DECOMPOSITIONS OF COMPLETE BIPARTITE GRAPHS AND COMPLETE GRAPHS INTO PATHS, STARS, AND CYCLES WITH FOUR EDGES EACH

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Abstract

Let G be either a complete graph of odd order or a complete bipartite graph in which each vertex partition has an even number of vertices. In this paper, we determine the set of triples \((p,q,r)\), with \(p, q, r > 0\), for which there exists a decomposition of \(G\) into \(p\) paths, \(q\) stars, and \(r\) cycles, each of which has 4 edges.

Keywords: complete graph, complete bipartite graph, path, star, cycle, decomposition.

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1. Introduction

All graphs considered here are finite and undirected, unless otherwise noted.

Let \(G, H, H_1,\ldots, H_r\) be graphs for some integer \(r\). A decomposition of \(G\) is a set of edge-disjoint subgraphs of \(G\) whose union is \(G\). An \(H\)-decomposition of \(G\) is a decomposition of \(G\) into copies of \(H\). If \(G\) has an \(H\)-decomposition, we say that \(G\) is \(H\)-decomposable. An \(\{H_1,\ldots, H_r\}\)-decomposition of \(G\) is a decomposition of \(G\) into copies of \(H_1,\ldots, H_r\) containing at least one copy of each \(H_i\), for each \(i = 1,\ldots, r\). If \(G\) has an \(\{H_1,\ldots, H_r\}\)-decomposition, we say that \(G\) is \(\{H_1,\ldots, H_r\}\)-decomposable. Moreover, if there is a decomposition of \(G\) containing precisely \(\alpha_i\) elements isomorphic to \(H_i\), then we say that \(G\) has an \(\{H_1^{\alpha_1},\ldots, H_r^{\alpha_r}\}\)-decomposition or \(G\) is \(\{H_1^{\alpha_1},\ldots, H_r^{\alpha_r}\}\)-decomposable. Let \(CD(G; H_1,\ldots, H_r)\) denote the set of all \(r\)-tuples \((\alpha_1,\ldots, \alpha_r)\) of positive integers...
such that $G$ is $\{H_1^{\alpha_1}, \ldots, H_r^{\alpha_r}\}$-decomposable. Obviously, if we can find an $r$-tuple in $CD(G; H_1, \ldots, H_r)$, then $G$ is $\{H_1, \ldots, H_r\}$-decomposable.

As usual, $K_n$ denotes the complete graph on $n$ vertices, and $K_{m,n}$ denotes the complete bipartite graph with vertex partitions of sizes $m$ and $n$. A $k$-path, denoted by $P_k$, is a path with $k$ edges; a $k$-star, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$; a $k$-cycle, denoted by $C_k$, is a cycle of length $k$.

Decompositions of graphs into isomorphic paths has attracted considerable attention (see [8, 12–14, 17–19, 28, 40, 42]). Besides, decompositions of graphs into $k$-stars have also attracted a fair share of interest (see [9, 25, 39, 41, 43, 44]). Moreover, decompositions of graphs into $k$-cycles have been a popular topic of research in graph theory (see [10, 27] for surveys of this topic).

The study of the $\{G, H\}$-decomposition was introduced by Abueida and Daven in [1]. In [2, 4], they investigated, respectively, the problem of $\{K_k, S_k\}$-decomposition of the complete graph $K_n$ and the problem of the $\{C_4, E_2\}$-decomposition of several graph products, where $E_2$ is a matching of size 2. Abueida and O’Neil [3] settled the existence problem for $\{C_k, S_{k-1}\}$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [29, 30] gave necessary and sufficient conditions for the existence of $\{G(n), H(n)\}$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$, where $G(n), H(n) \in \{C_n, P_{n-1}, S_{n-1}\}$.

Recently, Lee and Lin [20, 21, 23, 24] established necessary and sufficient conditions for the existence of $\{C_k, S_k\}$-decompositions of the complete bipartite graphs, the complete bipartite multigraphs, the complete bipartite graphs with a 1-factor removed, and the multicrows, respectively. Besides, Abueida, Lian [5], and Beggas et al. [7] investigated the problems of $\{C_k, S_k\}$-decompositions of the complete graph $K_n$ and $\lambda K_n$ respectively, giving some necessary or sufficient conditions for such decompositions to exist. In [22], Lee and Chu established necessary and sufficient conditions for the existence of $\{P_k, S_k\}$-decompositions of the balanced complete bipartite graphs. In 2016, Lin and Jou [26] established necessary and sufficient conditions for the existence of $\{P_k, C_k, S_k\}$-decompositions of the balanced complete bipartite graphs.

For the $\{G^p, H^q\}$-decompositions of a graph, Jeevadoss and Muthusamy [15, 16] determined the set of ordered pairs $(p, q)$ of positive integers for which there exists a $\{P_k^p, C_k^q\}$-decomposition of $\lambda K_{m,n}$ when $\lambda = 1$ and $k \equiv 0 \pmod{4}$; $\lambda = 2$ and $k \equiv 0 \pmod{2}$; for some positive integers $\lambda$, $m$, $n$, and $k$. Jeevadoss and Muthusamy [15] also determined the set of ordered pairs $(p, q)$ of positive integers for which there exists a $\{P_k^p, C_k^q\}$-decomposition of $K_n$ when $k$ is even and $n$ is odd with $n > 4k$. Fu et al. [11] determined the set of ordered pairs $(p, q)$ of positive integers for which there exists a $\{C_3^p, S_3^q\}$-decomposition of $K_n$. The author also determined the set of ordered pairs $(p, q)$ of positive integers for which there exists a $\{P_k^p, S_k^q\}$-decomposition of $K_n$ when $n \geq 4k$ [36]; there exists a $\{P_k^p, C_k^q\}$-decomposition of $K_n$ when $k$ is even, $n$ is odd, and $n > 5k$ [33]; there
exists a \(\{C_k^p, S_k^q\}\)-decomposition of \(K_n\) for some \(k\) and \(n\) [35]; there exists a \(\{P_k^p, S_k^q\}\)-decomposition of \(K_{m,n}\) when \(m > k\) and \(n \geq 3k\) [36]. In [37], the author also investigated the \(\{H^p, K^q\}\)-decomposition of the complete bipartite digraphs and the complete digraphs, where \(H\) and \(K\) are, respectively, directed paths and directed cycles with \(k\) edges each.

In this paper, we determine the set of triples \((p, q, r)\) of positive integers for which there exists a \(\{P_4^p, S_4^q, C_4^r\}\)-decomposition of \(K_n\) and \(K_{m,l}\) when \(n\) is odd, and both \(m\) and \(l\) are even.

2. Preliminaries

In this section we collect some needed terminologies and notations, and present some results which are useful for our discussions.

Let \(|V(G)|\) and \(e(G)\) denote, respectively, the order of a graph \(G\) and the number of edges in \(G\); and let us call a graph even if all its vertex degrees are even. Let \(G_1\) and \(G_2\) be graphs. The union \(G_1 \cup G_2\) of \(G_1\) and \(G_2\) is the graph with vertex set \(V(G_1) \cup V(G_2)\) and edge set \(E(G_1) \cup E(G_2)\).

The following theorem provides necessary conditions for the existence of a decomposition of an even graph into specified numbers of paths, cycles, and stars with the same number of edges each.

**Theorem 1.** Let \(G\) be an even graph and let \(k\), \(p\), \(q\), and \(r\) be positive integers with \(k \geq 3\). If \(G\) can be decomposed into \(p\) copies of \(P_k\), \(q\) copies of \(S_k\), and \(r\) copies of \(C_k\), then \(|V(G)| \geq k + 1\); \(k(p+q+r) = e(G)\) and \(p \geq \left\lceil \frac{k}{2} \right\rceil\) when \(q = 1\).

**Proof.** Conditions \(|V(G)| \geq k + 1\) and \(k(p+q+r) = e(G)\) are trivial. Assume \(\mathcal{D}\) is an arbitrary decomposition of \(G\) into \(p\) copies of \(P_k\), one copy of \(S_k\), and \(r\) copies of \(C_k\). Let \(H\) be the only \(S_k\) and \(C^{(1)} \cup \ldots \cup C^{(r)}\) denote those \(r\) copies of \(C_k\) in \(\mathcal{D}\). Then, there are \(2 \left\lceil \frac{k}{2} \right\rceil\) vertices with odd degree in \(G - E(\bigcup C^{(1)} \cup \ldots \cup C^{(r)})\). Since \(G - E(\bigcup C^{(1)} \cup \ldots \cup C^{(r)})\) has to decompose into \(p\) copies of \(P_k\), and there are exactly two vertices with odd degree in a path, \(p \geq \left\lceil \frac{k}{2} \right\rceil\).

Let \(\mathcal{D}(G; P_k, S_k, C_k)\) denote the set of all triples \((m, n, l)\) of non-negative integers such that a decomposition of a graph \(G\) into \(m\) copies of \(P_k\), \(n\) copies of \(S_k\), and \(l\) copies of \(C_k\) exists. Note that \((m, n, 0) \in \mathcal{D}(G; P_k, S_k, C_k)\) if \((m, n) \in CD(G; P_k, S_k)\); \((m, 0, l) \in \mathcal{D}(G; P_k, S_k, C_k)\) if \((m, l) \in CD(G; P_k, C_k)\); \((0, n, l) \in \mathcal{D}(G; P_k, S_k, C_k)\) if \((n, l) \in CD(G; S_k, C_k)\); \((\frac{e(G)}{k}, 0, 0), (0, \frac{e(G)}{k}, 0), (0, 0, \frac{e(G)}{k}) \in \mathcal{D}(G; P_k, S_k, C_k)\) if \(G\) can be decomposed into \(\frac{e(G)}{k}\) copies of \(P_k\) \((S_k, C_k)\).

Let \(G\) be an even graph, and let \(k\), \(p\), \(q\), and \(r\) be positive integers with \(k \geq 3\), \(|V(G)| \geq k + 1\), and \(k(p+q+r) = e(G)\). If \(k = 4\), by Theorem 1, \(p \geq \left\lceil \frac{k}{2} \right\rceil = 2\) if \(q = 1\), and hence \(CD(G; P_4, S_4, C_4) \subset \{ (p, q, r) : p, q, r > 0, p + q + r = \frac{e(G)}{4} \}\).
\[(p,q) \neq (1,1)\}. \text{ Note that both } CD(G; P_4, S_4, C_4) \text{ and } \{(p,q,r) : p,q,r > 0, p + q + r = \frac{e(G)}{4}, (p,q) \neq (1,1)\} \text{ are empty if } e(G) \text{ is not divisible by } 4. \text{ If we can prove that } CD(G; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0, p + q + r = \frac{e(G)}{4}, (p,q) \neq (1,1)\}, \text{ then } CD(G; P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0, p + q + r = \frac{e(G)}{4}, (p,q) \neq (1,1)\}, \text{ and hence we determine the set of triples } (p,q,r) \text{ of positive integers for which there exists a } \{P_4^b, S_4^b, C_4^b\}-\text{decomposition of } G.\]

If \(X_1, \ldots, X_n\) are \(n\) sets of triples of non-negative integers, then \(X_1 + \cdots + X_n\) denotes the set \(\{(p_1, q_1, r_1) + \cdots + (p_n, q_n, r_n) : (p_1, q_1, r_1) \in X_1, \ldots, (p_n, q_n, r_n) \in X_n\}\). The following two lemmas will be used for proving the main theorems.

**Lemma 2.** Let \(n, l, s\) be positive integers, and let \(X\) and \(Y\) be sets of triples of non-negative integers such that \(X \supset \{(p,q,r) : p,q,r > 0, p + q + r = s, (p,q) \neq (1,1)\}\) and \(Y \supset \{(al,bl,cl) : a,b,c \geq 0, a + b + c = n\}\). If \(l \geq 2\) and \(s \geq 3l\), then \(X + Y \supset \{(p,q,r) : p,q,r > 0, p + q + r = s + nl, (p,q) \neq (1,1)\}\).

**Proof.** Let \((p^*, q^*, r^*)\) be a triple of positive integers such that \(p^* + q^* + r^* = s + nl\) and \((p^*, q^*) \neq (1,1)\). Clearly, \((p^*, q^*, r^*) = (al+p^*, b'l+q^*, \gamma l+r^*)\) with \(1 \leq p', q', r' \leq l\) and \(\alpha, \beta, \gamma \geq 0\). It is not difficult to check that \(s = s' + nl\) where \(s' = p' + q' + r' \leq 3l \leq s\) and \(n' \geq 0\). Let \((al+p', b'l+q', \gamma l+r') = (\alpha' l+p', \beta' l+q', \gamma' l+r') + ((\alpha - \alpha') l, (\beta - \beta') l, (\gamma - \gamma') l)\), where \(\alpha' = \min\{\alpha, n'\}, \beta' = \min\{\beta, n' - \alpha'\}, \text{ and } \gamma' = n' - \alpha' - \beta'\). Clearly, \((\alpha' l+p') + (\beta' l+q') + (\gamma' l+r') = s\) and \((\alpha - \alpha') l, (\beta - \beta') l, (\gamma - \gamma') l) \in Y\).

It is left to show that \((\alpha' l+p', \beta' l+q') \neq (1,1)\). Assume for a contradiction that \(\alpha' l+p' = \beta' l+q' = 1\). It follows that \(p' = q' = 1\) and \(\alpha' = \beta' = 0\). Therefore, either \(n' = 0\) or \(\alpha = \beta = 0\). If \(n' = 0\), then \(s = s' = 3 + r' \leq 2 + l \leq 2 + \frac{s}{3}\), hence \(s \leq 3\) which is a contradiction since \(s \geq 6\). If \(\alpha = \beta = 0\), then \((p^*, q^*) = (p', q') = (1,1)\) which contradicts our assumption. Hence \((\alpha' l+p', \beta' l+q') \neq (1,1)\), thus \((p^*, q^*, r^*) \in X + Y\).

**Lemma 3.** Let \(s_1\) and \(s_2\) be positive integers with \(s_1, s_2 \geq 9\) and let \(X_1\) and \(X_2\) be sets of triples of non-negative integers such that \(X_1 \supset \{(a,b,c) : a,b,c \geq 0, a+b+c = s_1, (a,b,c) \neq (1,1,c), (1,0,c), (0,1,c) \text{ when } c \geq 1\}\) and \(X_2 \supset \{(p,q,r) : p,q,r > 0, p + q + r = s_2, (p,q) \neq (1,1)\}\). Then \(X_1 + X_2 \supset \{(p,q,r) : p,q,r > 0, p + q + r = s_1 + s_2, (p,q) \neq (1,1)\}\).

**Proof.** Let \((p^*, q^*, r^*)\) be a triple of positive integers such that \(p^* + q^* + r^* = s_1 + s_2\) and \((p^*, q^*) \neq (1,1)\). We consider three cases as follows.

**Case 1.** \(p^*, q^* \geq 3\). If \(r^* \geq s_2 - 3\), then let \((p^*, q^*, r^*) = (p^* - 1, q^* - 2, r^* - (s_2 - 3)) + (1,2,s_2 - 3)\). Clearly, \((p^* - 1, q^* - 2, r^* - (s_2 - 3)) \in X_1\) and \((1,2,s_2 - 3) \in X_2\). If \(r^* \leq s_2 - 4\), then \(p^* + q^* \geq s_1 + 4\). Since \(p^*, q^* \geq 3\) with \(p^* + q^* \geq s_1 + 4\), there exist positive integers \(p_1^*, p_2^*, q_1^*, q_2^*\) with \(p_1^* \geq 1, p_2^* \geq 2, q_1^* \geq 2, \text{ and } q_2^* \geq 1\) such that \(p^* = p_1^* + p_2^*, q^* = q_1^* + q_2^*, p_1^* + q_1^* = s_1, \text{ and}
Let \((p^*, q^*, r^*) = (p_1^*, q_1^*, 0) + \ldots\) be positive integers. \(K_{m,n}\) has a \(C_{2k}\)-decomposition if and only if \(m\) and \(n\) are even, \(k \geq 2\), \(m \geq k\), \(n \geq k\), and \(mn \equiv 0 \pmod{2^k}\).

Case 2. \(p^*, q^* \leq 2\). Let \((p^*, q^*, r^*) = (0, 0, s_1) + (p^*, q^*, r^* - s_1)\). In this case, \(r^* \geq s_1 + s_2 - 4\) and \((p^*, q^*) \neq (1, 1)\). It implies that \((0, 0, s_1) \in X_1\) and \((p^*, q^*, r^* - s_1) \in X_2\). Hence \((p^*, q^*, r^*) \in X_1 + X_2\).

Case 3. Either \(p^* \leq 2\), \(q^* \geq 3\) or \(p^* \geq 3\), \(q^* \leq 2\). Assume \(p^* \leq 2\) and \(q^* \geq 3\). If \(q^* \leq s_2 - 3\), then \(p^* + q^* \leq s_2 - 1\), and hence \(r^* \geq s_1 + 1\). Let \((p^*, q^*, r^*) = (0, 0, s_1) + (p^*, q^*, r^* - s_1)\). Clearly, \((0, 0, s_1) \in X_1\) and \((p^*, q^*, r^* - s_1) \in X_2\).

If \(q^* \geq s_2 - 2\) and \(r^* \geq 6\), then let \((p^*, q^*, r^*) = (0, s_1 - (r^* - 5), r^* - 5) + (p^*, s_2 - (p^* + 5), 5)\). Since \(1 \leq p^* \leq 2\), \(s_1 + s_2 - 2 \leq q^* + r^* \leq s_1 + s_2 - 1\). Moreover, since \(q^* \geq s_2 - 2\), \(r^* \leq s_1 + 1\), and hence \(s_1 - (r^* - 5) \geq 4\). Besides, \(s_2 - (p^* + 5) \geq 2\) since \(s_2 \geq 9\) and \(p^* \leq 2\). It implies that \((0, s_1 - (r^* - 5), r^* - 5) \in X_1\) and \((p^*, s_2 - (p^* + 5), 5) \in X_2\).

If \(q^* \geq s_2 - 2\) and \(r^* \leq 5\), then let \((p^*, q^*, r^*) = (0, s_1, 0) + (p^*, s_2 - (p^* + r^*), r^*)\). Since \(s_2 \geq 9\), \(p^* \leq 2\), and \(r^* \leq 5\), \(s_2 - (p^* + r^*) \geq 2\). Clearly, \((0, s_1, 0) \in X_1\) and \((p^*, s_2 - (p^* + r^*), r^*) \in X_2\). Hence \((p^*, q^*, r^*) \in X_1 + X_2\).

The case where \(p^* \geq 3\) and \(q^* \leq 2\) is similar to the case \(p^* \leq 2\) and \(q^* \geq 3\), therefore we omit its proof.

3. \(\{P_4^p, S_4^q, C_4^r\}\)-Decomposition of \(K_{m,n}\)

In this section we study the \(\{P_4^p, S_4^q, C_4^r\}\)-decomposition of \(K_{m,n}\) when both \(m\) and \(n\) are even. In particular, we prove that \(\mathcal{CD}(K_{m,n}; P_4, S_4, C_4) = \{(p, q, r) : p, q, r > 0; m + n \geq 6; 4(p + q + r) = mn; (p, q) \neq (1, 1); q\text{ is even when } m = 2; (p, q, r) \neq (1, 2, 1)\text{ when } m = n = 4\}\). We first recall three results on \(P_k\)-decomposition, \(S_k\)-decomposition, and \(C_k\)-decomposition of \(K_{m,n}\) as follows.

**Theorem 4** (Parker [28]). Let \(k, m,\) and \(n\) be positive integers. There exists a \(P_k\)-decomposition of \(K_{m,n}\) if and only if \(mn \equiv 0 \pmod{k}\) and one of cases in Table 1 occurs.

**Theorem 5** (Yamamoto et al. [44]). Let \(k, m,\) and \(n\) be positive integers with \(m \leq n\). There exists an \(S_k\)-decomposition of \(K_{m,n}\) if and only if one of the following conditions holds.

1. \(m \geq k\) and \(mn \equiv 0 \pmod{k}\);
2. \(m < k \leq n\) and \(n \equiv 0 \pmod{k}\).

**Theorem 6** (Sotteau [38]). Let \(k, m,\) and \(n\) be positive integers. \(K_{m,n}\) has a \(C_{2k}\)-decomposition if and only if \(m\) and \(n\) are even, \(k \geq 2\), \(m \geq k\), \(n \geq k\), and \(mn \equiv 0 \pmod{2k}\).
Table 1. Necessary and Sufficient Conditions for $P_k$-Decomposition of $K_{m,n}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$k$</th>
<th>$m$</th>
<th>$n$</th>
<th>Characterization</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>even</td>
<td>even</td>
<td>even</td>
<td>$k \leq 2m, k \leq 2n$, not both equalities</td>
</tr>
<tr>
<td>2.</td>
<td>even</td>
<td>even</td>
<td>odd</td>
<td>$k \leq 2m - 2, k \leq 2n$</td>
</tr>
<tr>
<td>3.</td>
<td>even</td>
<td>odd</td>
<td>even</td>
<td>$k \leq 2m, k \leq 2n - 2$</td>
</tr>
<tr>
<td>4.</td>
<td>odd</td>
<td>even</td>
<td>even</td>
<td>$k \leq 2m - 1, k \leq 2n - 1$</td>
</tr>
<tr>
<td>5.</td>
<td>odd</td>
<td>even</td>
<td>odd</td>
<td>$k \leq 2m - 1, k \leq n$</td>
</tr>
<tr>
<td>6.</td>
<td>odd</td>
<td>odd</td>
<td>even</td>
<td>$k \leq m, k \leq 2n - 1$</td>
</tr>
<tr>
<td>7.</td>
<td>odd</td>
<td>odd</td>
<td>odd</td>
<td>$k \leq m, k \leq n$</td>
</tr>
</tbody>
</table>

Before going into more detail, we need the following lemma.

**Lemma 7** ([36, Theorem 2.10]). Let $p$ and $q$ be non-negative integers, and let $k$, $m$, and $s$ be positive integers such that $k$ is even and $m < k$. There exists a decomposition of $K_{sk,m}$ into $p$ copies of $P_k$ and $q$ copies of $S_k$ if and only if $k(p + q) = e(K_{sk,m})$, and there is $t \in \{0, \ldots, s\}$ such that $\lceil \frac{tk}{2} \rceil \leq p \leq tm$.

Let $(x_1, \ldots, x_k)$ and $(x_1, \ldots, x_k, x_1)$ denote, respectively, the $k$-path and the $k$-cycle through vertices $x_1, \ldots, x_k$ in order, and let $(y; x_1, \ldots, x_k)$ denote the $k$-star with center $y$ and leaves $x_1, \ldots, x_k$. An internal vertex of a path is a vertex of degree 2. In the following lemma, we determine the set of ordered pairs $(p, q)$ of positive integers for which there exists a $\{P_1^p, S_4^q\}$-decomposition of $K_{2,2n}$.

**Lemma 8.** Let $n$, $p$, and $q$ be positive integers. $(p, q) \in CD(K_{2,2n}; P_1, S_4)$ if and only if $n \geq 2$, $p + q = n$, and $q$ is even.

**Proof.** Let $n$, $p$, and $q$ be positive integers. Assume that $(p, q) \in CD(K_{2,2n}; P_1, S_4)$. It is easily seen that $n \geq 2$ and $p + q = n$.

Let $D$ be an arbitrary decomposition of $K_{2,2n}$ into $p$ copies of $P_1$ and $q$ copies $S_4$. Let $(A, B)$ be the bipartition of $K_{2,2n}$ where $A = \{a_0, a_1\}$ and $B = \{b_0, b_1, \ldots, b_{2n-1}\}$. It is easily seen that each $S_4$ in $D$ has to center at either $a_0$ or $a_1$, and each $P_1$ in $D$ has to contain both $a_0$ and $a_1$ as its internal vertices. It implies that the number of copies of $S_4$ centered in $a_0$ in $D$ is the same as the number of copies of $S_4$ centered in $a_1$ in $D$, and hence $q$ is even.

Conversely, assume that $n \geq 2$, $p + q = n$ and $q$ is even. If $2n = 4s$ for some integer $s$, by Lemma 7, then $(p, n - p) \in CD(K_{2,2n}; P_1, S_4)$ for each $p \in \{2, 4, \ldots, 2s\}$ (i.e., $q = n - p \in \{2, 4, \ldots, 2s\}$). Assume $2n = 4s + 2$ for some integer $s$. For each $q \in \{2, 4, \ldots, 2(s - 1)\}$, the graph $K_{2,4s+2}$ is the edge-disjoint union of a copy $H_1^q$ of $K_{2,2q}$ and a copy $H_2^q$ of $K_{2,4s-2q+2}$. By Theorem 5, $H_1^q$ is $S_4$-decomposable, and by Theorem 4, $H_2^q$ is $P_4$-decomposable. If $q = 2s$, then let
$K_{2,4s+2}$ decompose into $K_{2,4s-4}$ and $K_{2,6}$. As mentioned above, $K_{2,4s-4}$ can be decomposed into $2s - 2$ copies of $S_4$. Besides, $K_{2,6}$ can be decomposed into one copy of $P_4$ and two copies of $S_4$ as follows: $(b_0, a_1, b_5, a_0, b_4), (a_0; b_0, b_1, b_2, b_3), (a_1; b_1, b_2, b_3, b_4)$.

In the following lemma, we determine the set of triples $(p, q, r)$ of positive integers for which there exists a $\{P_4, S_4^q, C_4^r\}$-decomposition of $K_{2,2n}$.

**Lemma 9.** Let $n$, $p$, $q$, and $r$ be positive integers with $n \geq 3$. $(p, q, r) \in CD(K_{2,2n}; P_4, S_4, C_4)$ if and only if $p + q + r = n$ and $q$ is even.

**Proof.** Let $n$, $p$, $q$, and $r$ be positive integers with $n \geq 3$. Assume that $(p, q, r) \in CD(K_{2,2n}; P_4, S_4, C_4)$. It is easily seen that $p + q + r = n$.

Let $D$ be an arbitrary decomposition of $K_{2,2n}$ into $p$ copies of $P_4$, $q$ copies of $S_4$, and $r$ copies of $C_4$, and let $C(1), \ldots, C(r)$ denote the $r$ copies of $C_4$ in $D$. It is easily seen that $K_{2,2n} - E(C(1) \cup \ldots \cup C(r)) \cong K_{2,2(n-r)}$. It implies that $K_{2,2(n-r)}$ can be decomposed into $p$ copies of $P_4$ and $q$ copies of $S_4$, and hence $q$ is even by Lemma 8.

Conversely, assume that $p + q + r = n$ and $q$ is even. Let $(A, B)$ be the bipartition of $K_{2,2n}$ where $A = \{a_0, a_1\}$ and $B = \{b_0, b_1, \ldots, b_{2n-1}\}$, and let $C(i) = (b_{2i-2}, a_0, b_{2i-1}, a_1, b_{2i-2})$ for each $i \in \{1, \ldots, r\}$. It clear that $C(i)$ is a $C_4$ and $K_{2,2n} - E(C(1) \cup \ldots \cup C(r)) \cong K_{2,2(n-r)}$. By Lemma 8, $K_{2,2(n-r)}$ is $\{P_4, S_4^q, C_4^r\}$-decomposable.

In the following lemma, we determine the set of triples $(p, q, r)$ of positive integers for which there exists a $\{P_4, S_4^q, C_4^r\}$-decomposition of $K_{4,4n}$.

**Lemma 10.** Let $n$, $p$, $q$, and $r$ be positive integers with $n \geq 2$. $(p, q, r) \in CD(K_{4,4n}; P_4, S_4, C_4)$ if and only if $p + q + r = 2n$ and $(p, q) \neq (1, 1); (p, q, r) \neq (1, 2, 1)$.

**Proof.** (Necessity) By Theorem 1, condition $p + q + r = 2n$ and $(p, q) \neq (1, 1)$ holds.

On the contrary, suppose $(1, 2, 1) \in CD(K_{4,4}; P_4, S_4, C_4)$. Let $D$ be an arbitrary decomposition of $K_{4,4}$ into one copy of $P_4$, two copies of $S_4$, and one copy of $C_4$; and let $S(1), S(2)$, and $C$ denote, respectively, the two copies of $S_4$ and the copy of $C_4$ in $D$. It is easily seen that $K_{4,4} - E(S(1) \cup S(2)) \cong K_{2,4}$ and $K_{2,4} - E(C) \cong K_{2,2}$. It follows that $K_{4,4} - E(S(1) \cup S(2) \cup C)$ is not $P_4$-decomposable, a contradiction.

(Sufficiency) By assumption, $CD(K_{4,2n}; P_4, S_4, C_4) \subset \{(p, q, r) : p, q, r > 0, p + q + r = 2n, (p, q) \neq (1, 1); (p, q, r) \neq (1, 2, 1)\}$, and hence $CD(K_{4,4}; P_4, S_4, C_4) \subset \{(2, 1, 1)\}$. Let $(A, B)$ be the bipartition of $K_{4,4}$ where $A = \{a_0, a_1, a_2, a_3\}$ and $B = \{b_0, b_1, b_2, b_3\}$. $K_{4,4}$ can be decomposed into two copies of $P_4$, one copy of $S_4$, and one copy of $C_4$ as follows: $(b_0, a_0, b_1, a_1, b_2), (a_0; b_1, a_2, b_2, a_0, b_3), (a_3; b_0, b_1, b_2, b_3), (b_0, a_1, b_3, a_2, b_0)$. 


Assume \( n = 3 \). We show that \( CD(K_{4,6}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p + q + r = 6, (p, q) \neq (1, 1)\} \) by Theorems 4, 5. We show that \( CD(K_{6,6}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p + q + r = 9, (p, q) \neq (1, 1)\} \) as follows.

We decompose \( K_{4,6} \) into one copy of \( K_{4,4} \) and one copy of \( K_{4,2} \). By Theorems 4, 5, and 6, \( K_{4,2} \) is \( P_4 \)-decomposable, \( S_4 \)-decomposable, and \( C_4 \)-decomposable, respectively. Since \( (2, 1, 1) \in CD(K_{4,4}; P_4, S_4, C_4) \), \( \{(2, 1, 1) + (0, 0, 0), (2, 1, 1) + (0, 2, 0), (2, 1, 1) + (0, 0, 2)\} = \{(4, 1, 1), (2, 3, 1), (2, 1, 3)\} \subset CD(K_{4,6}; P_4, S_4, C_4) \). Besides, it is easy to check that \( K_{4,4} \) is \( \{P_4^2, C_4^2\} \)-decomposable, \( \{P_4^3, C_4^1\} \)-decomposable, and \( \{P_4^3, S_4^1\} \)-decomposable, respectively. Thus \( \{(2, 0, 2) + (0, 2, 0), (3, 0, 1) + (0, 2, 0), (3, 1, 0) + (0, 0, 2)\} = \{(2, 2, 2), (3, 2, 1), (3, 1, 2)\} \subset CD(K_{4,6}; P_4, S_4, C_4) \). We now turn our attention to the case \((1,2,3)\). The graph \( K_{4,6} \) is the edge-disjoint union of two copies of \( K_{2,6} \). By Lemma 8 and Theorem 6, \( K_{2,6} \) is \( \{P_4^1, S_4^2\} \)-decomposable and \( C_4 \)-decomposable, respectively. Thus \((1,2,3) \in CD(K_{4,6}; P_4, S_4, C_4) \). Let \((A, B)\) be the bipartition of \( K_{4,6} \) where \( A = \{a_0, a_1, a_2, a_3\} \) and \( B = \{b_0, b_1, b_2, b_3, b_4, b_5\} \). We now show that \((1,4,1), (1,3,2) \in CD(K_{4,6}; P_4, S_4, C_4) \) as follows: \( (a_0; a_3, b_3, b_5), (a_0; b_0, b_1, b_2, b_3), (a_1; b_1, b_2, b_3, b_4), (a_2; b_1, b_2, b_3, b_4), (a_3; b_1, b_2, b_3, b_5), (b_0, a_1, b_2, b_3, b_4), (b_1, a_1, b_2, b_3, b_4), (b_2, a_0, b_2, b_3, b_4), (b_3, a_0, b_2, b_3, b_4), (b_4, a_0, b_2, b_3, b_4), (b_5, a_0, b_2, b_3, b_4) \) for \((1, 4, 1)\), \((1, 3, 2) \in CD(K_{4,6}; P_4, S_4, C_4) \) as follows: \( (1, 4, 1) \in CD(K_{4,6}; P_4, S_4, C_4) \) by Theorem 1, condition \( p + q + r = 3n \) and \( (p, q) \neq (1, 1) \) holds.

In the following lemma, we determine the set of triples \((p, q, r)\) of positive integers for which there exists a \( \{P_4^p, S_4^q, C_4^r\} \)-decomposition of \( K_{6,2n} \).

**Lemma 11.** Let \( n, p, q, \) and \( r \) be positive integers with \( n \geq 3 \). \((p, q, r) \in CD(K_{6,2n}; P_4, S_4, C_4) \) if and only if \( p + q + r = 3n \) and \((p, q) \neq (1, 1) \).

**Proof.** (Necessity) By Theorem 1, condition \( p + q + r = 3n \) and \((p, q) \neq (1, 1) \) holds.

(Sufficiency) Assume \( n = 3 \). It is easily seen that \( K_{6,6} \) can be decomposed into one copy of \( K_{4,6} \) and one copy of \( K_{2,6} \). By Lemma 10, \( CD(K_{4,6}; P_4, S_4, C_4) \) \( \supset \{(p, q, r) : p, q, r > 0, p + q + r = 6, (p, q) \neq (1, 1)\} \). By Theorem 4, 6 and Lemma 8, \( \{(3, 0, 0), (0, 0, 3), (1, 2, 0)\} \subset CD(K_{2,6}; P_4, S_4, C_4) \). Besides, \( K_{2,6} - E(C_4) \cong K_{2,4} \), hence \( \{(0, 2, 1) \subset CD(K_{2,6}; P_4, S_4, C_4) \) by Theorems 4, 5. We show that \( CD(K_{6,6}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p + q + r = 9, (p, q) \neq (1, 1)\} \) as follows.
Suppose \( q = 1 \) or \( 2 \). If \( p > r \), then let \( (p,q,r) = (p-3, q, r) + \ldots \) positive even integers with \( n \geq m \geq 8 \). \((p,q,r) \in \mathcal{D}(K_{m,n}; P_4, S_4, C_4)\) if and only if \( 4(p+q+r) = mn \) and \( (p,q) \neq (1,1) \).

Suppose \( q = 3 \), or \( 4 \), or \( 5 \). If \( r \geq 4 \), then let \( (p,q,r) = (p,q,r-3) + \ldots \) \((p,q,r) \in \mathcal{D}(K_{m,n}; P_4, S_4, C_4)\) if and only if \( 4(p+q+r) = mn \) and \( (p,q) \neq (1,1) \).

Suppose \( q = 6 \). In this case \( p + r = 3 \). If \( r = 2 \), then let \( (1,2,0) = (1,4,1) + (0,2,1) \), and if \( r = 1 \), then let \( (2,6,1) = (1,4,1) + (1,2,0) \). Since \( (1,4,1) \in \mathcal{D}(K_{4,6}; P_4, S_4, C_4)\) and \( (0,2,1), (1,2,0) \in \mathcal{D}(K_{2,6}; P_4, S_4, C_4) \), \((p,q,r) \in \mathcal{D}(K_{6,6}; P_4, S_4, C_4)\).

Suppose \( q = 7 \). Let \( (A,B) \) be the bipartition of \( K_{6,6} \) where \( A = \{a_0,a_1,a_2,a_3, a_4, a_5\} \) and \( B = \{b_0,b_1,b_2,b_3,b_4,b_5\} \). We show that \((1,7,1) \in \mathcal{D}(K_{6,6}; P_4, S_4, C_4)\) below: \((b_0,a_5,b_2,a_3,b_3), (a_0; b_0,b_1,b_2,b_3), (a_1; b_0,b_1,b_2,b_3), (a_2; b_0,b_1,b_2,b_3), (a_3; b_0,b_1,b_4,b_5), (a_4; b_0,b_2,b_4,b_5), (b_4; a_0,a_1,a_2,a_3), (b_5; a_0,a_1,a_2,a_5), (b_1,a_4, b_3,a_5,b_1)\).

Assume \( n \geq 4 \). If \( n \) is even, then write \( n = 2k \) for some integer \( k \) with \( k \geq 2 \). We decompose \( K_{6,4k} \) into one copy of \( K_{6,4} \) and one copy of \( K_{6,4(k-1)} \), and then we decompose \( K_{6,4(k-1)} \) into \( 3(k-1) \) copies of \( K_{2,4} \). By Theorems 4, 5 and 6, \( \{2,0,0\}, \{0,2,0\}, \{0,0,2\} \subset \mathcal{D}(K_{4,2}; P_4, S_4, C_4) \), and thus \( \mathcal{D}(K_{6,4(k-1)}; P_4, S_4, C_4) \supset \{(2a,2b,2c): a,b,c \geq 0, a+b+c = 3(k-1)\} \). By Lemma 10, \( \mathcal{D}(K_{6,4}; P_4, S_4, C_4) \supset \{(p,q,r): p,q,r > 0, p+q+r = 6, (p,q) \neq (1,1)\} \), and hence \( \mathcal{D}(K_{6,2n}; P_4, S_4, C_4) \supset \{(p,q,r): p,q,r > 0, p+q+r = 3n, (p,q) \neq (1,1)\} \) by Lemma 2.

If \( n \) is odd, then write \( n = 2k+1 \) for some integer \( k \) with \( k \geq 2 \), and thus \( 2n = 4k+2 = 4(k-1) + 6 \). We decompose \( K_{6,4k+2} \) into one copy of \( K_{6,6} \) and one copy of \( K_{6,4(k-1)} \), and then we decompose \( K_{6,4(k-1)} \) into \( 3(k-1) \) copies of \( K_{2,4} \). As mentioned above, \( \mathcal{D}(K_{6,4(k-1)}; P_4, S_4, C_4) \supset \{(2a,2b,2c): a,b,c \geq 0, a+b+c = 3(k-1)\} \). Since \( \mathcal{D}(K_{6,6}; P_4, S_4, C_4) = \{(p,q,r): p,q,r > 0, p+q+r = 9, (p,q) \neq (1,1)\} \), \( \mathcal{D}(K_{6,2n}; P_4, S_4, C_4) \supset \{(p,q,r): p,q,r > 0, p+q+r = 3n, (p,q) \neq (1,1)\} \), by Lemma 2.

In the following lemma, we determine the set of triples \((p,q,r)\) of positive integers for which there exists a \( \{P_3^p, S_4^q, C_4^r\}\)-decomposition of \( K_{m,n} \) when both \( m \) and \( n \) are positive even integers with \( n \geq m \geq 8 \).

**Lemma 12.** Let \( p, q, \) and \( r \) be positive integers, and let \( m \) and \( n \) be positive even integers with \( n \geq m \geq 8 \). \((p,q,r) \in \mathcal{D}(K_{m,n}; P_4, S_4, C_4)\) if and only if \( 4(p+q+r) = mn \) and \( (p,q) \neq (1,1) \).
Proof. (Necessity) By Theorem 1, condition \(4(p+q+r) = mn\) and \((p,q) \neq (1,1)\) holds.

(Sufficiency) We divided the proof into two cases as follows.

Case 1. \(m \equiv 0 \pmod{4}\). Write \(m = 4k\) for some integer \(k\) with \(k \geq 2\). We decompose \(K_{4k,n}\) into one copy of \(K_{4,n}\) and one copy of \(K_{4(k-1),n}\), and then we decompose \(K_{4(k-1),n}\) into \(\frac{n}{2}(k-1)\) copies of \(K_{4,2}\). By Theorems 4, 5 and 6, \([2,0,0),(0,2,0),(0,0,2)\] \(\subset D(K_{4,2}; P_4, S_4, C_4)\), and thus \(D(K_{4(k-1),n}; P_4, S_4, C_4) \supset\{(2a,2b,2c): a,b,c \geq 0, a+b+c = \frac{n}{2}(k-1)\}\). By Lemma 10, \(CD(K_{4,4}; P_4, S_4, C_4) = \{(p,q,r): p,q,r > 0, p+q+r = n, (p,q) \neq (1,1)\}\), and hence \(CD(K_{m,n}; P_4, S_4, C_4) \supset\{(p,q,r): p,q,r > 0, p+q+r = kn, (p,q) \neq (1,1)\}\) by Lemma 2.

Case 2. \(m \equiv 2 \pmod{4}\). Write \(m = 4k+2 = 4(k-1)+6\) for some integer \(k\) with \(k \geq 2\). We decompose \(K_{4k+2,n}\) into one copy of \(K_{4,n}\) and one copy of \(K_{4(k-1),n}\), and then we decompose \(K_{4(k-1),n}\) into \(\frac{n}{2}(k-1)\) copies of \(K_{4,2}\). As mentioned above, \(D(K_{4(k-1),n}; P_4, S_4, C_4) \supset\{(2a,2b,2c): a,b,c \geq 0, a+b+c = \frac{n}{2}(k-1)\}\). By Lemma 11, \(CD(K_{6,6}; P_4, S_4, C_4) = \{(p,q,r): p,q,r > 0, p+q+r = \frac{6n}{4}, (p,q) \neq (1,1)\}\), and hence \(CD(K_{4k+2,n}; P_4, S_4, C_4) \supset\{(p,q,r): p,q,r > 0, p+q+r = \frac{(4k+2)n}{4}, (p,q) \neq (1,1)\}\) by Lemma 2.

Now, we are ready for the main result of this section. It is obtained by combining Theorem 1 and Lemmas 9, 10, 11, and 12.

Theorem 13. Let \(m, n, p, q,\) and \(r\) be positive integers such that both \(m\) and \(n\) are even, and \(m \leq n\). \((p,q,r) \in CD(K_{m,n}; P_4, S_4, C_4)\) if and only if \(m + n \geq 6; 4(p+q+r) = mn; (p,q) \neq (1,1); q\) is even when \(m = 2; (p,q,r) \neq (1,2,1)\) when \(m = n = 4\).

4. \(\{P_4^p, S_4^q, C_4^r\}\)-decomposition of \(K_n\)

In this section, we study the \(\{P_4^p, S_4^q, C_4^r\}\)-decomposition of \(K_n\) when \(n\) is odd. In particular, we prove that \(CD(K_n; P_4, S_4, C_4) = \{(p,q,r): p,q,r > 0, 4(p+q+r) = \binom{n}{2}; (p,q) \neq (1,1)\}\). Let us begin with three well-known results on \(P_k\)-decomposition, \(S_k\)-decomposition, and \(C_k\)-decomposition of \(K_n\), respectively.

Theorem 14 (Tarsi [40]). Let \(k\) and \(n\) be positive integers. There exists a \(P_k\)-decomposition of \(K_n\) if and only if \(k + 1 \leq n\) and \(n(n-1) \equiv 0 \pmod{2k}\).

Theorem 15 (Tarsi [39] and Yamamoto et al. [44]). Let \(k\) and \(n\) be positive integers. There exists an \(S_k\)-decomposition of \(K_n\) if and only if \(2k \leq n\) and \(n(n-1) \equiv 0 \pmod{2k}\).
Theorem 16 (Alspach, Gavlas [6] and Šajna [31]). Let \( n \) and \( k \) be positive integers. \( K_n \) has a \( C_k \)-decomposition if and only if \( n \) is odd, \( 3 \leq k \leq n \), and \( n(n - 1) \equiv 0 \pmod{2k} \).

In the following, we will introduce three known results on \( \{P_4^p, C_4^r\} \)-decomposition, \( \{S_4^q, C_4^r\} \)-decomposition, and \( \{P_4^p, S_4^q\} \)-decomposition of \( K_n \), respectively.

Theorem 17 [33]. Let \( p \) and \( r \) be positive integers, and let \( n \) be a positive odd integer. \( (p, r) \in CD(K_n; P_4, C_4) \) if and only if \( 4(p + q) = e(K_n) \) and \( p \neq 1 \).

Theorem 18 [35]. Let \( q \) and \( r \) be positive integers, and let \( n \) be a positive odd integer. \( (q, r) \in CD(K_n; S_4, C_4) \) if and only if \( 4(p + q) = e(K_n) \) and \( q \neq 1 \).

Theorem 19 [36]. Let \( p, q, \) and \( n \) be positive integers with \( n \geq 16 \). \( (p, q) \in CD(K_n; P_4, S_4) \) if and only if \( 4(p + q) = e(K_n) \).

Theorem 19 determined the set of ordered pairs \((p, q)\) of positive integers for which there exists a \( \{P_4^p, S_4^q\} \)-decomposition of \( K_n \) when \( n \geq 16 \). In the following lemma, we determine the set of ordered pairs \((p, q)\) of positive integers for which there exists a \( \{P_4^p, S_4^q\} \)-decomposition of \( K_n \) when \( n < 16 \) and \( n \) is odd, thus we determine the set of ordered pairs \((p, q)\) of positive integers for which there exists a \( \{P_4^p, S_4^q\} \)-decomposition of \( K_n \) when \( n \) is odd.

Theorem 20. Let \( p \) and \( q \) be positive integers, and let \( n \) be a positive odd integer. \( (p, q) \in CD(K_n; P_4, S_4) \) if and only if \( 4(p + q) = e(K_n) \).

Proof. (Sufficiency) Condition \( 4(p + q) = e(K_n) \) is trivial.

(Sufficiency) Observe that \( 4 \equiv n(n - 1) \) implies \( 8 \mid (n - 1) \). It follows that \( n = 8m + 1 \) for some positive integer \( m \). By Theorem 19, we need only consider the case \( n = 9 \). Assume \( V(K_9) = \{1, \ldots, 9\} \). We show that \( CD(K_9; P_4, S_4) \supseteq \{(p, q) : p, q > 0, p + q = 9\} \).

Assume \((p, q) = (8, 1)\). \( K_9 \) can be decomposed into 8 copies of \( P_4 \) and one copy of \( S_4 \) as follows: \((3, 1, 9, 2, 4), (7, 5, 9, 6, 8), (4, 3, 2, 1, 5), (5, 2, 6, 1, 4), (5, 4, 6, 3, 7), (7, 4, 8, 3, 5), (1, 7, 2, 8, 5), (5, 6, 7, 8, 1), (9, 3, 4, 7, 8)\).

Assume \((p, q) = (7, 2)\). It is easily seen that \( K_9 \) is the edge-disjoint union of a copy \( H_{2}^g \) of \( K_8 \) and a copy \( H_{2}^f \) of \( S_8 \). By Theorem 14, \( H_{2}^f \) is \( P_4 \)-decomposable, and \( H_{2}^g \) can be decomposed into two copies of \( S_4 \). Hence the assertion follows.

Assume \((p, q) = (6, 3)\). \( K_9 \) can be decomposed into 6 copies of \( P_4 \) and 3 copies of \( S_4 \) as follows: \((5, 1, 3, 2, 6), (6, 3, 4, 2, 5), (3, 7, 4, 5, 6), (6, 4, 8, 5, 3), (6, 9, 5, 7, 1), (1, 8, 2, 7, 6), (1, 2, 4, 6, 9), (8, 3, 6, 7, 9), (9, 2, 3, 4, 7)\).

Assume \((p, q) = (5, 4)\). \( K_9 \) can be decomposed into 5 copies of \( P_4 \) and 4 copies of \( S_4 \) as follows: \((2, 4, 3, 6, 5), (5, 8, 4, 6, 7), (7, 5, 9, 6, 2), (2, 3, 1, 5, 4), (4, 7, 3, 5, 2), (1, 4, 6, 7, 8), (2, 1, 7, 8, 9), (8, 3, 6, 7, 9), (9, 1, 3, 4, 7)\).
Assume \((p,q) = (4,5)\). \(K_9\) can be decomposed into 4 copies of \(P_4\) and 5 copies of \(S_4\) as follows: 
\((5, 1, 3, 2, 6), (6, 3, 4, 2, 5), (7, 3, 5, 4, 8), (8, 5, 6, 4, 7), (1, 4, 6, 8, 9), (2, 1, 7, 8, 9), (7, 1, 5, 6, 9), (8, 3, 6, 7, 9), (9, 3, 4, 5, 6)\).

Assume \((p,q) = (3,6)\). \(K_9\) can be decomposed into 3 copies of \(P_4\) and 6 copies of \(S_4\) as follows: 
\((4,1,2,3,9),(9,4,3,8,7),(7,6,5,8,4),(1,3,5,7,9),(2,4,6,7,9),(5,2,3,4,9),(6,1,3,4,9),(7,3,4,5,9),(8,1,2,6,9)\).

Assume \((p,q) = (2,7)\). \(K_9\) can be decomposed into 2 copies of \(P_4\) and 7 copies of \(S_4\) as follows: 
\((1,2,3,4,5),(5,6,7,8,9),(1,3,4,8,9),(2,4,5,8,9),(3,6,7,8,9),(4,6,7,8,9),(5,1,3,8,9),(6,1,2,8,9),(7,1,2,5,9)\).

Assume \((p,q) = (1,8)\). \(K_9\) can be decomposed into one copy of \(P_4\) and 8 copies of \(S_4\) as follows: 
\((4,5,6,7,8),(1,2,3,4,5),(2,3,4,5,6),(3,4,6,7,8),(5,3,7,8,9),(6,1,4,8,9),(7,1,2,4,9),(8,1,2,4,9),(9,1,2,3,4)\).

The following lemma gives sufficient conditions for decomposing an edge-disjoint union of cycles of length \(k\) into copies of \(P_k\). In fact, the proof of the following lemma is essentially given in [33, Lemma 3.8]. We present it here for completeness.

**Lemma 21.** Let \(k\) and \(n\) be integers such that \(k \geq 3\) and \(n \geq 2\). For each \(i \in \{1,2,\ldots,n\}\), let \(C(i)\) denote the cycle of length \(k\), \((x_{i,1}, x_{i,2}, \ldots, x_{i,k}, x_{i,1})\). If \(x_{i,1} = x_{i,2} = \cdots = x_{i,n,1}, x_{i,n} \notin V(C(i))\) for each \(i \in \{2,3,\ldots,n\}\), and \(x_{i,n} \notin V(C(1))\), then \(\bigcup_{i=1}^{n} C(i)\) can be decomposed into \(n\) paths of length \(k\).

**Proof.** By assumptions, \(\bigcup_{i=1}^{n} C(i)\) can be decomposed into \(n\) paths of length \(k\) as follows: 
\((x_{2,2}, x_{2,3}, \ldots, x_{2,k}), (x_{1,2}, x_{1,1}), (x_{3,2}, x_{3,3}, \ldots, x_{3,k}), (x_{1,2}, x_{1,1}), \ldots, (x_{2,2}, x_{2,3}, \ldots, x_{2,k}), (x_{n,2}, x_{n,3}, \ldots, x_{n,k}), (x_{n,n}, x_{n-1,2})\).

In the following lemma, we determine the set of triples \((p,q,r)\) of positive integers for which there exists a \(\{P_4^p, S_4^q, C_4^r\}\)-decomposition of \(K_9\).

**Lemma 22.** Let \(p, q,\) and \(r\) be positive integers. \((p,q,r) \in CD(K_9; P_4, S_4, C_4)\) if and only if \(p+q+r = 9\) and \((p,q) \neq (1,1)\).

**Proof.** (Necessity) The assertion follows immediately from Theorem 1.

(Sufficiency) Let \(V(K_9) = \{1, \ldots, 9\}\). We split the proof into 7 cases according to the value of \(q\).

Assume \(q = 1\) (note that \(p \geq 2\) in this case). \(K_9\) can be decomposed into two copies of \(P_4\), one copy of \(S_4\), and 6 copies of \(C_4\) as follows: 
\((3,1,9,2,4),(7,5,9,6,8),(9,3,4,7,8), C(1) = (1,4,3,2,1), C(2) = (1,5,2,6,1), C(3) = (3,5,4,6,3), C(4) = (3,7,4,8,3), C(5) = (8,1,7,2,8), C(6) = (8,5,6,7,8)\).

Since \(1 \in V(C(1)) \cap V(C(2)), \quad 5 \notin V(C(1)), \quad \text{and} \quad 4 \notin V(C(2))\), \(C(1) \cup C(2)\) can be decomposed into two copies of \(P_4\), by Lemma 21. By the same argument, \(C(3) \cup C(4)\) and \(C(5) \cup C(6)\) can also be decomposed into two copies of \(P_4\). Hence \(CD(K_9; P_4, S_4, C_4) \supset \{(p,1,8-p) : 2 \leq p \leq 7 \text{ and } p \text{ is even}\}\).
On the other hand, $K_9$ can be decomposed into three copies of $P_4$, one copy of $S_4$, and 5 copies of $C_4$ as follows: $(3, 1, 9, 2, 4)$, $(9, 5, 8, 6, 7)$, $(9, 8, 7, 5, 6)$, $(9; 3, 4, 6, 7)$, $C(1) = (1, 4, 3, 2, 1)$, $C(2) = (1, 5, 2, 6, 1)$, $C(3) = (1, 7, 2, 8, 1)$, $C(4) = (3, 5, 4, 6, 3)$, $C(5) = (3, 7, 4, 8, 3)$. By the same argument mentioned above, $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of $P_4$. Thus $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 1, 8 - p) : 2 \leq p \leq 7 \text{ and } p \text{ is odd}\}.

Assume $q = 2$. $K_9$ can be decomposed into two copies of $S_4$, and 7 copies of $C_4$ as follows: $(1,3,4,8,9)$, $(2,3,4,8,9)$, $C(1) = (3,7,1,6,3)$, $C(2) = (3,5,6,4,3)$, $C(3) = (3,8,4,9,3)$, $C(4) = (2,7,5,1,2)$, $C(5) = (2,6,9,5,2)$, $C(6) = (7,6,8,9,7)$, $C(7) = (7,4,5,8,7)$. Since $3 \in V(C(1)) \cap V(C(2)) \cap V(C(3))$, $8 \notin V(C(1))$, $7 \notin V(C(2))$, and $5 \notin V(C(3))$, $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of $P_4$, by Lemma 21. By the same argument, $C(1) \cup C(2)$, $C(4) \cup C(5)$, and $C(6) \cup C(7)$ can also be decomposed into two copies of $P_4$. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 2, 7 - p) : p = 2, \ldots, 6\}$. Besides, $K_9$ can also be decomposed into one copy of $P_4$, two copies of $S_4$, and 6 copies of $C_4$ as follows: $(4,1,7,3,6)$, $(1,3,6,8,9)$, $(2,3,4,8,9)$, $(3,5,6,4,3)$, $(3,8,4,9,3)$, $(2,7,5,1,2)$, $(2,6,9,5,2)$, $(7,6,8,9,7)$. Thus $(1,2,6) \in \mathcal{CD}(K_9; P_4, S_4, C_4).

Assume $q = 3$. $K_9$ can be decomposed into three copies of $S_4$, and 6 copies of $C_4$ as follows: $(1,2,4,6,9)$, $(8,3,6,7,9)$, $(9,2,3,4,7)$, $C(1) = (2,5,1,3,2)$, $C(2) = (2,6,3,4,2)$, $C(3) = (4,7,3,5,4)$, $C(4) = (4,8,5,6,4)$, $C(5) = (7,6,9,5,7)$, $C(6) = (7,1,8,2,7)$. By Lemma 21, both $C(1) \cup C(2)$ and $C(3) \cup C(4)$ can be decomposed into two copies of $P_4$. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 3, 6 - p) : p = 2, 4\}$. Besides, $K_9$ can also be decomposed into one copy of $P_4$, three copies of $S_4$, and 5 copies of $C_4$ as follows: $(8,2,7,1,4)$, $(1,2,6,8,9)$, $(8,3,6,7,9)$, $(9,2,3,4,7)$, $C(1) = (3,1,5,2,3)$, $C(2) = (3,6,2,4,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (5,8,4,6,5)$, $C(5) = (5,9,6,7,5)$. By Lemma 21 again, both $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of $P_4$. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 3, 6 - p) : p = 1, 3, 5\}.

Assume $q = 4$. $K_9$ can be decomposed into 4 copies of $S_4$, and 5 copies of $C_4$ as follows: $(1,4,6,7,8)$, $(2,1,7,8,9)$, $(8,3,6,7,9)$, $(9,1,3,4,7)$, $C(1) = (3,1,5,2,3)$, $C(2) = (3,6,2,4,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (5,8,4,6,5)$, $C(5) = (5,9,6,7,5)$. By Lemma 21, both $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of $P_4$, and $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of $P_4$. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 4, 5 - p) : p = 2, 3, 4\}$. Besides, $K_9$ can also be decomposed into one copy of $P_4$, 4 copies of $S_4$, and 4 copies of $C_4$ as follows: $(3,2,5,1,4)$, $(1,3,6,7,8)$, $(2,1,7,8,9)$, $(8,3,6,7,9)$, $(9,1,3,4,7)$, $(2,6,3,4,2)$, $(3,7,4,5,3)$, $(4,8,5,6,4)$, $(5,9,6,7,5)$. Hence $(1, 4, 4) \in \mathcal{CD}(K_9; P_4, S_4, C_4).

Assume $q = 5$. $K_9$ can be decomposed into 5 copies of $S_4$, and 4 copies of $C_4$ as follows: $(1,4,6,8,9)$, $(2,1,7,8,9)$, $(7,1,5,6,9)$, $(8,3,6,7,9)$, $(9,3,4,5,6)$, $C(1) = (3,4,2,6,3)$, $C(2) = (3,1,5,2,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (4,8,5,$
6, 4). By Lemma 21, $C(1) \cup C(2)$ can be decomposed into two copies of $P_4$, and $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of $P_4$. Hence $CD(K_9; P_4, S_4, C_4) \supset \{(p, 5, 4-p) : p = 2, 3\}$. Besides, $K_9$ can also be decomposed into one copy of $P_4$, 5 copies of $S_4$, and three copies of $C_4$ as follows: $(4, 6, 5, 8, 3), (1; 4, 6, 8, 9), (2; 1, 7, 8, 9), (7; 1, 5, 6, 9), (8; 4, 6, 7, 9), (9; 3, 4, 5, 6), (1; 5, 2, 3, 1), (2, 6, 3, 4, 2), (3, 7, 4, 5, 3)$. Hence $(1, 5, 3) \in CD(K_9; P_4, S_4, C_4)$.

Assume $q = 6$. $K_9$ can be decomposed into two copies of $P_4$, 6 copies of $S_4$, and one copy of $C_4$ as follows: $(4, 9, 3, 8, 7), (7; 6, 5, 8, 4), (1; 3, 5, 7, 9), (2; 4, 6, 7, 9), (5; 2, 3, 4, 9), (6; 1, 3, 4, 9), (7; 3, 4, 5, 9), (8; 1, 2, 6, 9), (1; 2, 3, 4, 1)$. Besides, $K_9$ can also be decomposed into one copy of $P_4$, 6 copies of $S_4$, and two copies of $C_4$ as follows: $(1; 8, 5, 6, 7), (1; 3, 5, 7, 9), (2; 4, 6, 7, 9), (5; 2, 3, 4, 9), (6; 1, 3, 4, 9), (7; 3, 4, 5, 9), (8; 2, 6, 7, 9), (1; 2, 3, 4, 1), (3; 9, 4, 8, 3)$. Thus $CD(K_9; P_4, S_4, C_4) \supset \{(2, 6, 1), (1, 6, 2)\}$.

Assume $q = 7$. $K_9$ can be decomposed into one copy of $P_4$, 7 copies of $S_4$, and one copy of $C_4$ as follows: $(2; 8, 3, 9, 4), (1; 3, 7, 8, 9), (2; 5, 6, 7, 9), (4; 2, 5, 6, 7), (5; 1, 3, 8, 9), (6; 1, 3, 5, 9), (7; 3, 5, 6, 9), (8; 4, 6, 7, 9), (1; 2, 3, 4, 1)$. Thus $CD(K_9; P_4, S_4, C_4) \supset \{(1, 7, 1)\}.

Now, we prove the main result of this section.

**Theorem 23.** Let $p$, $q$, and $r$ be positive integers, and let $n$ be a positive odd integer. $(p, q, r) \in CD(K_n; P_4, S_4, C_4)$ if and only if $4(p + q + r) = \binom{n}{2}$ and $(p, q) \neq (1, 1)$.

**Proof.** *(Necessity)* The assertion follows immediately from Theorem 1.

*(Sufficiency)* Observe that $4 \mid \frac{n(n-1)}{2}$ implies $8 \mid (n - 1)$. It follows that $n = 8m + 1$ for some positive integer $m$. The proof is by induction on $m$. By Lemma 22, the assertion holds for $m = 1$. Assume $m \geq 2$. When $m$ is even, write $m = 2k$ for some integer $k$. It is easily seen that $K_{16k+1}$ can be decomposed into two copies of $K_{8k+1}$ and a copy of $K_{8k,8k}$. By the induction hypotheses, $CD(K_{8k+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p + q + r = k(8k+1), (p, q) \neq (1, 1)\}$. By Theorems 14, 15, 16, 17, 18, and 20, $D(K_{8k+1}; P_4, S_4, C_4) \supset \{(a, b, c) : a, b, c \geq 0 \text{ with at least one of } a, b, c \text{ is 0}, a + b + c = k(8k + 1), (a, b, c) \neq (1, 0, c), (0, 1, c) \text{ when } c \geq 1\}$. Therefore, $D(K_{8k,8k}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p + q + r = 2k(8k + 1), (p, q) \neq (1, 1)\}$. Besides, $K_{8k,8k}$ can be decomposed into $8k^2$ copies of $K_{2,4}$, and by Theorems 4, 5, and 6, $\{(0, 2, 0), (0, 0, 2)\} \subset D(K_{2,4}; P_4, S_4, C_4)$. Hence $D(K_{8k,8k}; P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \geq 0, a + b + c = 8k^2\}$. By Lemma 2, $CD(K_{8k,8k}; K_{8k,8k} \cup K_{8k+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, 4(p + q + r) = (16k+1), (p, q) \neq (1, 1)\}$, that is, $CD(K_{8m+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, 4(p + q + r) = \binom{8m+1}{2}, (p, q) \neq (1, 1)\}$. The proof is complete.
When $m$ is odd, write $m = 2k + 1$ for some integer $k$. It is easily seen that $K_{16k+9}$ can be decomposed into one copy of $K_{8k+1}$, one copy of $K_{8k,8(k+1)}$, and one copy of $K_{8k,9}$. Besides, $K_{8k,8(k+1)}$ can be decomposed into $8k(k + 1)$ copies of $K_{2,4}$. The case where $m = 2k + 1$ is similar to the case $m = 2k$, therefore we omit its proof.

**Remark.** As mentioned on page 3, $\mathcal{D}(K_n; P_4, S_4, C_4)$ denote the set of all triples $(a, b, c)$ of non-negative integers such that a decomposition of $K_n$ into $a$ copies of $P_4$, $b$ copies of $S_4$, and $c$ copies of $C_4$ exists. In fact, when $n$ is odd, all triples in $\mathcal{D}(K_n; P_4, S_4, C_4)$ can be determined by combining Theorems 14, 15, 16, 17, 18, 20 and 23.

For the set $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$, we can also determine all triples in $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$ when both $m$ and $n$ are even. Let $p$, $q$, and $r$ be positive even integers, and let $m$ and $n$ be positive even integers with $m \leq n$. Jeevadoss and Muthusamy [15] showed that $\mathcal{CD}(K_{m,n}; P_4, C_4) = \{(p, r) : m \geq 2$ and $n \geq 4; 4(p + r) = mn$ and $p \neq 1\}$. Besides, we proved that $\mathcal{CD}(K_{m,n}; S_4, C_4) = \{(q, r) : m \geq 2$ and $n \geq 4; 4(q + r) = mn$ and $q \neq 1; q$ is even when $m = 2; r \neq 1$ when $m = 4\}$ and $\mathcal{CD}(K_{m,n}; P_4, S_4) = \{(p, q) : m \geq 2$ and $n \geq 4; 4(p + q) = mn; q$ is even when $m = 2; p \neq 1$ when $m = 4}\}$. Because the proofs are rather lengthy and the arguments are similar to the proofs of Lemmas 8, 9, 10, 11, and 12, we omit the proofs here. Thus all triples in $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$ can be determined by combining Theorems 4, 5, 6, and 13, $\mathcal{CD}(K_{m,n}; P_4, S_4)$, $\mathcal{CD}(K_{m,n}; P_4, C_4)$, and $\mathcal{CD}(K_{m,n}; S_4, C_4)$.

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\{P_4^p, S_4^q, C_4^r\}-decomposition of $K_{m,n}$ and $K_n$


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