v_2u_2$, we can color $v_2$ linearly since $|L(v_2) \setminus \{c(v), c(u_2), c_2(u_2)\}| \geq 1$. When $c(v) = c(u_2)$, the number of available colors for $v_2$ is at least $|L(v_2) \setminus \{c(u_2), c_2(u_2)\}| \geq 2$. If there is an available color $\alpha \notin \{c(v_1), c(v_3)\}$ for $v_2$, then we color $v_2$ with $\alpha$. Now we assume that the available colors for $v_2$ are exactly $c(v_1)$ and $c(v_3)$. Notice that $|c_2(u_2)| = \left\lceil \frac{M-1}{2} \right\rceil$ and $|c_1(u_2)| \leq 1$ now. To avoid bi-colored cycle created, the number of forbidden colors for $v_2$ is at most $|c(u_2), c(N(v_2))| = 1 + |c_2(u_2)| + |c_1(u_2)| \leq 1 + \left\lceil \frac{M-1}{2} \right\rceil + 1 = \left\lceil \frac{M}{2} \right\rceil + 1$, so we can color $v_2$ linearly. Thus, we get a linear $L$-coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$. A contradiction.

Now, we prove the inequality. Suppose to the contrary that, we have
\[
\left\lfloor \frac{d(v_2)+d(v_3)}{2} \right\rfloor \leq \left\lceil \frac{M}{2} \right\rceil,
\]
and $u_1$ is the neighbor of $v_1$ other than $v$. Let $G' = G - v_1$, then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$.

Case 1. $c(v_2) \neq c(v_3)$. If $c(v) \neq c(u_1)$, then we can extend the coloring $c$ to $v_1$ to get a linear $L$-coloring of $G$ since $|L(v_1) \setminus \{c(v), c(u_1), c_2(u_1)\}| \geq 1$. If $c(v) = c(u_1)$, the number of available colors for $v_1$ is at least $|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq 2$. If there is a color $\alpha \notin \{c(v_2), c(v_3)\}$ available for $v$, then we can extend $c$ from $G'$ to $G$ by coloring $v_1$ with $\alpha$. Now we assume that $L(v_1) \setminus \{c(v), c_2(u_1)\} = \{c(v_2), c(v_3)\}$. Notice that $|c_2(u_1)| = \left\lceil \frac{M-1}{2} \right\rceil$ now. Then $c(v_2)$ and $c(v_3)$ appears at most once in $N(u_1)$, but both of them could not appear in $N(u_1)$ at the same time (otherwise $|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq \left\lceil \frac{M}{2} \right\rceil + 2 - (1 + \left\lceil \frac{M-3}{2} \right\rceil) \geq 3$). So we color $v_1$ with $c(v_3)$ if $c(v_2)$ appears in $N(u_1)$, otherwise color $v_1$ with $c(v_2)$. Then there will be no bi-colored cycle created. Thus, we get a linear $L$-coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$.

Case 2. $c(v_2) = c(v_3)$. If $c(v) = c(u_1)$, then we can color $v_1$ linearly since $|L(v_1) \setminus \{c(v_2), c(v_1), c_2(v_1)\}| \geq 1$, and no bi-colored cycle created. Now, suppose $c(v) \neq c(u_1)$. We erase the color of $v$ first, then we can extend the list coloring $c$ to $v_1$ since $|L(v_1) \setminus \{c(v_2), c(u_1), c_2(u_1)\}| \geq 1$. To avoid bi-colored cycle created, the number of forbidden colors for $v$ is at most $2 + |c(v_2) \cup c_2(v_3)| + |c_1(v_2) \cap c_1(v_3)| \leq 2 + \left\lceil \frac{d(v_2)+d(v_3)}{2} \right\rceil = 1 + \left\lceil \frac{d(v_2)+d(v_3)}{2} \right\rceil \leq \left\lceil \frac{M}{2} \right\rceil + 1$. We also can extend the linear $L$-coloring $c$ of $G'$ to $G$. A contradiction.

Lemma 20. Let $v$ be a 4-vertex in $G$. Then $n_2(v) \leq 3$.

Proof. Let $N(v) = \{v_1, \ldots, v_4\}$. Suppose to the contrary, let $n_2(v) = 4$, and $u_i$ be the neighbor of $v_i$ other than $v$ for $i = 1, \ldots, 4$. Let $G' = G - N[v]$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. Since $|L(v_i) \setminus \{c(u_i), c_2(u_i)\}| \geq 2$, we can color $v_i$ linearly with at least two different colors for $i = 1, 2, 3, 4$. Finally, we can color $v$ with a color in $L(v) \setminus c(N(v))$ if $|c(N(v))| = 4$, or in
$L(v) \{c(N(v)), c(u_i)\}$ if $c(v_i) = c(v_j)$ for $1 \leq i < j \leq 4$. And no bi-colored cycle appears in this process. Thus, we get a linear list coloring of $G$ extended from the linear $L$-coloring $c$ of $G'$, a contradiction.

**Lemma 21.** Let $v$ be a 4-vertex with $N(v) = \{v_1, \ldots, v_4\}$. If $d(v_1) = d(v_2) = d(v_3) = 2$ and $v_3$ is a $(4, 5)$-vertex, then $v_4$ must be a $4^+$-vertex.

**Proof.** Suppose to the contrary, let $v_4$ be a $3^-$-vertex, and $u_i$ be the neighbor of $v_i$ other than $v$ for $i = 1, 2, 3$. Let $G' = G - \{v, v_1, v_2, v_3\}$, then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the coloring $c$ of $G'$ to $v_1$ with $c(v_1) \neq c(v_4)$ since $|L(v_1) \{c(u_1), c(u_2), c(v_4)\}| \geq 1$. Then there are at least $|L(v_2) \{c(u_2), c_2(u_3)\}| \geq 2$ colors available for $v_2$.

If there is a color $\alpha \notin \{c(v_1), c(v_4)\}$ available for $v_2$, let $c(v_2) = \alpha$. If $|c_2(u_3)| = 1$, we can choose a color for $v$ in $L(v) \{c(v_1), c(v_2), c(v_4), c_2(u_3)\}$, and there will be no bi-colored cycle created passing $v_4$. Then, we color $v_3$ with a color in $L(v_3) \{c(v), c(u_3), c_2(u_3)\}$ if $c(v) \neq c(u_3)$. When $c(v) = c(u_3)$, in order to color $v_3$ linearly (no bi-colored cycle created), we must forbidden $c(v), c_2(N(u_3))$ and $(c(v_1), c(v_3), c(v_4)) \cap c_1(N(u_3))$. Notice that $d(u_3) = 5$, then $|c_2(N(u_3)) \cup \{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))| \leq 3$. So we can color $v_3$ linearly with a color in $L(v_3) \{c(v), c_2(N(u_3)), c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))$. Thus, we get a linear list coloring of $G$.

Suppose the available color set for $v_2$ is exactly $\{c(v_1), c(v_4)\}$. Notice that $|c_2(u_3)| = \left\lceil \frac{M - 1}{2} \right\rceil$ now. We color $v_2$ with $c(v_4)$ first. If $|c_2(u_3)| = 2$, we color $v_3$ with a color in $L(v_3) \{c(v_1), c(u_3), c_2(u_3)\}$. If $|c_2(u_3)| \leq 1$, we color $v_3$ with a color in $L(v_3) \{c(v_1), c(v_3), c_2(u_3)\}$. Notice $d(u_3) = 5$, then no bi-colored cycle passes $v_3$. Finally, we can color $v$ with a color in $L(v) \{c(v_1), c(v_3), c(v_4)\}$ if $|c_2(u_3)| = 1$, or in $L(v) \{c(v_1), c(v_3), c(v_4), c_2(u_2)\}$ if $|c_2(u_3)| = 0$. Clearly, there will be no bi-colored cycle passing $v_4$. Then we extend the linear $L$-coloring $c$ of $G'$ to $G$, a contradiction.

**Lemma 22.** Let $v$ be a 4-vertex with $N(v) = \{v_1, \ldots, v_4\}$. If $d(v_1) = d(v_2) = d(v_3) = 2$ and $v_2, v_3$ are $(4, 5)$-vertices, then $v_4$ must be a $5^+$-vertex.

**Proof.** Suppose to the contrary, let $v_4$ be a $4^-$-vertex, and $u_i$ be the neighbor of $v_i$ other than $v$ for $i = 1, 2, 3$. Let $G' = G - \{v, v_1, v_2, v_3\}$. Then $G'$ has a linear $L$-coloring $c$ by the minimality of $G$. We can extend the coloring $c$ of $G'$ to $v_1$ with $c(v_1) \neq c(v_4)$ since $|L(v_1) \{c(u_1), c(u_2), c(v_4)\}| \geq 1$. Then there are at least $|L(v_2) \{c(u_2), c_2(u_3)\}| \geq 2$ colors available for $v_2$.

If there is an available color $\alpha \notin \{c(v_1), c(u_3)\}$ for $v_2$, let $c(v_2) = \alpha$. If $|c_2(u_3)| = 1$, we color $v$ with a color in $L(v) \{c(v_1), c(v_2), c(v_4), c_2(u_3)\}$. Then, we color $v_3$ with a color in $L(v_3) \{c(v), c(u_3), c_2(u_3)\}$ if $c(v) \neq c(u_3)$. If $c(v) = c(u_3)$, we color $v_3$ with a color in $L(v_3) \{c(u_3), c_2(u_3)\}, L(v_3) \{c(u_3), c(v_1), c(v_2), c(v_4)\}$ or $L(v_3) \{c(u_3), c(v_1), c(v_2), c(v_4)\}$ when $|c_2(u_3)| = 2$, $|c_2(u_3)| = 1$ or $|c_2(u_3)| = 0$.
respectively. Notice that \(v_3\) is a \((4, 5)\)-vertex, it means \(d(u_3) = 5\), then there will be no bi-colored cycle passing \(v_3u_3\). Then we get a linear list coloring of \(G\). If \(|c_2(v_4)| = 0\), we choose a color for \(v\) in \(L(v)\{c(v_1), c(v_2), c(v_3), c(u_3)\}\), then we can color \(v_3\) linearly since \(|L(v_3)\{c(v), c(u_3), c_2(u_3)\}| \geq 1\). We also get a linear list coloring of \(G\).

When the available color set for \(v_2\) is exactly \(\{c(v_1), c(v_4)\}\) (notice that \(|c_2(v_2)| = 2\), and there will be no bi-colored cycle passing \(v_2u_2\), we can color \(v_2\) with \(c(v_4)\). If \(|c_2(u_3)| = 2\), we color \(v_3\) with a color in \(L(v_3)\{c(v_4), c(u_3), c_2(u_3)\}\); if \(|c_2(u_3)| \leq 1\), we color \(v_3\) with a color in \(L(v_3)\{c(v_4), c(v_1), c(u_3), c_2(u_3)\}\). Notice \(d(u_3) = 5\), there will be no bi-colored cycle passing \(v_3u_3\). Finally, we can color \(v\) with a color in \(L(v)\{c(v_1), c(v_3), c(v_4), c_2(v_4)\}\) if \(|c_2(v_4)| = 1\), or in \(L(v)\{c(v_1), c(v_3), c(v_4)\}\) if \(|c_2(v_4)| = 0\). Then we extend the linear \(L\)-coloring \(c\) of \(G'\) to \(G\), a contradiction.

To complete our proof of Theorem 17, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function \(\omega\) on \(V(G)\) by \(\omega(v) = d(v) - \frac{20}{7}\) for every \(v \in V(G)\). The discharging rules are as follows.

**R1.** Every \(5^+\)-vertex sends \(\frac{d(v) - 20}{d(v)}\) to each adjacent vertex.

**R2.** Every \(4^-\)-vertex sends \(\frac{2}{7}\) to each adjacent \((4, 5)\)-vertex, \(\frac{1}{3}\) to each adjacent \((4, 6)\)-vertex, \(\frac{1}{4}\) to each adjacent \(3^-\)-vertex;

**R3.** Every \(3^-\)-vertex sends \(\frac{3}{7}\) to each adjacent \(2^-\)-vertex (if it has one).

Now we are going to show that \(\omega'(v) \geq 0\) for all \(v \in V(G)\). We only need to check the final charges of \(4^-\)-vertices from the discharging rules.

Let \(v\) be a \(4^-\)-vertex in \(G\). Then \(n_2(v) \leq 3\) by Lemma 20. If \(n_2(v) \leq 2\), then \(\omega'(v) \geq \omega(v) - 2 \times \frac{2}{7} - 2 \times \frac{1}{4} = 0\) by R2. When \(n_2(v) = 3\), if there are three \((4, 6)\)-vertices in \(N(v)\), then \(\omega'(v) \geq \omega(v) - 3 \times \frac{1}{3} - \frac{1}{7} = 0\); if there is only one \((4, 5)\)-vertex in \(N(v)\), then \(\omega'(v) \geq \omega(v) - 2 \times \frac{4}{7} - \frac{3}{7} > 0\) by Lemma 21 and R2; if there are two or more \((4, 5)\)-vertices in \(N(v)\), we have \(\omega'(v) \geq \omega(v) - 3 \times \frac{2}{7} + \frac{3}{7} > 0\) by Lemma 22 and R2.

Let \(v\) be a \(3^-\)-vertex in \(G\). Then \(n_2(v) \leq 1\) by Lemma 19. If \(n_2(v) = 0\), then \(\omega'(v) \geq \omega(v) - \frac{2}{7} > 0\). When \(n_2(v) = 1\), if there is a \(3^-\)-vertex in \(N(v)\), we have \(\omega'(v) \geq \omega(v) - \frac{4}{7} + \frac{3}{7} > 0\) by Lemma 19 and R3; if there are two \(4^-\)-vertices in \(N(v)\), then \(\omega'(v) \geq \omega(v) - \frac{2}{7} + 2 \times \frac{1}{4} = 0\).

Let \(v\) be a \(2^-\)-vertex with \(N(v) = \{x, y\}\) and \(d(x) \leq d(y)\). Clearly, \(d(x) \geq 3\) by Lemma 18. If \(d(x) = 3\), then \(d(y) \geq 5\) by Lemma 19, so \(\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0\) by R3 and R1. If \(d(x) = 4\), then \(d(y) \geq 5\) by Lemma 18, so \(\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0\), or \(\omega'(v) \geq \omega(v) + \frac{11}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0\). Otherwise, \(d(x) \geq 5\) and \(d(y) \geq 5\), we have \(\omega'(v) \geq \omega(v) + \frac{2}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0\).

In summary, the proof of Theorem 3 is completed.
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References

    doi:10.1016/j.disc.2011.05.017

    doi:10.1007/s11425-010-3073-0

    doi:10.1016/j.dam.2014.03.019

    doi:10.1016/j.disc.2007.07.112

    doi:10.1016/j.disc.2010.10.023

    doi:10.1016/j.ejc.2013.02.008


    doi:10.1016/S0012-365X(97)00209-4

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