DECOMPOSITIONS OF CUBIC TRACEABLE GRAPHS

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Abstract

A traceable graph is a graph with a Hamilton path. The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching. We prove the conjecture for cubic traceable graphs.

Keywords: decomposition, cubic traceable graph, spanning tree, matching, 2-regular graph.

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1. Introduction

In the paper all graphs are finite and simple. The reader can refer to [3, 18] for concepts not defined here. A graph $G$ is cubic if every vertex in $G$ is of degree 3. A spanning tree of $G$ is an acyclic connected subgraph containing all vertices of $G$. A graph that consists of pairwise disjoint edges is called a matching. A $k$-regular spanning subgraph of $G$ is called a $k$-factor. A 1-factor of $G$ is also called a perfect matching. An edge $e$ of $G$ is called a chord of a cycle $C$ in $G$ if the two endpoints of $e$ are on $C$ but $e$ is not itself an edge of $C$. A cycle $C$ is separating in a cubic graph $G$ if either $C$ has a chord, or $G - V(C)$ is disconnected; otherwise, non-separating. A Hamilton cycle is a cycle in $G$ containing all vertices of $G$. A graph with a Hamilton cycle is called a Hamiltonian graph. A Hamilton path...
is a path in $G$ containing all vertices of $G$. A graph with a Hamilton path is
called a traceable graph. Assume that $H$ is a Hamilton path in $G$. Each edge $e \in E(G) \setminus E(H)$ is called a chord of $H$. For every chord $e = vu$ of $H$, there
exists a unique cycle $C_e$ consisting of $e$ and the subpath $vHu$. We call $C_e$ the
associated cycle of $e$. A chord $e = st$ of $H$ is minimal if there is no other chord
of $H$ whose two endpoints are on the subpath $sHt$.

A decomposition of a graph $G$ consists of pairwise edge-disjoint subgraphs
whose union is $G$. It is a canonical problem in structural graph theory to decom-
pose cubic graphs into subgraphs with certain properties. Such a problem can be
traced back to the Petersen Theorem [16] that every bridgeless cubic graph has a
1-factor, which implies that each bridgeless cubic graph can be decomposed into
a 1-factor and a 2-factor. The Vizing Theorem [17] on proper edge-coloring shows
that every cubic graph admits a decomposition consisting of four matchings.

Decompositions of cubic graphs into paths are related to the Fan-Raspaud
conjecture [9] that every 2-edge-connected cubic graph contains three perfect
matchings with empty intersection. It is interesting to decompose a cubic graph
into a spanning tree and other subgraphs. Malkevitch [14] asked which cubic
graphs admit a decomposition into a spanning tree and a 2-regular subgraph, that
is, a decomposition with a HIST (a homeomorphically irreducible spanning tree
is a spanning tree without a 2-degree vertex). Many researchers characterized
graphs with a HIST (see [1, 2, 5, 6, 7]). Douglas [8] proved that it is NP-
complete to decide whether a graph with maximum degree 3 contains a HIST,
which positively solves the problem presented by Albertson, Berman, Hutchinson
and Thomassen [2]. It is clear that the complete graph $K_4$ can be decomposed
into a HIST (a star) and a 2-regular subgraph (a triangle) while the cube $Q_3$ has
no HIST. However, we can decompose $Q_3$ into a spanning tree (with two 2-degree
vertices), a 2-regular subgraph (a 4-cycle) and a matching (an edge). See Figure
1. Relating the restriction that the spanning tree does not contain a vertex of
degree 2, Hoffmann-Ostenhof presented the following conjecture.

![Figure 1](image-url)

**Figure 1.** A decomposition of $K_4$ with a star (thin line) and a triangle (dot line) in (a)
while a decomposition of $Q_3$ with a spanning tree (thin line), a 4-cycle (dot line) and a
matching (thick line) in (b).
Conjecture 1 (3-Decomposition Conjecture). Every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching.

Conjecture 1 was first posed in [10] (see also [4, Problem BCC 22.12] and [13]). Ozeki and Ye [15] showed that Conjecture 1 holds for 3-connected cubic graphs on the plane and the projective plane. Hoffmann-Ostenhof, Kaiser and Ozeki [12] proved that Conjecture 1 holds for all connected planar cubic graphs. In [1, 11] it was proved that a cubic Hamiltonian graph admits such a desired decomposition. It was informed that Ye [19] showed Conjecture 1 for 3-connected cubic graphs on the Klein bottle and the torus. In the paper, we prove Conjecture 1 for traceable cubic graphs.

Theorem 2. Every traceable cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching.

The proof of Theorem 2 consists of four cases (see Section 2). The first case discusses cubic Hamiltonian graphs. The second and third cases are more extensive analyses than the first case. A new technique is used to deal with the fourth case.

2. Proof of Theorem 2

Assume that $G$ is a cubic graph with a Hamilton path $H$. Let the vertices $v_1$ and $v_6$ be the two endpoints of $H$. Then $v_1$ and $v_6$ are incident with two chords of $H$, every other vertex on $H$ is incident with only one chord. If $v_1$ is adjacent to $v_6$ by a chord of $H$, then let the vertex $v_2$ be a neighbor of $v_1$ and $v_5$ be a neighbor of $v_6$ such that the two pairs of vertices are jointed by chords of $H$, respectively. Otherwise, let the vertices $v_2, v_3$ be two neighbors of $v_1$ jointed by chords of $H$ such that these vertices are ordered as $v_1, v_2, v_3$ on $H$, and let the vertices $v_4, v_5$ be two neighbors of $v_6$ jointed by chords of $H$ with the order as $v_4, v_5, v_6$ on $H$.

Lemma 3. Assume that $C$ is a 2-regular non-separating subgraph of $G$ that is the union of associated cycles of chords of $H$, and assume that each of $v_1$ and $v_6$ is joined by a chord of $H$ to at least one vertex of $V(C) \cup \{v_1, v_6\}$. Then there is a decomposition of $G - E(C)$ into a spanning tree of $G$ and a matching.

Proof. Since $C$ is a 2-regular non-separating subgraph of $G$, $G - E(C)$ is connected and has a spanning tree. Let $T$ be a spanning tree of $G - E(C)$ that contains the forest $H - E(C)$, and let $M$ be the subgraph of $G$ induced by $E(G - E(C \cup T))$. Then $M$ is a matching of $G - E(C)$. Thus $G - E(C)$ admits a decomposition consisting of the spanning tree $T$ and the matching $M$. □
Proof of Theorem 2. Let $G$ and $H$ be defined as above. Considering the symmetry of the position of the vertex $v_i$ ($i = 1, 2, \ldots, 6$) on $H$, we have the following four cases.

Case 1. $v_1$ is adjacent to $v_6$ by a chord of $H$.

Case 2. $v_4$ is on the subpath $v_1Hv_2$.

Case 3. $v_4$ is on the subpath $v_2Hv_3$.

Case 4. $v_4$ is on the subpath $v_3Hv_5$.

It is sufficient to show that each case admits a desired decomposition of $G$. See Figure 2.

Case 1. $v_1$ is adjacent to $v_6$ by a chord of $H$. In this case, $G$ is a Hamiltonian cubic graph. For completeness we give a proof similar to [1, 11].

Since $G$ is a simple cubic graph, there are other chords of $H$ besides the chord $v_1v_6$. Then there exists a minimal chord $e$ of $H$. Let $C_e$ be the associated cycle of $e$. Then $C_e$ is a non-separating cycle. From Lemma 3, $G - E(C_e)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. So there is a decomposition of $G$ with the 2-regular subgraph $C_e$, the spanning tree $T$ and the matching $M$.

Case 2. $v_4$ is on the subpath $v_1Hv_2$. Let $C_1^2 = v_1Hv_4v_6Hv_3v_1$ and $C_2^2 = v_1v_2Hv_3v_1$ be the cycles (see (2) of Figure 2).
Suppose that $C^2_1$ is a non-separating cycle of $G$. From Lemma 3, $G - E(C^2_1)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. Thus we can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C^2_1$ and the matching $M$. Otherwise, $C^2_1$ is a separating cycle of $G$. Then there is at least one chord of $C^2_1$ (and of $H$ also) locating on the subpath $v_1Hv_4$, locating on the subpath $v_3Hv_6$, or linking the subpaths $v_1Hv_4$ and $v_3Hv_6$.

Further suppose that $C^2_2$ is a non-separating cycle of $G$. Let $M$ be a set of all chords of $H$ whose two endpoints are not both on $C^2_2$ except the chord $v_4v_6$, and let $T = G - E(C^2_2) - M$. Then $M$ and $T$ are a matching and a spanning tree of $G$ respectively. $T \cup M \cup C^2_2$ forms a desired decomposition of $G$. Otherwise, $C^2_2$ is a separating cycle of $G$. Then there is at least one chord of $C^2_2$ on the subpath $v_2Hv_3$. Now, we discuss three subcases as follows.

Subcase 2.1. There is at least one chord of $C^2_1$ on the subpath $v_1Hv_4$. Since there is at least one chord of $C^2_1$ on the subpath $v_1Hv_4$, we can pick a minimal chord $e_1 = u_1u_2$ of $H$ such that the right endpoint $u_2$ of $e_1$ is the closest to the vertex $v_4$ among all minimal chords of $H$ on the subpath $v_1Hv_4$. Let $C_{e_1}$ be the associated cycle of $e_1$. Similarly, since there is at least one chord of $C^2_2$ on the subpath $v_2Hv_3$, we have a minimal chord $e_2 = v_3u_4$ of $H$ such that the left endpoint $u_3$ of $e_2$ is the closest to the vertex $v_2$ among all minimal chords of $H$ on the subpath $v_2Hv_3$. Let $C_{e_2}$ be the associated cycle of $e_2$. Suppose that there is no chord of $H$ which links $C_{e_1}$ and $C_{e_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_1}$ and on $C_{e_2}$ except the chords $v_1v_3$ and $v_4v_6$. Thus $M$ becomes a matching of $G$. Let $T = G - E(C_{e_1} \cup C_{e_2}) - E(M)$. Then $T$ is a spanning tree of $G$. We can give a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_1} \cup C_{e_2}$, and the matching $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{e_1}$ and $C_{e_2}$. Let $e_3 = u_5u_6$ be such a chord of $H$, and let $C_{e_3}$ be the associated cycle of $e_3$. Suppose that $C_{e_3}$ is a non-separating cycle of $G$. From Lemma 3, $G - E(C_{e_3})$ has a decomposition consisting of a spanning tree $T$ and a matching $M$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{e_3}$ and the matching $M$. Otherwise, $C_{e_3}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u_5Hu_6$ other than $e_3$. So there must be a minimal chord of $H$ on the subpath $u_5Hu_6$.

Let $e_4$ be a minimal chord of $H$ on the subpath $u_5Hu_6$, and let $C_{e_4}$ be the associated cycle of $e_4$. If the vertex $v_4$ is on $C_{e_4}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_4}$ except the chord $v_1v_3$. Let $T = G - E(C_{e_4}) - E(M)$. So we obtain a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_4}$, and the matching $M$. If the vertex $v_2$ is on $C_{e_4}$ and the vertex $v_4$ not on $C_{e_4}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_4}$ except the chord $v_4v_6$. Let $T = G - E(C_{e_4}) - E(M)$. Thus we can decompose $G$ into the spanning tree $T$,
the 2-regular subgraph $C_{e_4}$, and the matching $M$. See Figure 3.

![Figure 3](image-url)  

Figure 3. $v_2$ is on $C_{e_4}$ and $v_4$ not on $C_{e_4}$.

So we suppose that neither $v_2$ nor $v_4$ is on $C_{e_4}$. According to the choices of $e_1$ and $e_2$, we deduce that $e_4$ must locate on the subpath $v_4Hv_2$. Thus there is at least one minimal chord on the subpath $v_4Hv_2$ (for example, the minimal chord $e_4$). We pick up a minimal chord, denoted by $e_4^*$, on the subpath $v_4Hv_2$ such that the right endpoint $u^*$ of $e_4^*$ is the closest to the vertex $v_2$ among all minimal chords of $H$ on the subpath $v_4Hv_2$. Let $C_{e_4'}$ be the associated cycle of $e_4^*$. Further suppose that there is no chord of $H$ which links $C_{e_4'}$ and $C_{e_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_4'}$ and on $C_{e_2}$ except the chords $v_1v_2$ and $v_4v_6$. Let $T = G - E(C_{e_4'} \cup C_{e_2}) - E(M)$. So we obtain a desired decomposition of $G$ with $T$, $C_{e_4'} \cup C_{e_2}$, and $M$. Otherwise, there is at least one chord of $H$ which links $C_{e_4'}$ and $C_{e_2}$. Since neither the subpath $u^*Hv_2$ nor the subpath $v_2Hv_3$ has any chord, there must exist a minimal chord $e_5$ of $H$ such that its associated cycle $C_{e_5}$ contains the vertex $v_2$. We can employ the same means to get a desired decomposition of $G$ as the case that $v_2$ is on $C_{e_4}$ and $v_4$ not on $C_{e_4}$. See Figure 4.

**Subcase 2.2.** There is at least one chord of $C_2^2$ on the subpath $v_3Hv_6$. Since there is at least one chord of $C_2^2$ on the subpath $v_3Hv_6$, we choose a minimal chord $e_1' = u'_1u'_2$ of $H$ such that the left endpoint $u'_1$ of $e_1'$ is the closest to the vertex $v_3$ among all minimal chords of $H$ on the subpath $v_3Hv_6$. Let $C_{e_1'}$ be the associated cycle of $e_1'$. Similarly, since there is at least one chord of $C_2^2$ on the subpath $v_2Hv_3$, there exists a minimal chord $e_2' = u'_3u'_4$ of $H$ such that the right endpoint $u'_3$ of $e_2'$ is the closest to the vertex $v_3$ among all minimal chords of $H$ on the subpath $v_2Hv_3$. Let $C_{e_2'}$ be the associated cycle of $e_2'$. Suppose that there is no chord of $H$ which links $C_{e_1'}$ and $C_{e_2'}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_1'}$ and on $C_{e_2'}$ except the chords $v_1v_3$ and $v_4v_6$. Thus $M$ becomes a matching of $G$. Let $T = G - E(C_{e_1'} \cup C_{e_2'}) - E(M)$. Then $T$ is a spanning tree of $G$. We obtain a desired decomposition of $G$ with
T, $C_{e'_1} \cup C_{e'_2}$, and $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{e'_1}$ and $C_{e'_2}$. Let $e'_3 = u'_6u'_6$ be such a chord of $H$, and let $C_{e'_3}$ be the associated cycle of $e'_3$. Suppose that $C_{e'_3}$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e'_3}$ except the chord $v_4v_6$, and let $T = G - E(C_{e'_3}) - E(M)$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{e'_4}$ and the matching $M$. Otherwise, $C_{e'_3}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u'_6Hu'_6$ other than $e'_3$. So there must be a minimal chord of $H$ on the subpath $u'_6Hu'_6$. Let $e'_4$ be a minimal chord of $H$ on the subpath $u'_6Hu'_6$, and let $C_{e'_4}$ be the associated cycle of $e'_4$. According to the definitions of $e'_4$ and $e'_2$, we deduce that $e'_4$ is incident with the vertex $v_3$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e'_4}$ except the chord $v_4v_6$, and let $T = G - E(C_{e'_4}) - E(M)$. So $G$ has the decomposition with the spanning tree $T$, 2-regular subgraph $C_{e'_4}$ and the matching $M$.

Subcase 2.3. There exists at least one chord of $C_3^2$ which links the subpaths $v_1Hu_4$ and $v_3Hu_6$. From Subcase 2.1 and Subcase 2.2, we only need to consider that neither the subpath $v_1Hu_4$ nor the subpath $v_3Hu_6$ has any chord of $C_3^2$ in the subcase. Since there exists at least one chord of $C_3^2$ which links the subpaths $v_1Hu_4$ and $v_3Hu_6$, we can pick a chord $e_6 = u_7u_8$ whose left endpoint $u_7$ is the closest to the vertex $v_1$ among all chords of $C_3^2$ which link the subpaths $v_1Hu_4$ and $v_3Hu_6$. Since neither the subpath $v_1Hu_4$ nor the subpath $v_3Hu_6$ has any chord of $C_3^2$, so do the subpaths $v_1Hu_7$ and $v_3Hu_8$. Then, we can deduce the cycle $C_3^2 = v_1Hu_7u_8Hv_3u_1$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3^2$ except the chord $v_4v_6$. Let $T = G - E(C_3^2) - E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. So we get a desired decomposition of $G$ with $T$, $C_3^2$, and $M$, see Figure 5.
Case 3. $v_4$ is on the subpath $v_2Hv_3$. Suppose that there exists a minimal chord $f$ of $H$ on the subpath $v_1Hv_3$ such that its associated cycle $C_f$ contains the vertex $v_4$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_f$ except the chord $v_1v_3$, and let $T = G - E(C_f) - E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we have a desired decomposition of $G$ with $T$, $C_f$ and $M$. Otherwise it is sufficient to consider that (3.0) the associated cycle of any minimal chord of $H$ on the subpath $v_1Hv_3$ does not contain $v_4$.

Since there is a chord of $H$ on the subpath $v_1Hv_2$ (for example, the chord $v_1v_2$), we can pick a minimal chord $f_1 = t_1t_2$ of $H$ such that the right endpoint $t_2$ is the closest to the vertex $v_2$ among all minimal chords of $H$ on the subpath $v_1Hv_2$. Note if $f_1$ is the chord $v_1v_2$, then let $t_1 = v_i$ ($i = 1, 2$). Let $C_{f_1}$ be the associated cycle of $f_1$. Similar to the subpath $v_5Hv_6$, we can pick a minimal chord $f_2 = t_3t_4$ of $H$ such that the left endpoint $t_3$ is the closest to the vertex $v_5$ among all minimal chords of $H$ on the subpath $v_5Hv_6$. If $f_2$ is the chord $v_5v_6$, then let $t_3 = v_5$ and $t_4 = v_6$. Let $C_{f_2}$ be the associated cycle of $f_2$. Suppose that there is no chord of $H$ which links the cycles $C_{f_1}$ and $C_{f_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_1}$ and on $C_{f_2}$ except the chords $v_1v_3$ and $v_4v_6$. Then $M$ is a matching of $G$. Let $T = G - E(C_{f_1} \cup C_{f_2}) - E(M)$. $T$ is a spanning tree of $G$. So we can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{f_1} \cup C_{f_2}$, and the matching $M$. Otherwise, there exists at least one chord of $H$ which links $C_{f_1}$ and $C_{f_2}$. We can assume that a chord $f_3 = t_5t_6$ of $H$ links $C_{f_1}$ and $C_{f_2}$ and $t_5$ is the left endpoint of $f_3$. Let $C_3 = v_1v_2Hv_3v_1$. If $C_3$ is a non-separating cycle of $G$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3$ except the chord $f_3$, and $T = G - E(C_3) - E(M)$. It is clear that $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we obtain a desired decomposition of $G$ with $T$, $C_3$ and $M$. Otherwise, $C_3$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $v_2Hv_3$. Let $f_4$ be any minimal chord of $H$ on the subpath $v_2Hv_3$, and let $C_{f_4}$ be
the associated cycle of \( f_4 \). From (3.0), \( C_{f_4} \) does not contain the vertex \( v_4 \).

Suppose that there is not any chord of \( H \) which links the cycles \( C_{f_4} \) and \( C_{f_2} \). Let \( M \) be the set of all chords of \( H \) none of whose endpoints are on \( C_{f_4} \) and on \( C_{f_2} \) except the chords \( v_1v_3 \) and \( v_1v_6 \). Let \( T = G - E( C_{f_4} \cup C_{f_2} ) - E(M) \). Then \( G \) has the desired decomposition \( \{ T, C_{f_4} \cup C_{f_2}, M \} \). Otherwise, there is a chord of \( H \) which links \( C_{f_4} \) and \( C_{f_2} \). Of course, there is at least one chord of \( H \) which links the subpath \( t_5Hv_3 \) and \( C_{f_2} \). Let \( f_5 = t_7t_8 \) be a chord of \( H \) linking the subpath \( t_5Hv_3 \) and \( C_{f_2} \) such that the left endpoint \( t_7 \) is the closest to the vertex \( t_5 \) among all chords of \( H \) linking the subpath \( t_5Hv_3 \) and \( C_{f_2} \). Let \( C_2^3 = t_5Ht_7t_8Ht_6t_5 \). If \( C_2^3 \) is a non-separating cycle of \( G \), then let \( M \) be the set of all chords of \( H \) none of whose two endpoints are on \( C_2^3 \) except the chords \( v_1v_3 \) and \( v_5v_6 \). Let \( T = G - E( C_2^3 ) - E(M) \). So we get a desired decomposition of \( G \) with \( T, C_2^3 \) and \( M \). Otherwise, \( C_2^3 \) is a separating cycle of \( G \). Then there must be at least one chord of \( H \) on the subpath \( t_5Ht_7 \). Let \( f_6 \) be a minimal chord of \( H \) on the subpath \( t_5Ht_7 \), and let \( C_{f_6} \) be the associated cycle of \( f_6 \). From (3.0), we have that \( C_{f_6} \) does not contain the vertex \( v_4 \). According to the choice of \( f_5 \), there is no chord of \( H \) which links \( C_{f_6} \) and \( C_{f_2} \). Let \( M \) be the set of all chords of \( H \) none of whose two endpoints are on \( C_{f_6} \) and on \( C_{f_2} \) except the chords \( v_1v_3 \) and \( v_4v_6 \). Let \( T = G - E( C_{f_6} \cup C_{f_2} ) - E(M) \). Thus we have a desired decomposition of \( G \) with the spanning tree \( T \), the 2-regular subgraph \( C_{f_6} \cup C_{f_2} \), and the matching \( M \), see Figure 6.

![Figure 6](image-url)

**Figure 6.** Case 3 is illustrated.

**Case 4.** \( v_4 \) is on the subpath \( v_3Hv_5 \). Since there are chords of \( H \) on the subpath \( v_1Hv_3 \) (for example, the chords \( v_1v_2 \) and \( v_1v_3 \)), we can choose a minimal chord \( g_1 = s_1s_2 \) of \( H \) on the subpath \( v_1Hv_3 \). If \( g_1 \) is the chord \( v_1v_2 \), then \( s_i = v_i \) \((i = 1, 2)\). Let \( C_{g_1} \) be the associated cycle of \( g_1 \). Similarly, let \( g_2 = s_3s_4 \) be a minimal chord of \( H \) on the subpath \( v_3Hv_6 \). If \( g_2 \) is the chord \( v_5v_6 \), then \( s_3 = v_5 \) and \( s_4 = v_6 \). Let \( C_{g_2} \) be the associated cycle of \( g_2 \). If there is no chord of \( H \)
which links the cycles $C_{g_1}$ and $C_{g_2}$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{g_1}$ and on $C_{g_2}$ except the chords $v_1v_3$ and $v_4v_6$. Let $T = G - E(C_{g_1} \cup C_{g_2}) - E(M)$. Thus we have a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_1} \cup C_{g_2}$ and the matching $M$. Otherwise, we suppose that

(4.0) the associated cycle of any minimal chord of $H$ on the subpath $v_1Hv_3$ is linked by a chord of $H$ with the associated cycle of each minimal chord of $H$ on the subpath $v_4Hv_6$.

Since the subpath $v_1Hv_2$ has at least one chord of $H$, there is a minimal chord $g_3$ of $H$. If the subpath $v_1Hv_2$ only has the chord $v_1v_2$, then $g_3 = v_1v_2$. Let $C_{g_3}$ be the associated cycle of $g_3$. Similarly, there exists a minimal chord $g_4$ of $H$ on the subpath $v_5Hv_6$. If the subpath $v_5Hv_6$ only has the chord $v_5v_6$, then $g_4 = v_5v_6$. Let $C_{g_4}$ be the associated cycle of $g_4$. From (4.0), there is at least one chord of $H$ which links $C_{g_3}$ and $C_{g_4}$. Let $g_5 = s_5s_6$ be such a chord of $H$. We discuss the following two subcases.

**Subcase 4.1.** There are at least two chords of $H$ which link the subpaths $v_1Hv_3$ and $v_4Hv_6$. Let $g_6 = s_7s_8$ be a chord of $H$ linking the subpaths $v_1Hv_3$ and $v_4Hv_6$ different from $g_5$ such that the left endpoint $s_7$ is the closest to the vertex $s_5$ among all chords of $H$ linking such two subpaths. Suppose that the cycle $C_4^1 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_4^1$ except the chords $v_1v_3$ and $v_4v_6$, and let $T = G - E(C_4^1) - E(M)$. Thus $G$ can be decomposed into the spanning tree $T$, the 2-regular subgraph $C_4^1$ and the matching $M$, see Figure 7.

![Figure 7. The cycle $C_4^1 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of $G$ (dot line).](image)

Otherwise, $C_4^1$ is a separating cycle of $G$. From (4.0) and the choice of $g_6$, we can deduce that there exists at least one chord of $H$ on the subpath $s_8Hs_6$. 


Then we pick a minimal chord $g_7$ of $H$ on the subpath $s_8Hs_6$ such that the right endpoint of $g_7$ is the closest to the vertex $s_6$ among all chords of $H$ on the subpath $s_8Hs_6$. Let $C_{g_7}$ be the associated cycle of $g_7$. According to (4.0), there is at least one chord of $H$ which links the cycles $C_{g_3}$ and $C_{g_7}$. Let $g_8 = s_9s_{10}$ be a chord of $H$ linking $C_{g_3}$ and $C_{g_7}$ such that the right endpoint $s_{10}$ is the closest to the vertex $s_6$ among such all chords of $H$. Then, the cycle $C_2^4 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle. Since both $s_5$ and $s_9$ are on the associated cycle $C_{g_3}$ of the minimal chord $g_3$, there is no chord of $H$ on the subpath $s_9Hs_5$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_4^3$ except the chords $v_1v_3$ and $v_4v_6$, and let $T = G - E(C_4^3) - E(M)$. So we have a desired decomposition of $G$ with $T$, $C_4^3$ and $M$, see Figure 8.

![Figure 8](image_url)

**Figure 8.** The cycle $C_4^3 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle of $G$ (dot line).

**Subcase 4.2.** The chord $g_5$ is the only one chord of $H$ which links the subpaths $v_1Hv_3$ and $v_4Hv_6$. According to (4.0), it can be deduced that the vertex $s_5$ locates between two endpoints of any minimal chord of $H$ on the subpath $v_1Hv_3$. If not, there is a minimal chord of $H$ such that its associate cycle is not incident with $s_5$. Then there is a chord of $H$ different from $g_5$ which links the associated cycle of this minimal chord and $C_{g_4}$, contradiction. Further we can obtain that $s_5$ locates between two endpoints of each chord of $H$ on the subpath $v_1Hv_3$.

Only for convenience, we give a drawing of the graph $G$ here. Except that the chord $g_5$ is arranged on one side of $H$, all chords of $H$ are arranged on the other side of $H$. We discuss two cases as follows.

**Subcase 4.2.1.** There exists a chord $g$ of $H$ such that $g$ intersects at least two chords of $H$ on the subpath $v_1Hv_3$. Let $g_9 = s_{11}s_{12}$ and $g_{10} = s_{13}s_{14}$ be two chords of $H$ intersecting $g$ such that the left endpoint $s_{11}$ of $g_9$ is the closest to the left endpoint $s_{13}$ of $g_{10}$ among such all chords of $H$ on the subpath $v_1Hv_3$. Let the cycle $C_3^4 = s_{11}s_{12}Hs_{14}s_{13}Hs_{11}$. Then $C_3^4$ is a non-separating cycle of $G$. If
none of $g_9$ and $g_{10}$ is the chord $v_1v_3$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C^4_3$ and on $C_{g_4}$ except the chords $v_1v_3$, $v_4v_6$, and $g$; otherwise, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C^4_3$ and on $C_{g_4}$ except the chords $v_4v_6$ and $g$. Let $T = G - E(C^4_3 \cup C_{g_4}) - E(M)$. Thus the graph $G$ can be decomposed into the spanning tree $T$, the 2-regular subgraph $C^4_3 \cup C_{g_4}$ and the matching $M$, see Figure 9.

**Figure 9.** (1) $g$ intersects the chords $g_9$ and $g_{10}$ on the subpath $v_1Hv_3$, and none of $g_9$ and $g_{10}$ is the chord $v_1v_3$; (2) $g$ intersects the chords $g_9$ and $g_{10}$ on the subpath $v_1Hv_3$, and one of $g_9$ and $g_{10}$ is the chord $v_1v_3$.

**Subcase 4.2.2.** There is not any chord of $H$ that intersects two chords on the subpath $v_1Hv_3$. Suppose that there is a chord $g^1 = s^1s^2$ of $H$ such that the endpoint $s^1$ is on the subpath $v_1Hs_5$ and the endpoint $s^2$ is on the subpath $v_2Hv_3$ (the former case for short). On the subpath $v_1Hs^2$, we start from the second edge and choose every other edge along the direction from $v_1$ to $s^2$. Otherwise, there is not any chord of $H$ one of which endpoints is on the subpath $v_1Hs_5$ and the other on the subpath $v_2Hv_3$ (the latter case for short). On the subpath $v_1Hv_2$, we start from the second edge and choose every other edge along the direction from $v_1$ to $v_2$. Let $M_0$ be the set of the chosen edges in both cases. Then $M_0$ is a matching of $G$. Let $V$ be the set of vertices on the subpath $v_1Hs^2$ for the former case or the set of vertices on the subpath $v_1Hv_2$ for the latter case. We first prove the following claim.

**Claim.** Let $M_0$, $V$, the former case, and the latter case be defined as above. Then the subgraph $G[V] - E(M_0)$ is a path, where $G[V]$ is a subgraph of $G$ induced by $V$.

**Proof.** Let $G_1 = G[V] - E(M_0)$. Since the vertex $s_5$ locates between the two endpoints of each chord of $H$ on the subpath $v_1Hv_3$, $V$ insists of $s_5$ and the union of the two endpoints of each chord on the subpath $v_1Hs^2$ for the former case or on the subpath $v_1Hv_2$ for the latter case. $|V|$ is odd. Both the subpath $v_1Hs^2$ and the subpath $v_1Hv_2$ have an even number of edges. According to the choice of $M_0$, all vertices of $G_1$ are of degree 2 except two 1-degree vertices $s_5$ and $s^2$ for
Figure 10. (1) the former case that there exists a chord $g^1 = s^1 s^2$ of $H$ such that $s^1$ is on the subpath $v_1 H s_5$ and $s^2$ is on the subpath $v_2 H v_3$; (2) the latter case that there is not any chord of $H$ like $g^1$.

the former case or two 1-degree vertices $s_5$ and $v_2$ for the latter case. It suffices to prove that $G_1$ is connected.

Suppose that $G_1$ is disconnected. The components of $G_1$ consist of one path and some cycles according to the degree condition of $G_1$. Let $C$ be a component of $G_1$ which is a cycle. In $G_1$, $s_5$ is not incident with $C$ since $s_5$ is of degree 1. Let $t_1$ and $t_2$ be two vertices of $C$ such that $t_1$ is the closest to $s_5$ among all vertices of $C$ which locate on the left side of $s_5$ and $t_2$ is the closest to $s_5$ among all vertices of $C$ which locate on the right side of $s_5$. Let the path $P = t_1 H s_5 H t_2$. Then the edges incident with $t_1$ and $t_2$ on $P$ are edges of $M_0$. So $P$ has an odd number of edges and an even number of vertices according to the choice of $M_0$. We can deduce that there is a chord $g^*$ of $H$ such that it only has one endpoint on $P$. The endpoint of $g^*$ not on $P$ can not be on $C$ according to the choice of $M_0$. Then $g^*$ intersects at least two edges of $C$ which are chords of $H$ on the subpath $v_1 H v_3$, contradiction with assumptions in Subcase 4.2.2. So $G_1$ is connected, and is a path.

Let the subpath $P_1 = s^2 H v_6$ for the former case or $P_1 = v_2 H v_6$ for the latter case. Let $M_1$ be the set of all chords of $H$ on $P_1$ none of whose two endpoints are on $C_{g_4}$ except the chord $v_4 v_6$. Let $M = M_0 \cup M_1 \cup v_1 v_3$, and let $T = G - E(C_{g_4}) - E(M)$. Thus we get a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_4}$ and the matching $M$, see Figure 10.

From Theorem 2, we have the following corollary.

**Corollary 4.** Let $G$ be a connected cubic graph with $n$ vertices and girth at least $(n - 1)$. Then $G$ can be decomposed into a spanning tree, a 2-regular graph and a matching.
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